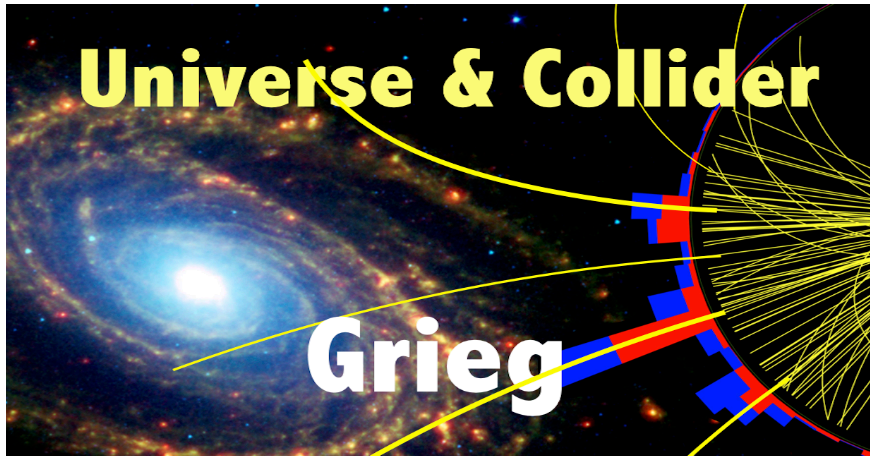
Quantum theory of dark matter scattering

Ayuki Kamada (University of Warsaw)



Based on
AK, Hee Jung Kim and Takumi Kuwahara, JHEP, 2020
AK, Takumi Kuwahara and Ami Patel, arXiv:2303.17961

June 15, 2023 @ PPC 2023

Contents

Dark matter phenomenology

- long-range force
- Sommerfeld enhancement and self-scattering

Scattering state of quantum mechanics

- different limits of single state determine the above two
- tight correlation is expected and indeed found

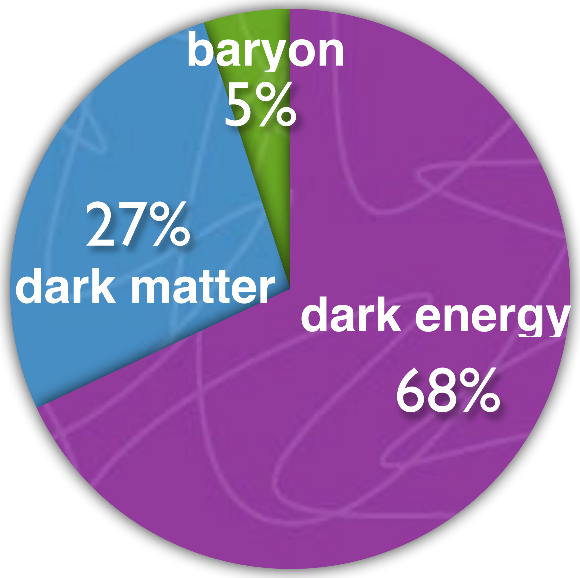
Formulation of the correlation

- Watson’s theorem and Omnès solution
- effective range theory around resonances
- Levinson’s theorem

Dark matter

Dark matter

- evident from cosmological observations
    - cosmic microwave background (CMB)…
- one of the biggest mysteries
    - astronomy, cosmology, particle physics…



cosmic energy budget

Long-range force

- mediator lighter than the dark matter
- electroweak-scale or lighter dark matter
    - new dark force (e.g., dark photon)
- TeV-scale dark matter (e.g., weak multiplet)
    - weak force

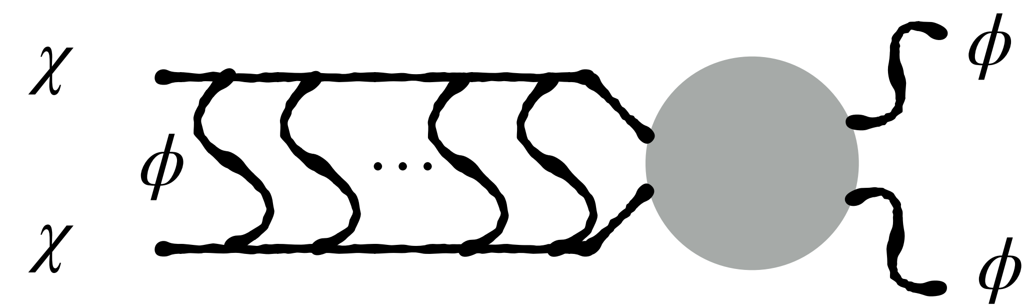
$$V = -\frac{\alpha_\chi}{r} e^{-m_\phi r}$$

- Yukawa potential

Sommerfeld enhancement

Distortion of wave function

- multiple exchanges of a mediator
- non-perturbative but described by the Schrödinger equation (later)



Enhanced annihilation

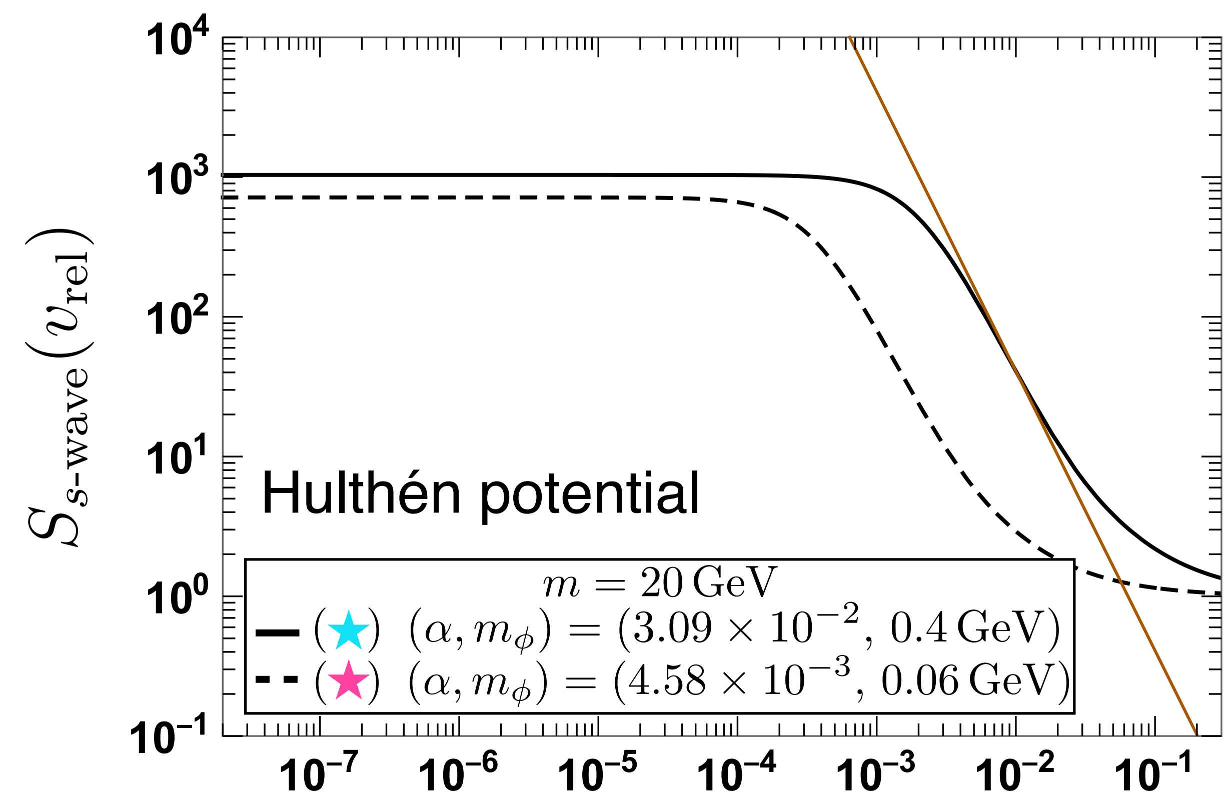
- annihilation cross section is enhanced at low velocity

