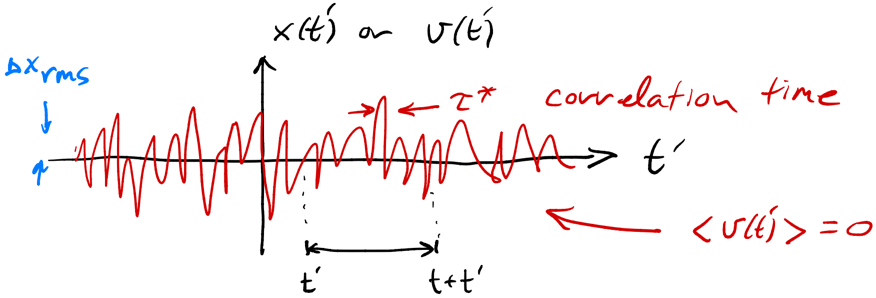
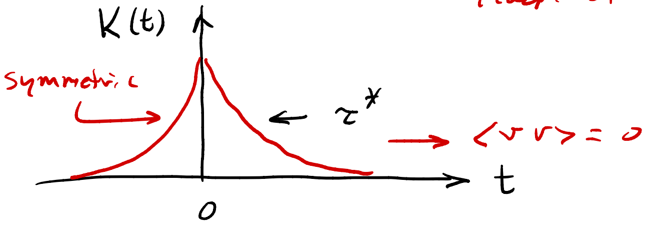
Noise & Noisy signals :



(auto) correlation function : $\langle v(t')\, v(t'+t) \rangle_0 \equiv K(t)$

indep. of t'

eq. ave



Diffusion

assume $x(0)=0$

position

$$x(t) - \cancel{x(0)}^{\,0} = \int_0^t v(t')\,dt'$$

velocity

$$\langle \Delta x^2(t)\rangle = \langle x^2(t)\rangle = \int_0^t dt' \int_0^t dt'' \langle v(t')v(t'')\rangle$$

$$= 2\int_0^t dt' \int_0^{t'} dt'' \langle v(t')v(t'')\rangle$$

$$\Rightarrow \frac{d}{dt}\langle x^2(t)\rangle = 2\int_0^t dt'' \langle v(t)v(t'')\rangle$$

$t'' \equiv t - t'''$

$t \gg \tau^*$

$$= 2\underbrace{\int_0^\infty dt''' \langle v(t)v(t-t''')\rangle_0}_{\text{const.} = D = \frac{kT}{6\pi\eta a}}$$

radius

viscosity

i.e., $\langle x^2(t)\rangle = 2Dt$ (1D)

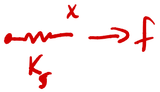
$\langle r^2(t)\rangle = 6Dt$ (3D)

Conceptual Problem:

x or $v \propto f$ the force acting on particle

e.g., $v = \mu f$ $\quad \mu = \frac{1}{6\pi\eta a}$ "mobility"

or $x = \frac{1}{K_s} f$



$x = \underbrace{\frac{1}{K_s}}_{\text{compliance}} f$

$\Rightarrow$ expect $x^2 \propto \frac{1}{\eta^2} f_B^2$

$\langle x^2(t) \rangle \propto \frac{1}{\eta} k_B T$ $\quad$ i.e. $\langle f_B^2 \rangle \propto \eta kT$

Note, if the viscosity is time/frequence dependent, then even Brownian forces are also time/frequence dependent.

Linear response:

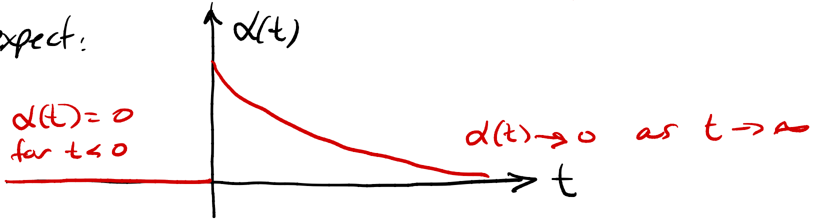
eg., $v = \mu f$ applied to ptcl.

in general $v(t) = \int_{-\infty}^{t \to \infty} \underbrace{\alpha(t-t')}_{} f(t')\, dt'$

response or "memory" fcn.