$$(\sigma_{\rm ann} v_{\rm rel}) = S(\sigma^{(0)}_{\rm ann} v_{\rm rel})$$

- without potential

- Sommerfeld enhancement factor

- larger cross section in the late Universe than the thermal one



AK, Kim and Kuwahara, JHEP, 2020 $v_{\rm rel}/c$

Indirect detection

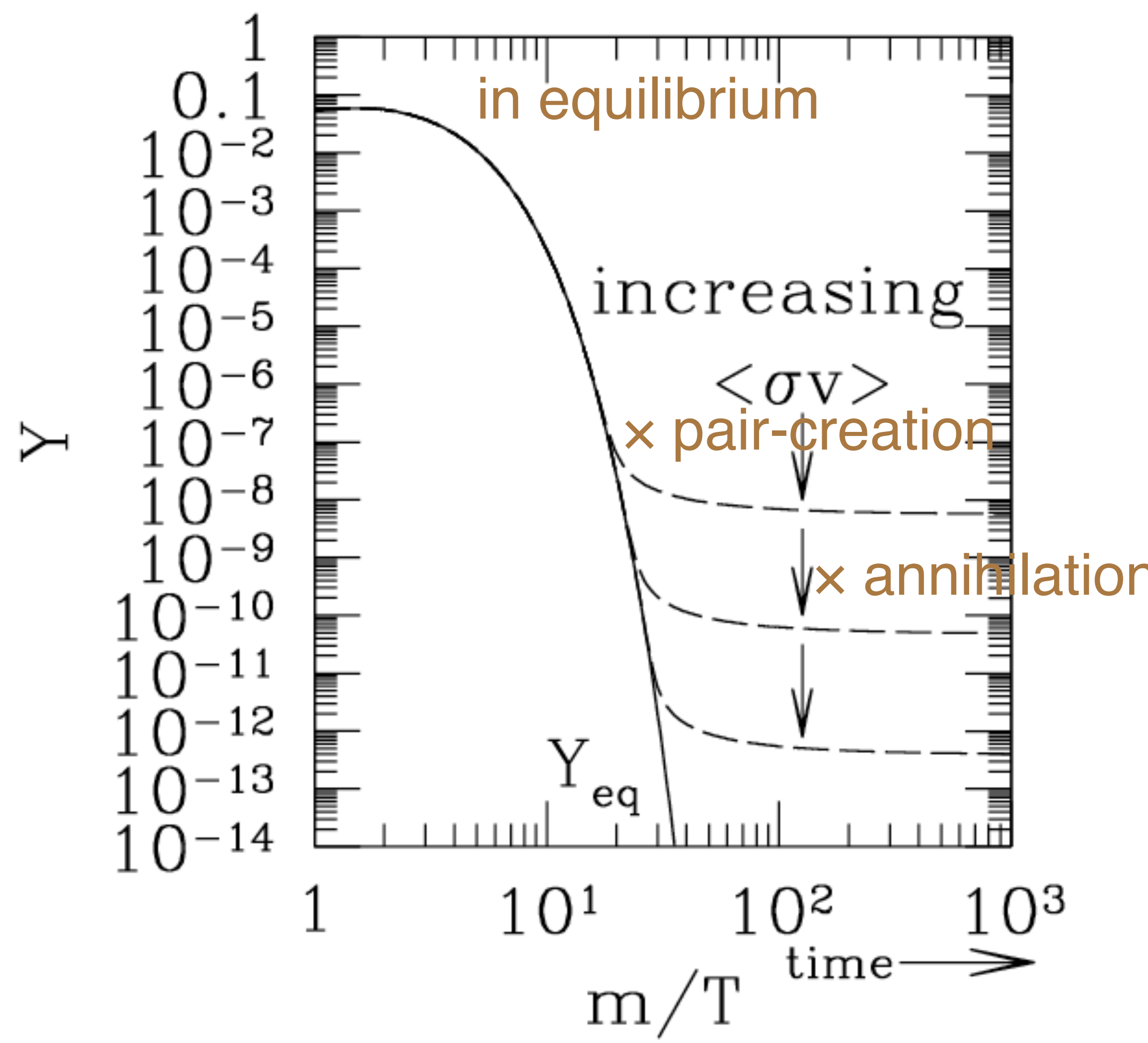
Canonical cross section

- thermal freeze-out (annihilation in the early Universe) $v_{\rm rel} \simeq 1/2$

$$\Omega h^2 = 0.1 \times \frac{3 \times 10^{-26}\,{\rm cm}^3/{\rm s}}{\langle \sigma_{\rm ann} v \rangle}$$

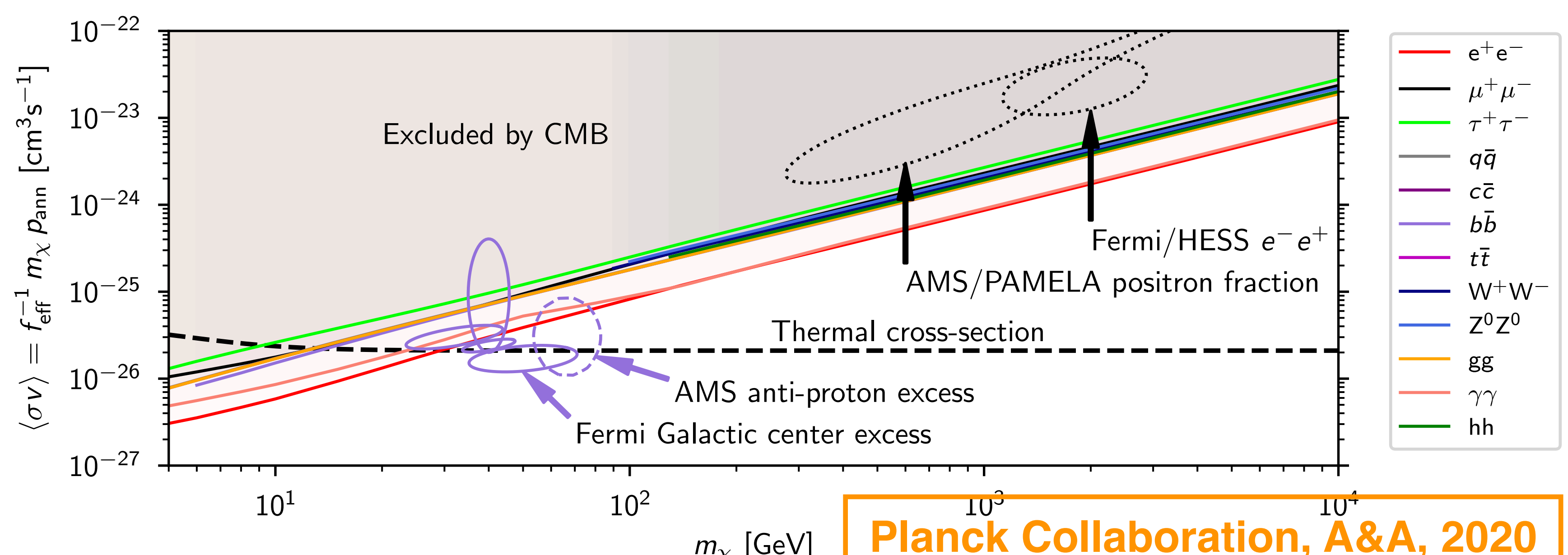
- requires a weak-scale annihilation cross section

$$\langle \sigma_{\rm ann} v \rangle \simeq 1\,{\rm pb} \times c$$



CMB constraints

- energy deposit around the last scattering



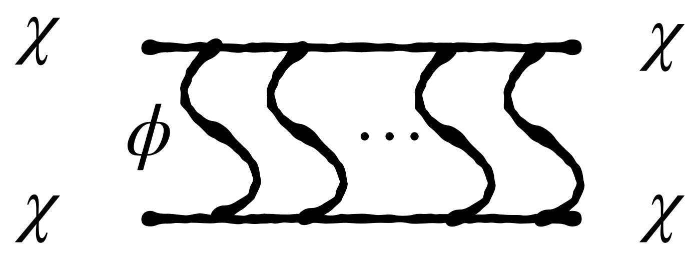
Planck Collaboration, A&A, 2020

Self-scattering

The same light mediator

- non-perturbative (multiple exchanges) when the distortion of wave function is significant

- again described by the Schrödinger equation (later)

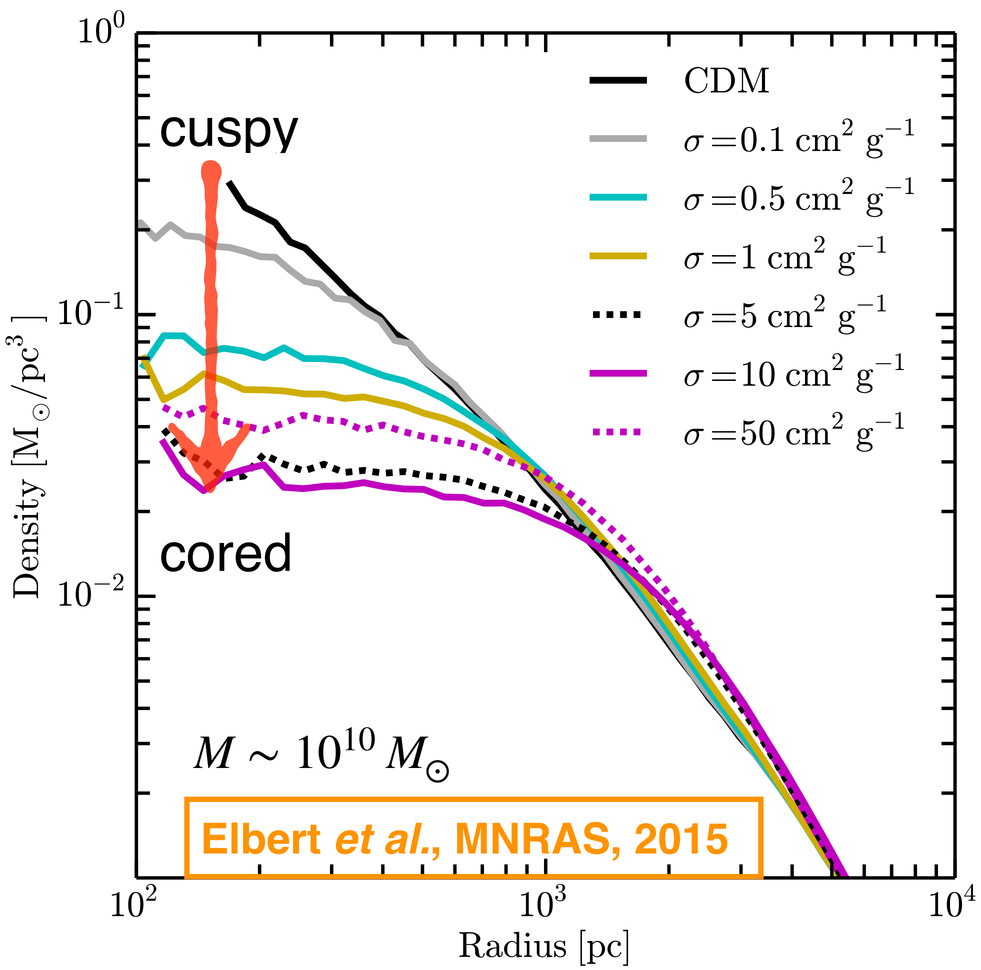


Self-interacting dark matter

- interactions among dark matter particles

$$\sigma/m \sim 1\,\mathrm{cm}^2/\mathrm{g} \sim 1\,\mathrm{barn}/\mathrm{GeV}$$

- dark matter density profile inside a halo turns from cuspy to cored



Velocity dependence

Self-interacting dark matter

- cored profile “appear to” provide better fit to astronomical data

- “data” points from astrophysical observations of various size halos

Light mediator

- introduce a velocity dependence, which is compatible with “data”