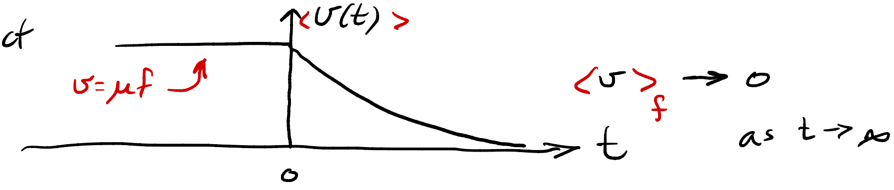
only earlier times

expect:



Example: consider $f(t') = \begin{cases} f & \text{for } t' < 0 \\ 0 & \text{for } t > 0 \end{cases}$ &

expect



In general, $f = f_a + f_B$

Brownian/thermal

active

Fourier transforms

$$x(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\,\tilde{x}(\omega)\,\frac{d\omega}{2\pi}$$

and

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\,x(t)\,dt$$

convolution

$$v(t) = \int_{-\infty}^{\infty} \alpha(t-t')\,f(t')\,dt'$$

$$\Rightarrow \quad \tilde{v}(\omega) = \tilde{\alpha}(\omega)\,\tilde{f}(\omega)$$

$$\tilde{\alpha}(\omega) = \alpha'(\omega) + i\alpha''(\omega)$$

power spectrum

purely $\mathbb{R}$

power spectrum

$$\tilde{v}(\omega)\,\tilde{v}^{*}(\omega) \in \mathbb{R}$$

and $$\tilde{K}_v(\omega) = \int e^{i\omega t}\,\underbrace{\langle v(t)\,v(0)\rangle_0}_{K_v(t)\ \leftarrow\ \mathbb{R}\ \&\ \text{even}}\,dt \;\propto\; \langle \overbrace{|\tilde{v}(\omega)|}^{}{}^{2} \rangle_0$$

$$\tilde{K}_f(\omega) \propto \langle |\tilde{f}(\omega)|^2 \rangle$$

$$\tilde{K}_v(\omega) = |\tilde{\alpha}_v(\omega)|^2\,\tilde{K}_f(\omega)$$

ie, $$\tilde{K}_f(\omega) = \frac{1}{|\tilde{\alpha}_v(\omega)|^2} \tilde{K}_v(\omega)$$

↗ spectrum of forces

not α_v

also, $$x(t) = \int \alpha_x(t-t') f(t') dt'$$

$$\tilde{x}(\omega) = \tilde{\alpha}_x(\omega) \tilde{f}(\omega)$$

$$\Rightarrow \boxed{\tilde{K}_f(\omega) = \frac{1}{|\alpha_x(\omega)|^2} \tilde{K}_x(\omega)}$$

But, note that

$$\int_{-\infty}^{\infty} e^{-i\omega t} \tilde{x}(\omega) \frac{d\omega}{2\pi}$$

since $v(t) = \frac{d}{dt} x(t)$, $\underline{\tilde{v}(\omega) = -i\omega \tilde{x}(\omega)}$

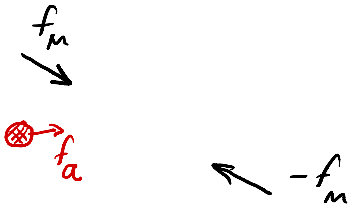
similarly,

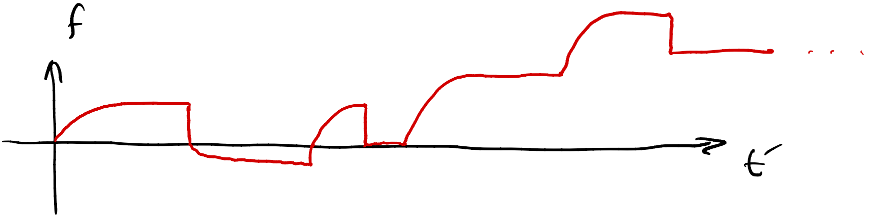
$$v(t) = \int \underbrace{\frac{d}{dt}\alpha_x(t-t')}_{\alpha_v(t-t')} f(t') dt'$$