- MW satellites

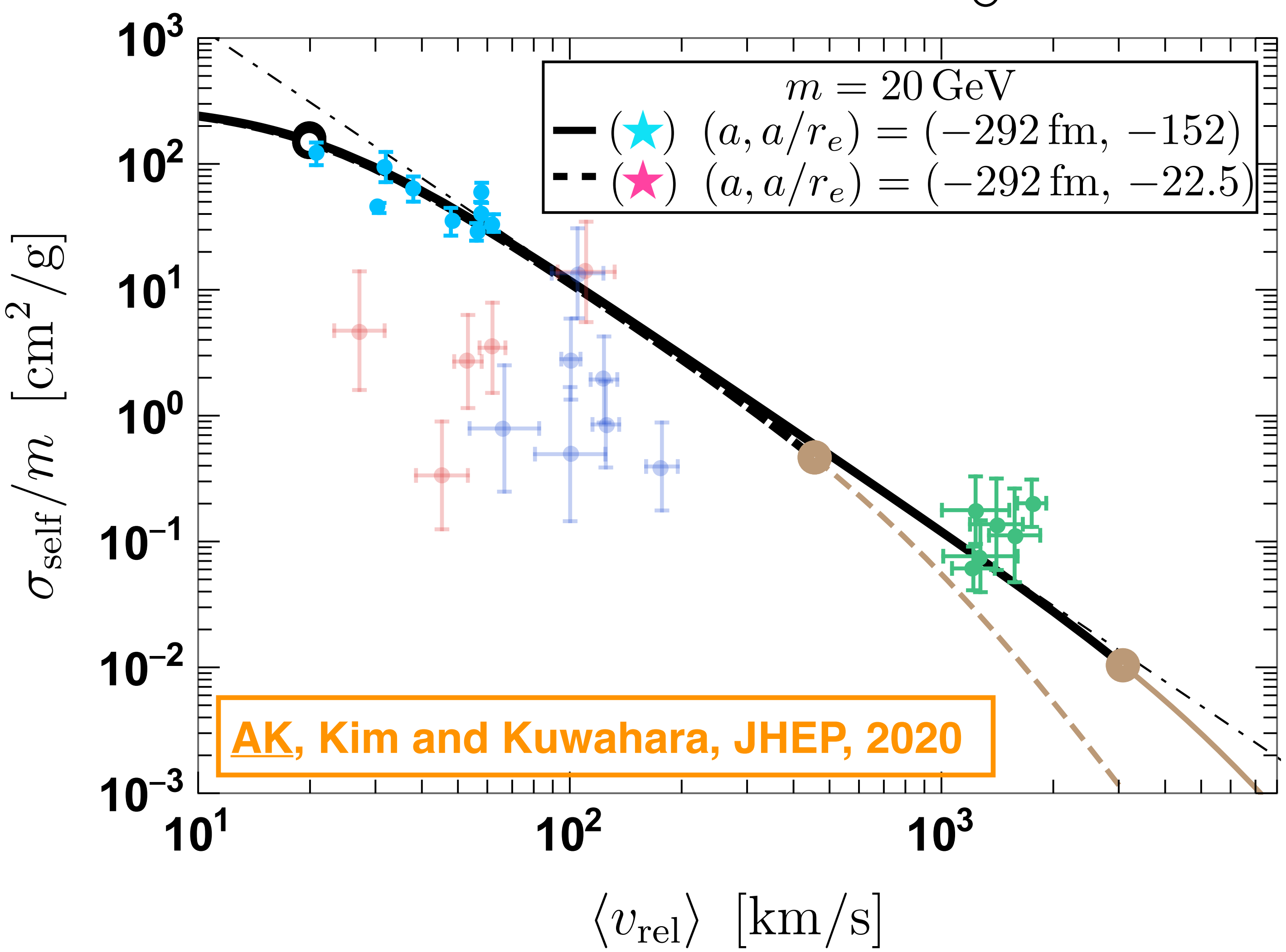
$M_{\rm infall} \sim 10^9\, M_\odot$

- dwarf spiral galaxies

$M \sim 10^{11}\, M_\odot$

- galaxy clusters

$M \sim 10^{14}\, M_\odot$

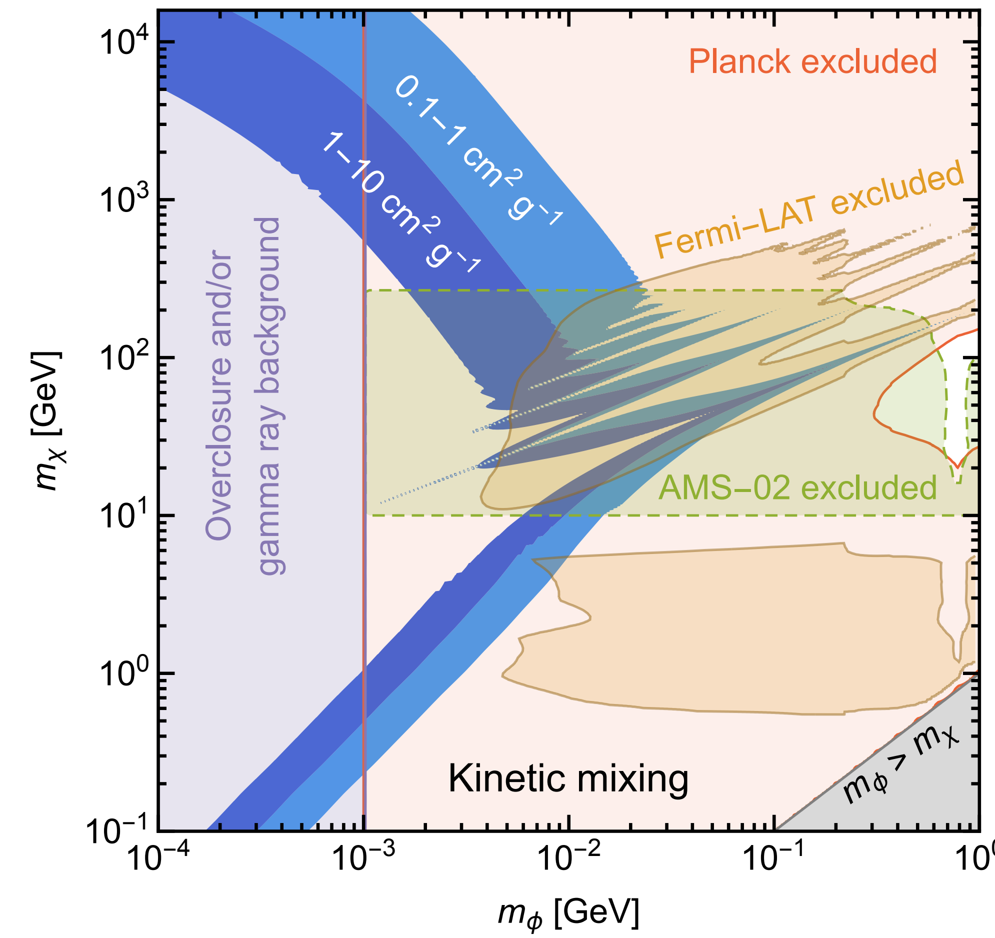


Correlation

Bringmann, Kahlhoefer, Schmidt-Hoberg and Walia, JHEP, 2020

Sommerfeld enhancement and self-scattering

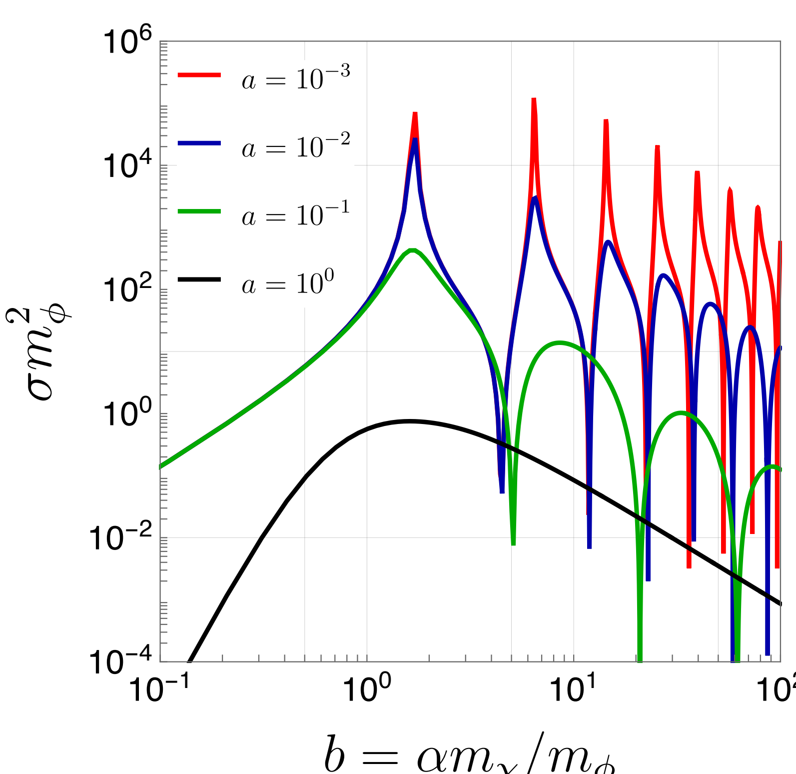
- some correlation is known
    - main obstacle in SIDM model building
    - resonant enhancement occurs at the same parameter point

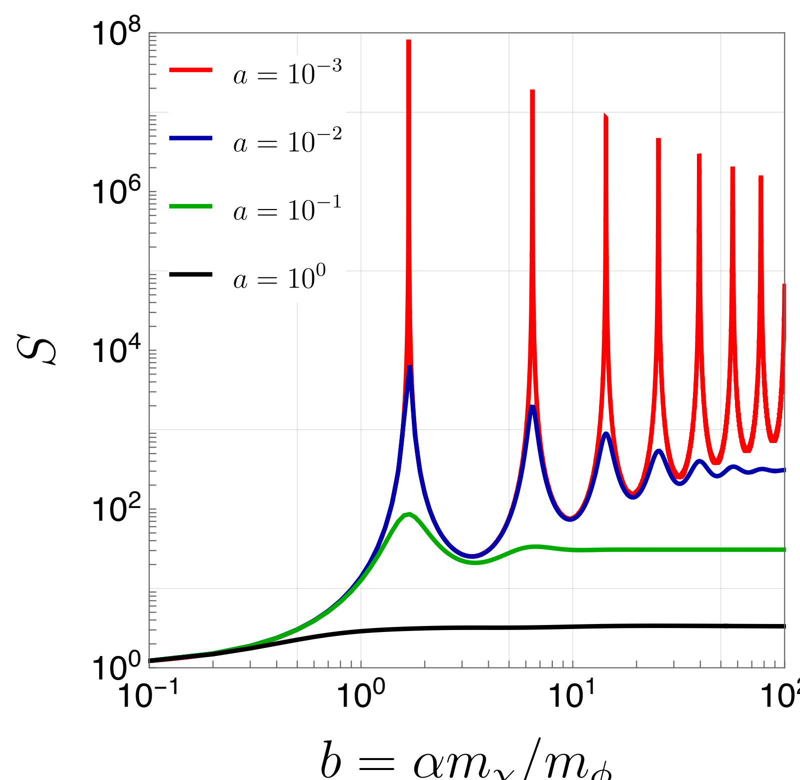


- dark photon

$$a = \frac{v_{\rm rel}}{2\alpha_\chi}$$

$$b = \frac{\alpha_\chi m_\chi}{m_\phi}$$





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Scattering in quantum mechanics

Schrödinger equation

Weinberg, "Lectures on Quantum Mechanics"

$$\left[-\frac{1}{2\mu}\nabla^2 + V(r)\right]\psi_k(\vec{x}) = E\psi_k(\vec{x}) \qquad E = \frac{k^2}{2\mu} \qquad k = \mu v_{\rm rel}$$

- potential from long-range force

- reduced mass ($\mu = m/2$ for identical particle)

- scattering state (energy-eigenstate of Schrödinger equation)

$$\psi_k(\vec{x}) \to e^{ikz} + f(k,\theta)\frac{e^{ikr}}{r} \qquad r \to \infty$$

- (in-coming) plane wave
- scattering amplitude
- out-going spherical wave

Partial-wave decomposition

$$e^{ikz} = \sum_{\ell=0}^{\infty} \frac{1}{2ikr}(2\ell+1)R_{k,\ell}(r)P_\ell(\cos\theta)\left(e^{ikr} - e^{-i(kr-\ell\pi)}\right)$$

$$\psi_k(\vec{x}) = \sum_{\ell=0}^{\infty} \frac{1}{k} e^{i\left(\frac{1}{2}\ell\pi + \delta_\ell\right)}(2\ell+1)R_{k,\ell}(r)P_\ell(\cos\theta)$$

- radial Schrödinger equation

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2} - 2\mu V(r)\right]R_{k,\ell}(r) = 0$$

Sommerfeld enhancement and self-scattering

Scattering phase

- radial wave function at infinity

$$R_{k,\ell}(r) \to \frac{\sin(kr - \frac{1}{2}\ell\pi + \delta_\ell)}{r} \quad r \to \infty$$

$$f(k,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(k) P_\ell(\cos\theta) \qquad f_\ell(k) = \frac{e^{2i\delta_\ell} - 1}{2ik}$$

$$\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell \qquad \sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell(k)$$

- diagonalized S-matrix $S_\ell = e^{2i\delta_\ell}$

Sommerfeld enhancement

Iengo, JHEP, 2009 | Cassel, J.Phys.G, 2010

- radial wave function around the origin

- annihilation through the contact interaction (delta function potential)

$$S_{k,\ell} = \left| \frac{R_{k,\ell}(r)}{R^{(0)}_{k,\ell}(r)} \right|^2 \quad r \to 0$$

- without potential

Correlation

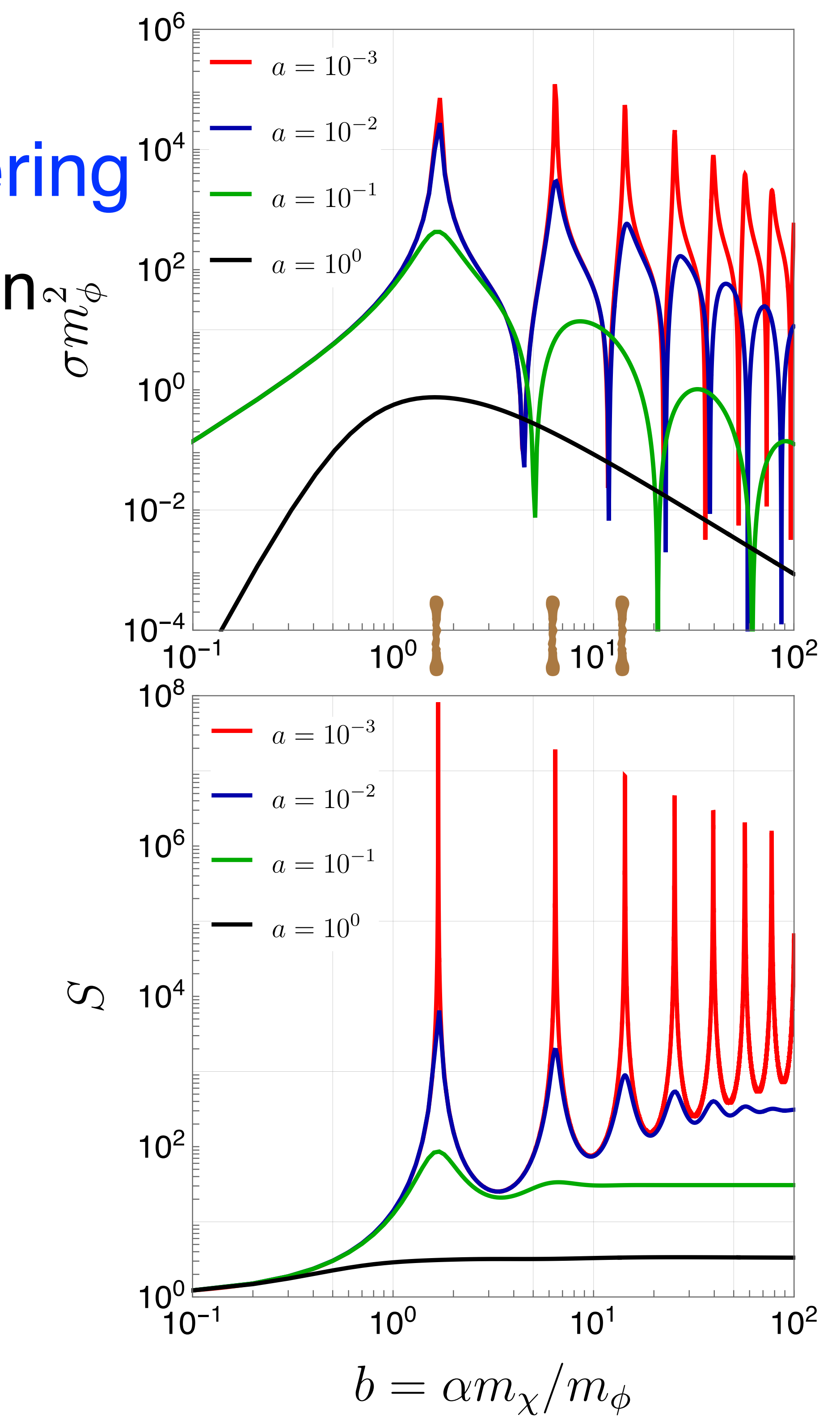
Sommerfeld enhancement and self-scattering

- determined by a single radial wave function

- not surprising that we see a correlation

    - resonances for the same parameter

- **still we want to formulate the direct relation**



Remarks

- hereafter ignore a contact interaction in the Schrödinger equation

    - not a problem unless the wave function is localized around the origin

    - at resonances (later) and quite small velocities, we need to take it into account; otherwise Unitarity is violated **Blum, Sato and Slatyer JHEP, 2016**

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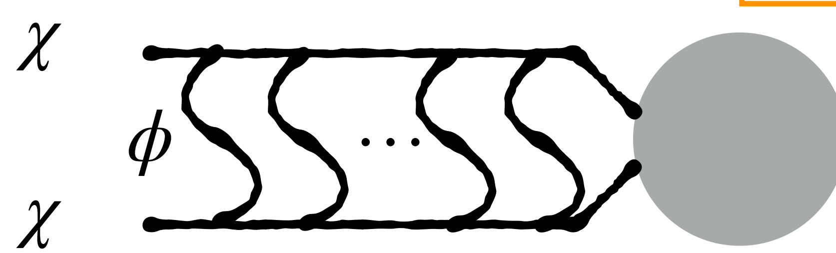
Watson theorem

Oller, "A Brief Introduction to Dispersion Relations"

Annihilation matrix element

$$\Gamma_\alpha(k^2 + i\epsilon) = \langle 0 | \Theta_\chi | \psi^+_{\alpha,k} \rangle$$



- in-state as a whole

- inserting out states

$$\Gamma_\alpha(k^2 + i\epsilon) = \sum_\beta S_{\beta\alpha}(k) \langle 0 | \Theta_\chi | \psi^-_{\beta,k} \rangle = \sum_\beta S_{\beta\alpha}(k) \Gamma_\beta(k^2 - i\epsilon)$$

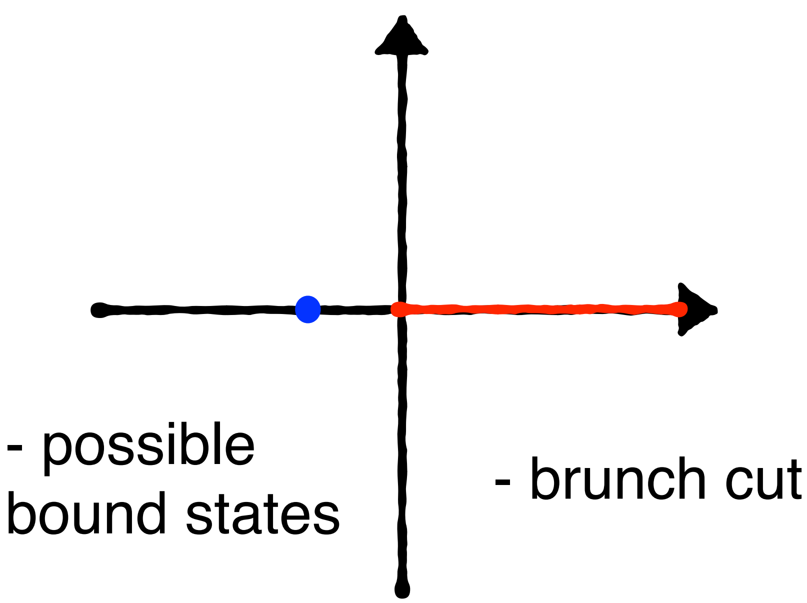
- assuming the real matrix element (T-invariance)

$$\Gamma_\alpha(k^2 + i\epsilon) = \sum_\beta S_{\beta\alpha}(k) \Gamma_\beta(k^2 + i\epsilon)^*$$

- for a partial wave

$$\Gamma_\ell(k^2 + i\epsilon) = e^{2i\delta_\ell} \Gamma_\ell(k^2 + i\epsilon)^*$$

- complex k^2 plane



Omnès solution

Omnès function

$$\Omega_\ell(k^2) = \exp[\omega_\ell(k^2)] \qquad \omega(k^2) = \frac{1}{\pi}\int_0^\infty dq^2 \frac{\delta_\ell(q)}{q^2 - k^2}$$

- principal value

- computed by phase shift and reproduce the brunch cut

$$F_\ell(k^2) = \prod_{b_\ell} \frac{k^2}{k^2 + \kappa_{b,\ell}^2}$$

- rational function reproducing bound-state poles (Levinson theorem)

$$\Gamma_\ell(k^2) = \Omega_\ell(k^2) F_\ell(k^2)$$

- from Liouville theorem

- we normalize $\delta_\ell(k) \to 0 \quad \Gamma_\ell(k^2) \to 1 \;\; k^2 \to \infty$