& $$\underline{\tilde{\alpha}_v(\omega) = -i\omega \tilde{\alpha}_x(\omega)}$$

Model of Active (motor) force fluctuations

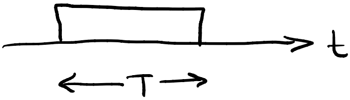
(w/ A. Levine)





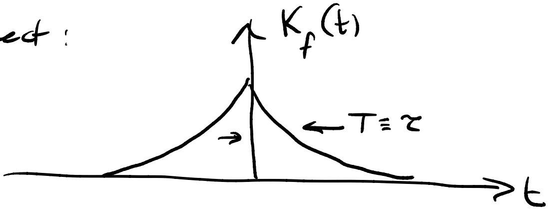
stochastic sum of individual events

$\simeq$



period of activity

Expect:



$$K_f(t) \simeq K_f^{(0)} e^{-|t|/\tau}$$

$$\tilde{K}_f(\omega) \simeq \frac{\tilde{K}_f(0)}{1+(\omega\tau)^2}$$

active only

In general, $f = \underbrace{f_a}_{\text{active}} + \overbrace{f_B}^{\text{Brownian}}$

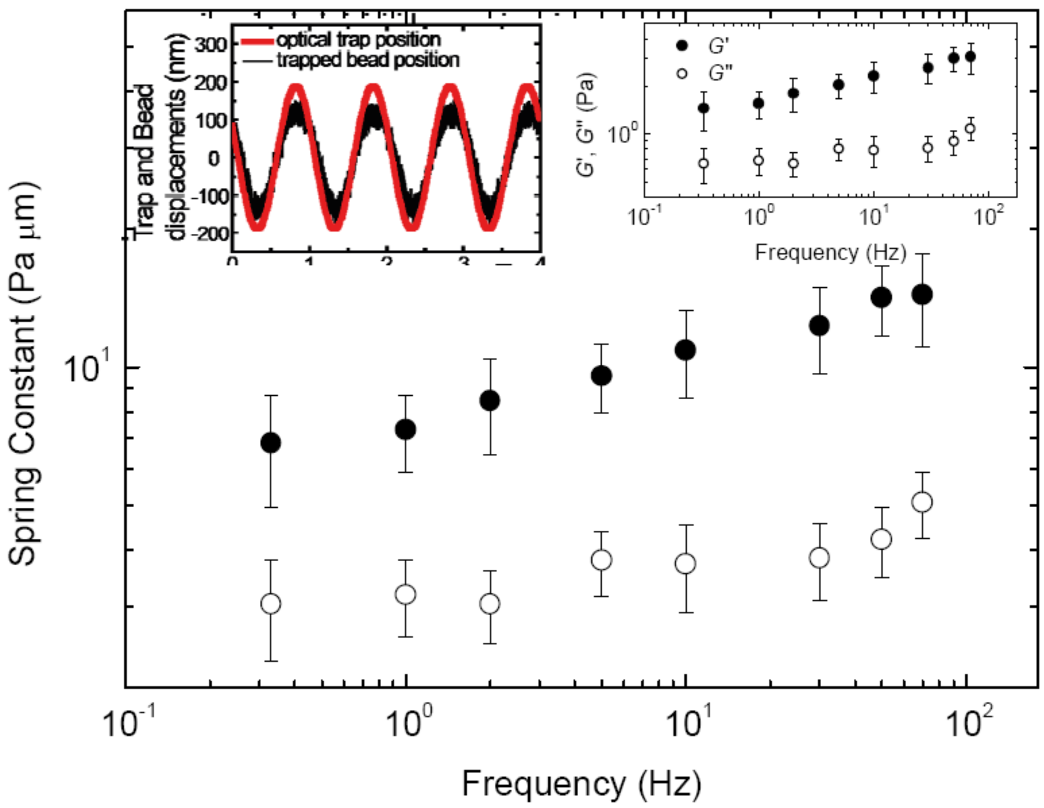
$$\langle f(t) f(0)\rangle = \langle f_a(t) f_a(0)\rangle + \langle \cancelto{0}{f_a(t) f_B(0)} \rangle$$

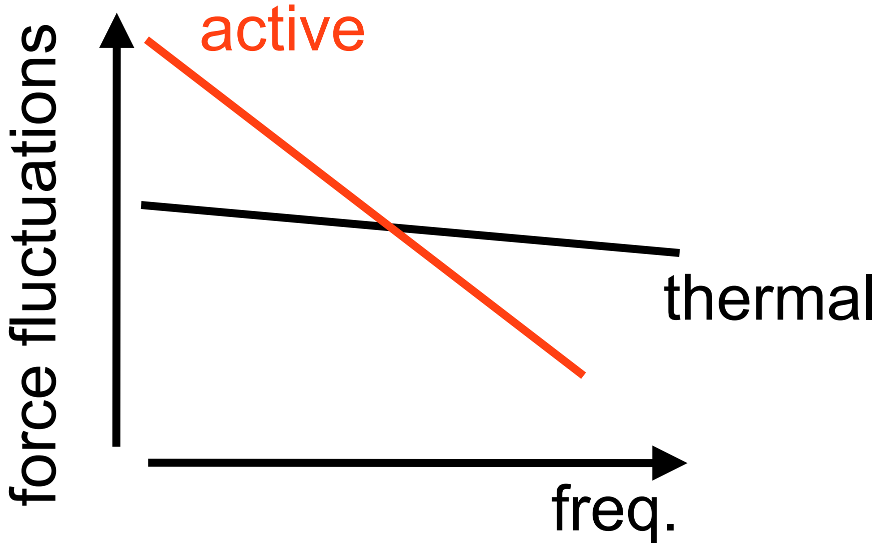
assume no correlation

$$+ \langle \cancelto{0}{f_B f_a} \rangle + \langle f_B(t) f_B(0)\rangle$$

$$\Rightarrow \quad \tilde{K}_f(\omega) = \tilde{K}_{f_a}(\omega) + \tilde{K}_{f_B}(\omega)$$

Cell activity & mechanics probed by injected beads





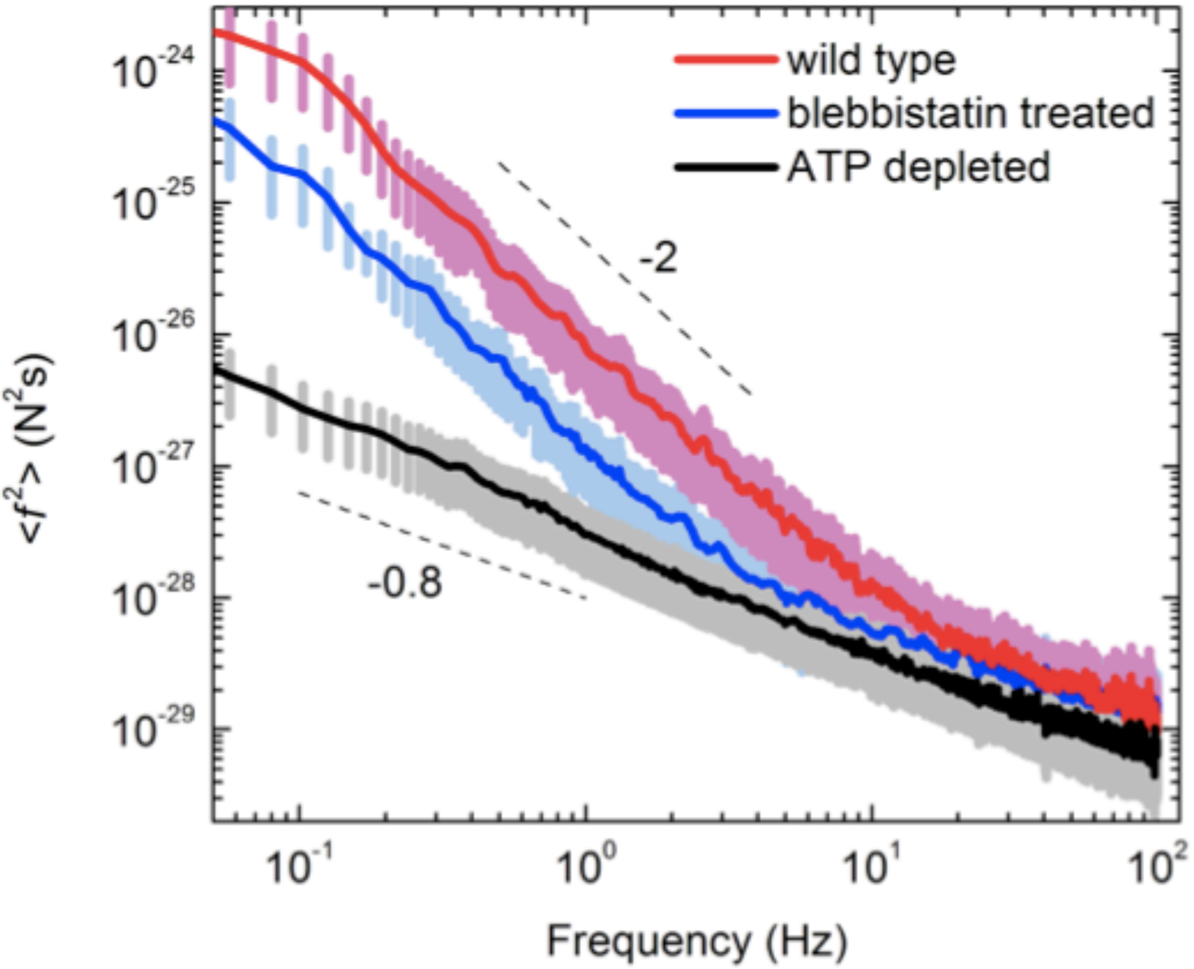
$$\langle |f_\omega|^2 \rangle \sim 1/\omega^2$$