- scattering phase and Sommerfeld enhancement are negligible at high velocity

Sommerfeld enhancement

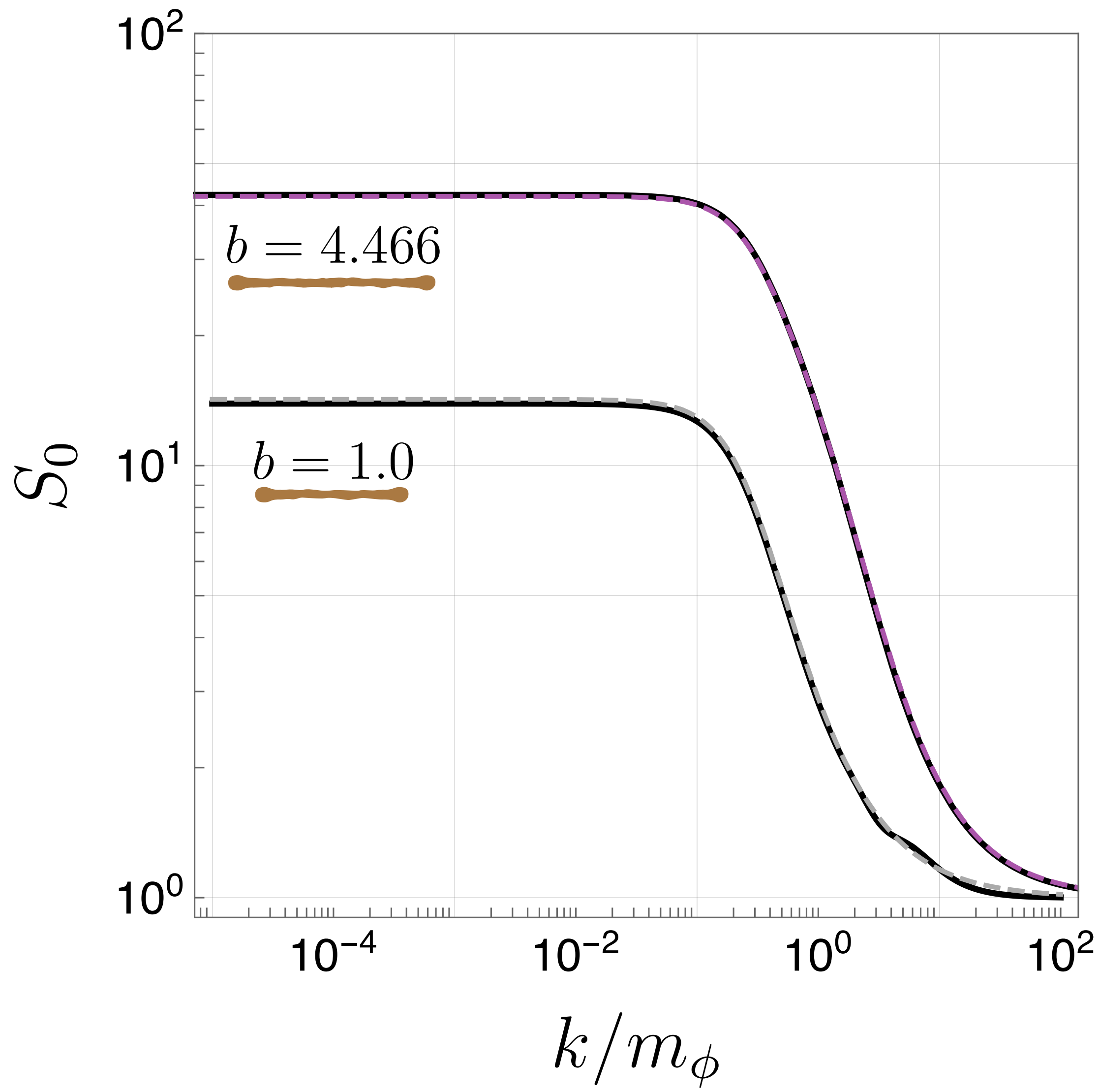
$$S_\ell = |\Gamma_\ell(k^2)|^2$$

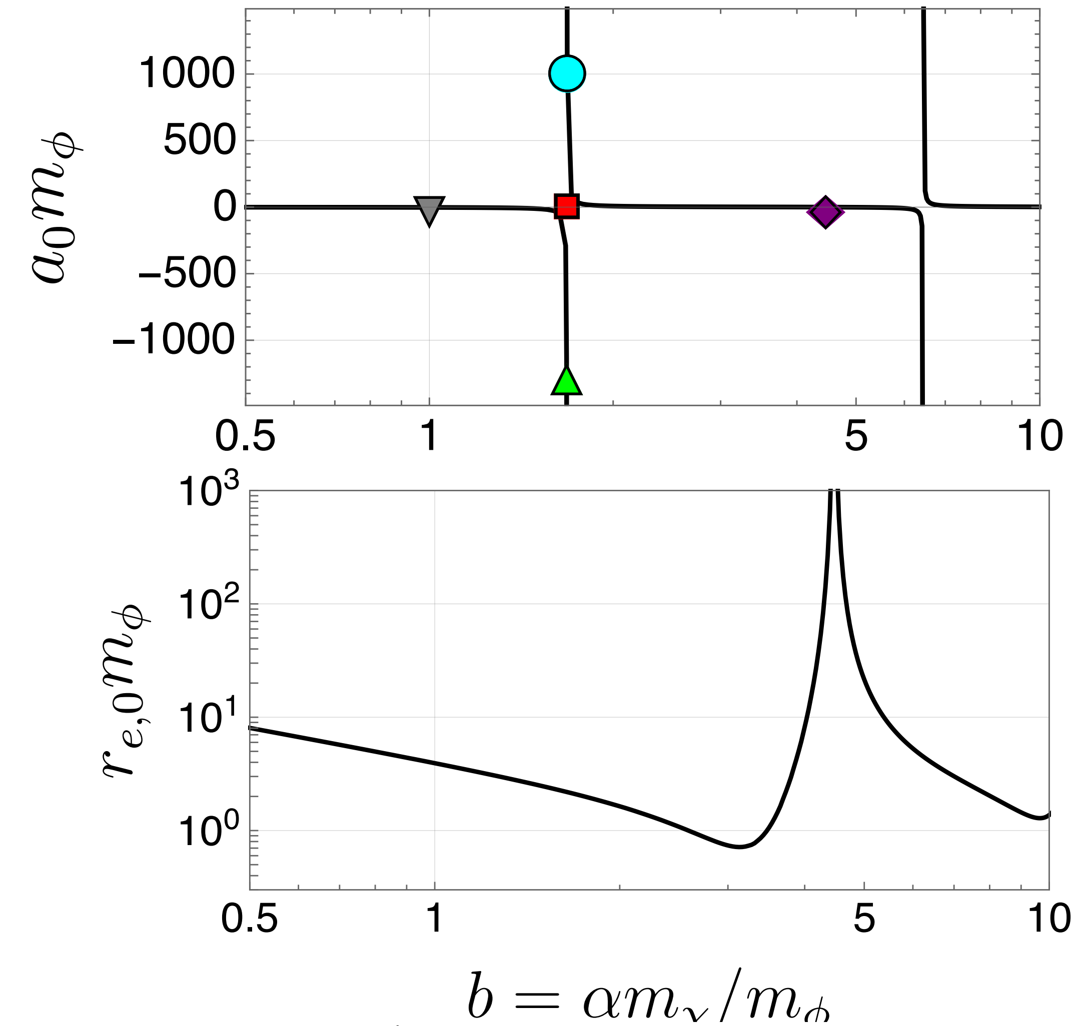
Omnès solution

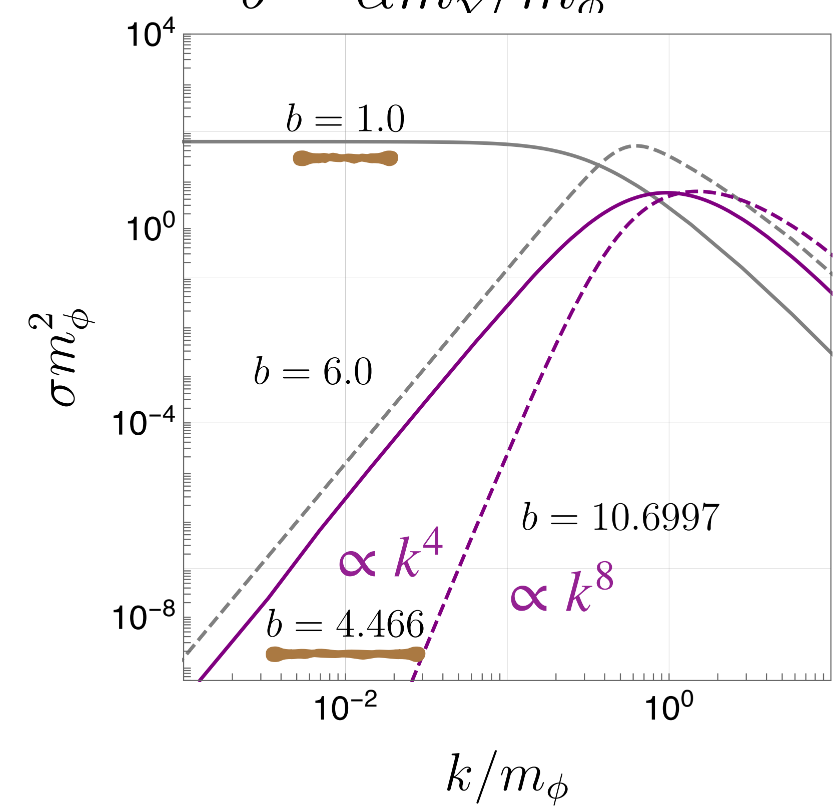
Yukawa potential

AK, Kuwahara and Patel, arXiv:2303.17961

- s-wave







- Omnès solution agrees with direct computation from scattering state

- with proper $F_0(k^2)$ (later)

Around resonances

AK, Kuwahara and Patel, arXiv:2303.17961

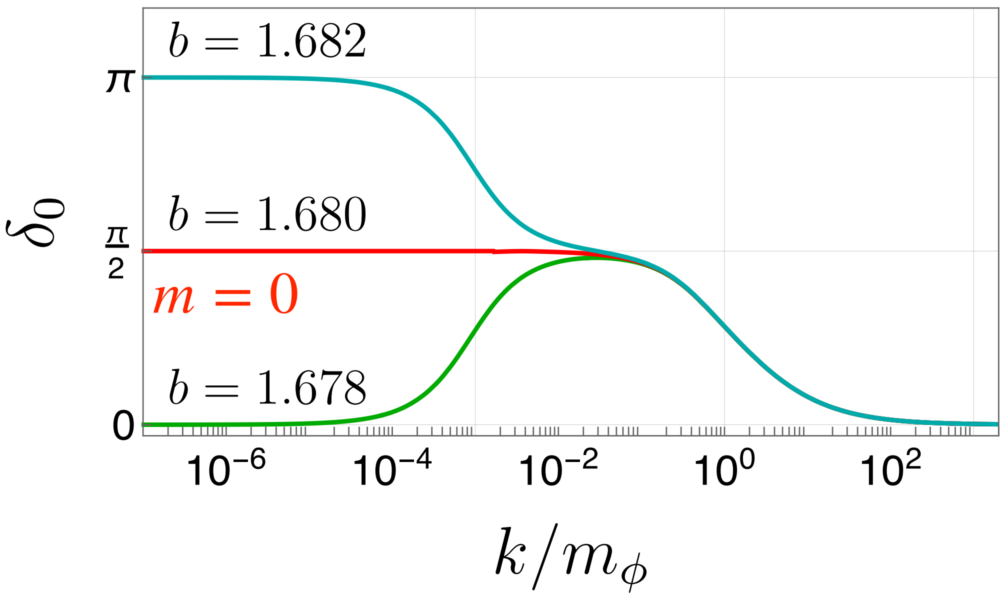
Effective range theory

- s-wave resonances

$$k \to 0 \quad k\cot\delta_0 \to -\frac{1}{a_0} + \frac{r_{e0}}{2}k^2$$

$$a_0 \to \infty$$

$$k \to 0 \quad \delta_0 \to \left(\frac{1}{2} + m\right)\pi \quad m = 0, 1, 2\ldots$$