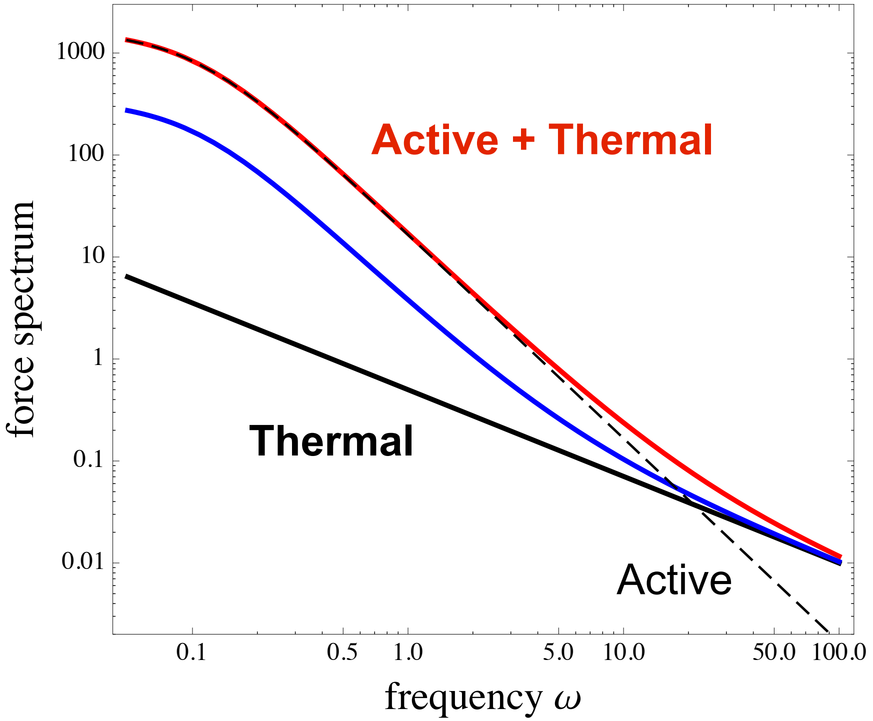
with A Levine, *PRL* 2008; *JPC* 2009

with Guo and Weitz, *Cell* 2014

see also Lau et al., *PRL* 2003 and Mizuno et al., *PRL* 2009.