- Omnès function

for m=0 (later)

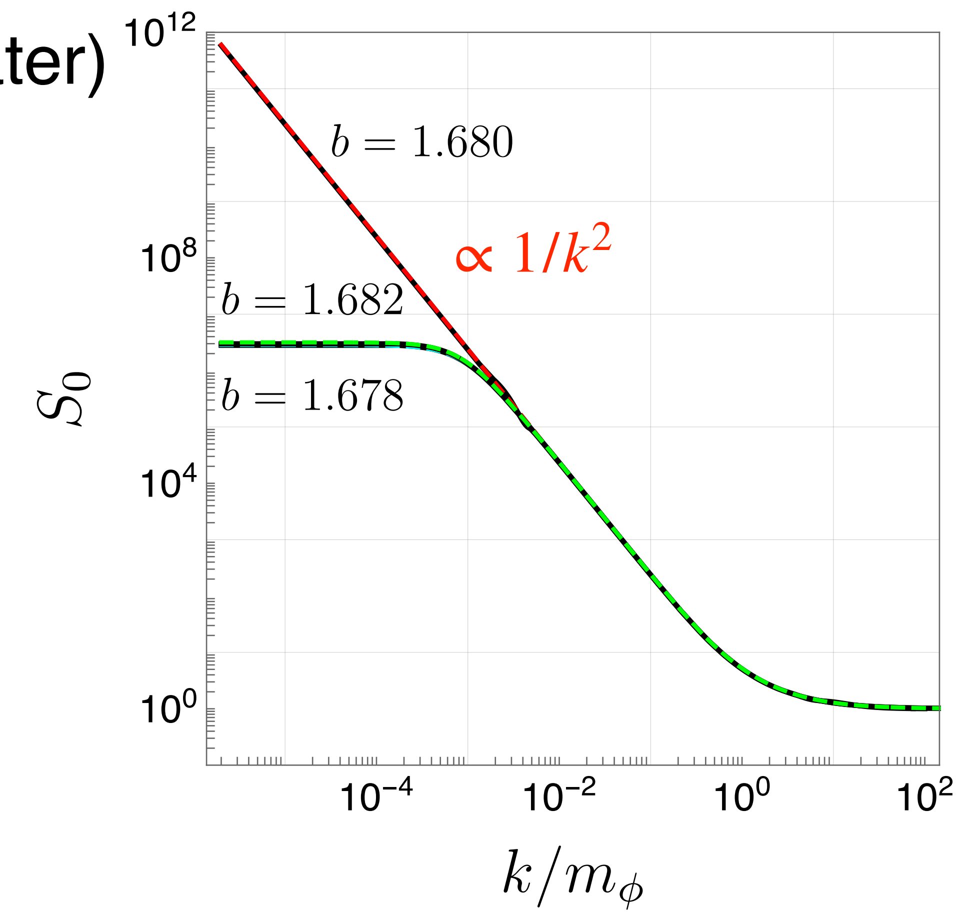
$$\omega_0(k^2) = \frac{1}{\pi}\int_0^\infty dq^2 \frac{\delta_0(q)}{q^2 - k^2}$$

$$F_0(k^2) = 1$$

$$k \to 0 \quad \to -\left(\frac{1}{2} + m\right)\ln(r_{e,0}^2 k^2)$$

$$\Gamma_0(k^2) = \exp[\omega_\ell(k^2)]F_0(k^2) \qquad S_0 = |\Gamma_0(k^2)|^2$$

$$k \to 0 \quad \to \frac{F_0(k^2)}{k^{1+2m}}$$



Summary

Long-range force of dark matter

- Sommerfeld enhancement and self-scattering cross section
  - indirect detection and structure formation
- the two are known to be correlated
  - they are determined by a single wave function

This talk

- we formulate the direct relation between the two
  - Watson theorem and Omnès solution
- we discuss how we can understand the velocity dependence around the resonances by using our formulation
  - effective range theory and Levinson theorem

Thank you

Effective range theory

Analyticity of scattering amplitude

Chu, Garcia-Cely and Murayama, JCAP, 2020

$$f_\ell = \frac{1}{k \cot \delta_\ell - ik}$$

- effective range theory

$$k \to 0 \quad k^{2\ell+1} \cot \delta_\ell \to -\frac{1}{a_\ell^{2\ell+1}} + \frac{1}{2 r_{e,\ell}^{2\ell-1}} k^2$$

- scattering length
- effective range

- $f_\ell \propto k^\ell k^\ell$ to make $f(\vec{k})$ an analytic function around $k = 0$
  - initial $\quad \ell = 1 \quad k \cos\theta = k_z$
  - final