Cell activity & mechanics probed by injected beads

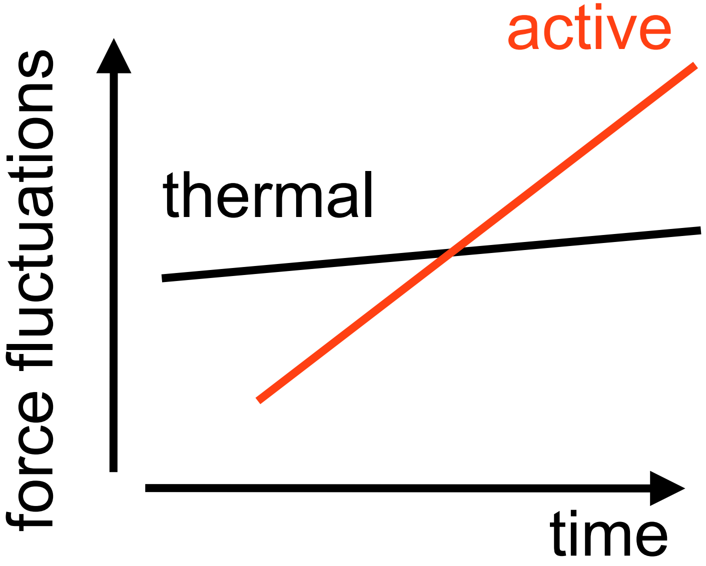




with Guo and Weitz, *Cell* 2014

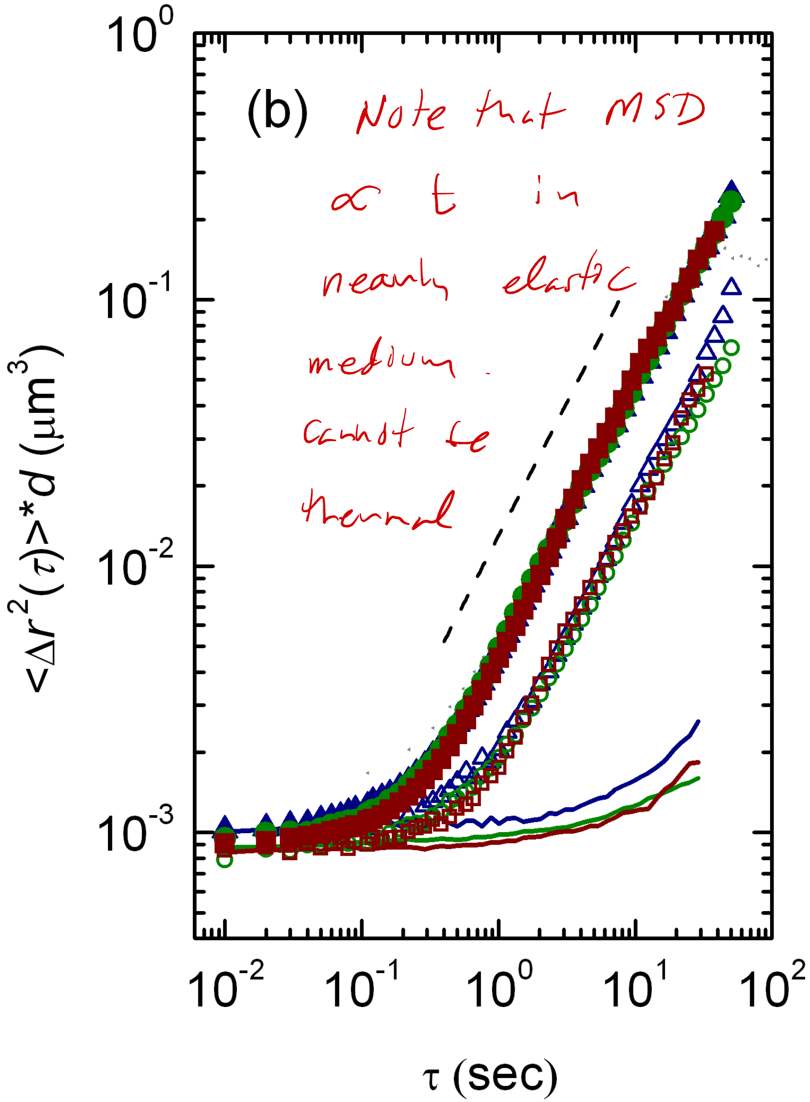
see also Lau et al., *PRL* 2003 and Mizuno et al., *PRL* 2009.

Cell activity & mechanics probed by injected beads



MSD

mean-square displacement



Note that MSD $\propto t$ in nearly elastic medium. cannot be thermal

with Guo and Weitz, *Cell* 2014

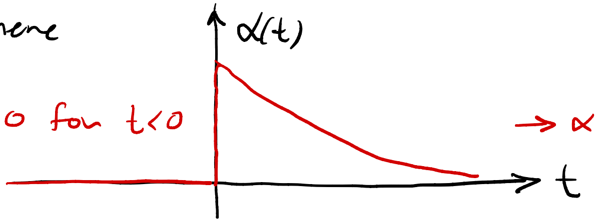
see also Lau et al., *PRL* 2003 and Mizuno et al., *PRL* 2009.

Back to linear response – thermal fluctuations

in general $$v(t) = \int_{-\infty}^{\infty} \alpha(t-t') f(t') \, dt' ,$$

where

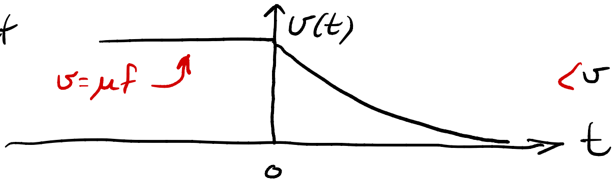
$\alpha \equiv 0$ for $t<0$



$\rightarrow \alpha \rightarrow 0$ for $t \rightarrow \infty$

Example: consider $$f(t') = \begin{cases} f & \text{for } t'<0 \\ 0 & \text{for } t>0 \end{cases} \quad \&$$

expect



$\langle v \rangle_f \rightarrow 0$ as $t \rightarrow \infty$

At $t=0^-$, $v = \mu f = \int_{-\infty}^{0} \alpha(0-t') \underset{\text{const.}}{f} \, dt' = f \int_0^{\infty} \alpha(t'') \, dt''$

$$\Rightarrow \quad \mu = \frac{1}{6\pi\eta a} = \int_0^{\infty} \alpha(t) \, dt$$

But, recall that $\int_0^\infty \langle v(t)\, v(0)\rangle_0 \, dt = D = \mu kT$

Thus, $$\int_0^\infty \alpha(t)\, dt = \frac{1}{kT} \int_0^\infty \langle v(t) v(0)\rangle_0 \, dt$$

response to $f \neq 0$
"perturbation"

fluctuations
in eq., with
$f = 0$

<u>Onsager regression hypothesis</u>:

following a perturbation, return to equilibrium is governed by <u>same dynamics</u> as fluctuations about equilibrium

Question: is $\alpha_v(t) = \frac{1}{kT} \underbrace{\langle v(t) v(0)\rangle_0}_{K(t)}$?

not quite:

$$\underbrace{\alpha_v(t) = \begin{cases} \frac{1}{k_B T} K(t) & t>0 \\ 0 & t<0 \end{cases}}_{FDT}$$

<u>f</u>luctuation <u>d</u>issipation <u>t</u>heorem

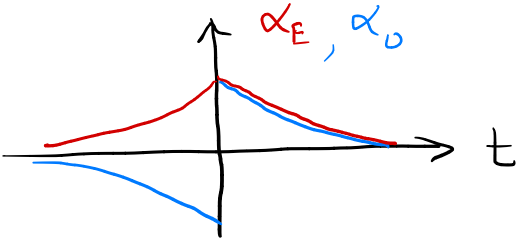
define $\alpha_E(t) = \frac{1}{2}\left(\alpha(t) + \alpha(-t)\right)$

$\alpha_O(t) = \frac{1}{2}\left(\alpha(t) - \alpha(-t)\right)$

"even"

"odd"

(recall $\alpha(t) = 0$ for $t<0$)



Note $\alpha(t) = \alpha_E + \alpha_O$

for $t>0$ $\quad 2\alpha_E(t) = \alpha(t) = \frac{1}{kT}K(t)$

In fact, $\quad 2\alpha_E(t) = \frac{1}{kT}K(t)$

for all t, since K is even

$$\Rightarrow \quad 2kT\tilde{\alpha}_E(\omega) = \tilde{K}(\omega)$$

Properties of Fourier transforms

if $f(t)$ is $\mathbb{R}$ and even,

then $\tilde{f}(\omega)$ is purely real

if $f(t)$ is $\mathbb{R}$ odd, then

$\tilde{f}(\omega)$ is purely imaginary

I.e., $\tilde{\alpha}_E(\omega) = \alpha'(\omega)$ some $\mathbb{R}$ fcn. α'

and $\tilde{\alpha}_O(\omega) = i\underbrace{\alpha''}(\omega)$ some $\mathbb{R}$ fcn α''

thus, $\tilde{\alpha}(\omega) = \alpha'(\omega) + i\alpha''(\omega)$

and

$$\underbrace{2kT\tilde{\alpha}'(\underset{v}{\omega}) = \tilde{K}(\underset{v}{\omega})}_{FDT}$$

if thermal

Careful: this is not the same for α_x and K_x.

As noted above, $\tilde{v} = -i\omega\tilde{x}$,

so $\tilde{K}_v(\omega) = \omega^2 \tilde{K}_x(\omega)$

and $\tilde{\alpha}_v = -i\omega\tilde{\alpha}_x = \alpha'_v + i\alpha''_v$

Thus, $\alpha'_v(\omega) = \omega\alpha''_x(\omega)$, where

$$\tilde{\alpha}_x = \underbrace{\alpha'_x}_{\text{real part}} + i\underbrace{\alpha''_x}_{\text{imaginary part}}$$

$$\Rightarrow \underbrace{\frac{2kT\,\alpha''_x(\omega)}{\omega} = \tilde{K}_x(\omega)}_{\text{equivalent expression of FDT}}$$

What about f_B (Brownian) ?

$$2kT\,\alpha_v'(\omega) = \tilde{K}_v(\omega) = |\tilde{\alpha}_v(\omega)|^2\,\tilde{K}_{f_B}(\omega)$$

from above

$$\Rightarrow \quad \tilde{K}_{f_B}(\omega) = 2kT\,\frac{\alpha_v'(\omega)}{|\tilde{\alpha}_v(\omega)|^2} \propto kT$$

as expected, but this is ω-dependent in general

$$= 2kT \cdot \frac{\alpha_x''(\omega)}{\omega\,|\tilde{\alpha}_x(\omega)|^2}$$

Consider a Brownian particle in a harmonic potential and viscous liquid:

$$\cancelto{0}{m\frac{d^2}{dt^2}}x(t) = -K_s\,x - \zeta\frac{d}{dt}x + f$$

ignore inertia that is only relevant at short times

spring constant

drag coefficient

$\zeta = 6\pi\eta a$ for radius a

applied force

$$\Rightarrow \quad (K_s - i\omega\zeta)\tilde{x} = \tilde{f} \qquad \text{after Fourier trans.}$$

i.e.,
$$\tilde{\alpha}_x(\omega) = \frac{1}{K_s - i\omega\zeta} = \frac{K_s + i\omega\zeta}{K_s^2 + \omega^2\zeta^2}$$

Thus,
$$\alpha_x''(\omega) = \frac{\omega\zeta}{K_s^2 + \omega^2\zeta^2} = \omega\zeta\,|\tilde{\alpha}_x(\omega)|^2$$

&
$$\tilde{K}_{f_B}(\omega) = 2kT\zeta = 12\pi\eta a kT$$

Note that this is independent of K_s and ω!

Also,
$$\langle f_B(t) f_B(0)\rangle = 12\pi\eta a kT \cdot \delta(t)$$
$\nearrow$ δ-fcn.

This force spectrum is "white noise",

but for a general viscoelastic material,

$\tilde{K}_{f_B}$ can be expected to be ω-dependent

or "colored noise".

References :

- For Linear response theory and the Fluctuation Dissipation Theorem, see "Intro. to Modern Statistical Mechanics" by David Chandler

  (most other Physics refs. prove this using Quantum Mechanics which is not necessary)

  Also, I have lecture notes from a class (with Daan Frenkel) I can send.

- For models of cytoskeletal activity, see
  - Lau et al., PRL 91 : 198101 (2003).
  - MacKintosh & Levine, PRL 100 : 018104 (2008)
  - Levine & MacKintosh, J Phys Chem, 113 : 3820 (2009)

- For the experiments presented above, see:
  - Guo et al., Cell <u>158</u>: 822 (2014)
- Related work can also be found in:
  - Mizuno et al., Science <u>315</u>: 370 (2007)
  - Mizuno et al., PRL <u>102</u>: 168102 (2009)
  - Fakhri et al., Science <u>344</u>: 1031 (2014)