- higher partial-wave is suppressed at low energy

Effective range theory

Yukawa potential

AK, Kuwahara and Patel, in preparation

- s-wave

$$k \cot \delta_0 \to -\frac{1}{a_0} + \frac{r_{e,0}}{2} k^2$$

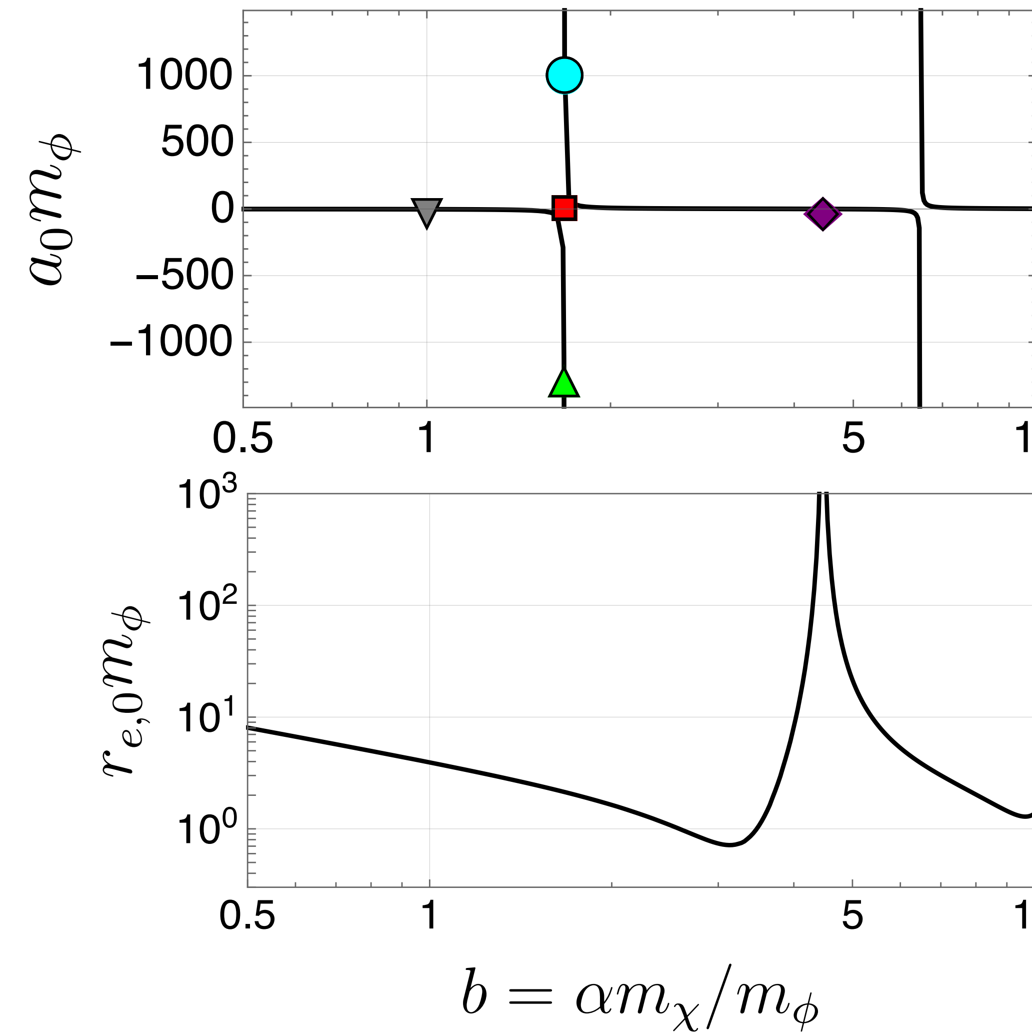
- on resonance $\quad a_0 \to \infty$

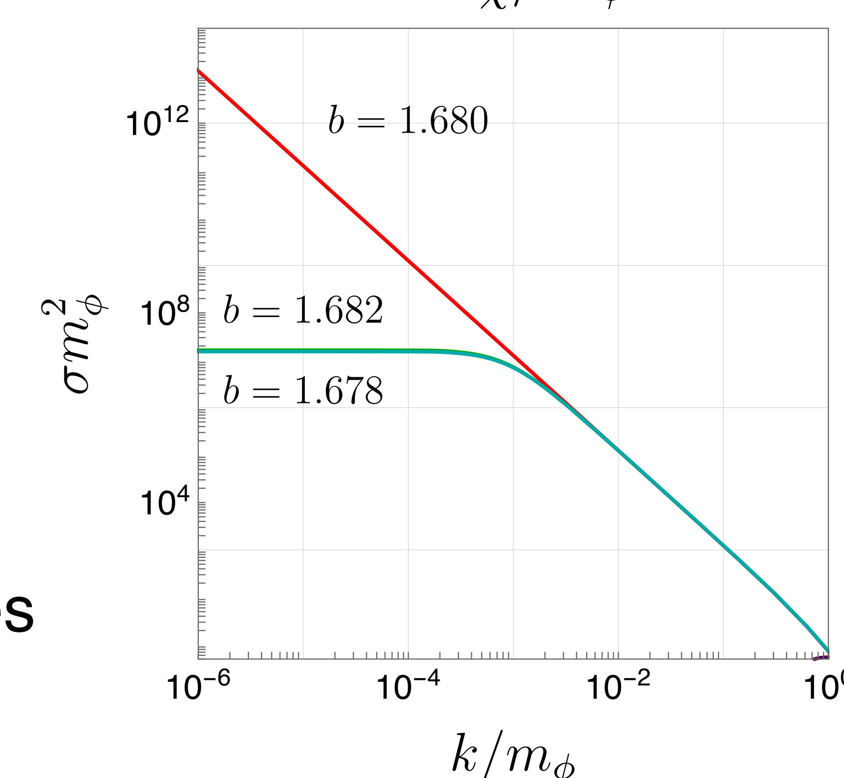
- around resonances

    - shallow virtual state
        - non-normalizable $\quad \kappa_b < 0$
    - shallow bound state
        - pole of scattering amplitude $\quad k = i\kappa_b$

$$\kappa_{b0} \approx \frac{1}{a_0} \quad \text{- s-wave} \qquad \kappa_b > 0$$

$$\kappa_{b\ell}^2 \approx -\frac{2r_{e,\ell}^{2\ell-1}}{a_\ell^{2\ell+1}} \quad \text{- higher-partial waves}$$





Watson theorem

Weinberg, "Lectures on Quantum Mechanics"

"In" and "out" states

- Lippmann-Schwinger equation $|\psi^{\pm}> = |\phi> + \frac{1}{E - H_0 \pm i\epsilon} V |\psi^{\pm}>$

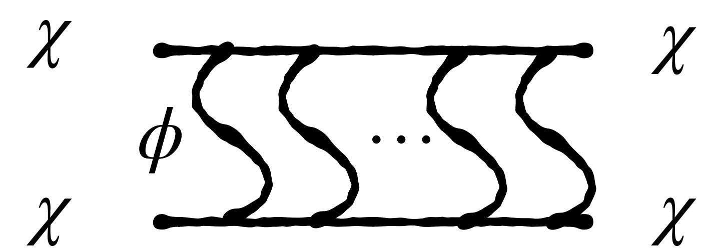
  - free $\qquad \epsilon > 0$

  - "+": in (far future)
    "-": out (far past)

- S-matrix relates these two $|\psi_\alpha^{+}> = \int d\beta S_{\beta\alpha} |\psi_\beta^{-}>$

$$S_{\beta\alpha} = \langle \psi_\beta^{-} | \psi_\alpha^{+} \rangle = \delta(\beta - \alpha) - 2i\pi\delta(E_\alpha - E_\beta) T_{\beta\alpha}$$

$$V|\psi_\alpha^{+}> = \int d\beta T_{\beta\alpha} |\phi_\beta>$$



- in-state as a whole

- for partial waves $S_{\ell,\beta\alpha} = I_{\beta\alpha} + 2i\sqrt{\sigma_\alpha \sigma_\beta} T_{\ell,\beta\alpha}$ $\qquad \sigma_\alpha = \beta_\alpha \theta(k^2 - k_\alpha^2)$

  - velocity

- $\psi_k^{+}(\vec{x}) \to e^{ikz} + f(k,\theta)\frac{e^{ikr}}{r}$ is actually in-state

  $r \to \infty$

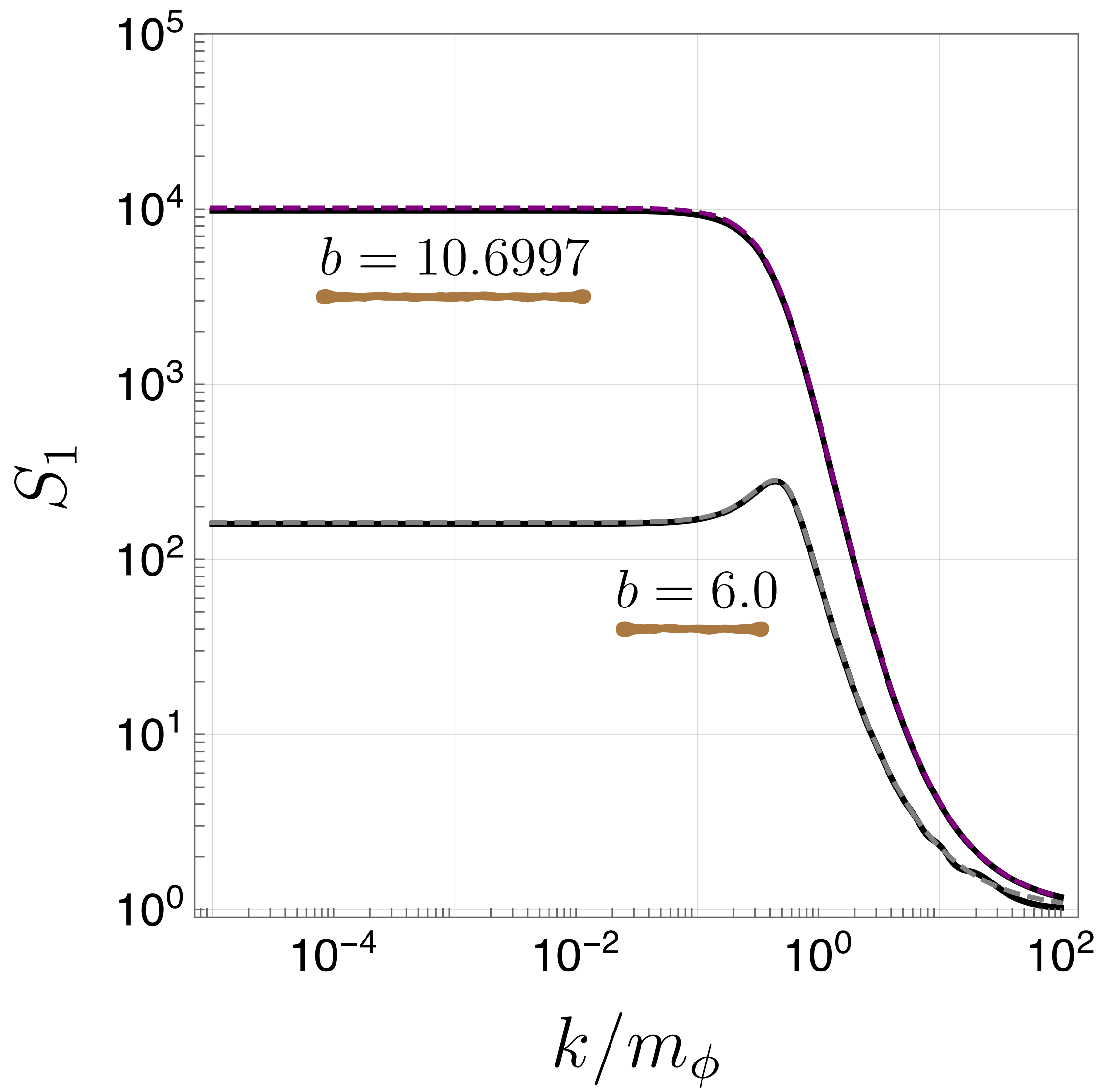
- out-state $\qquad \psi_k^{-}(\vec{x}) \to e^{ikz} + f(k,\theta)^* \frac{e^{-ikr}}{r} \qquad r \to \infty$

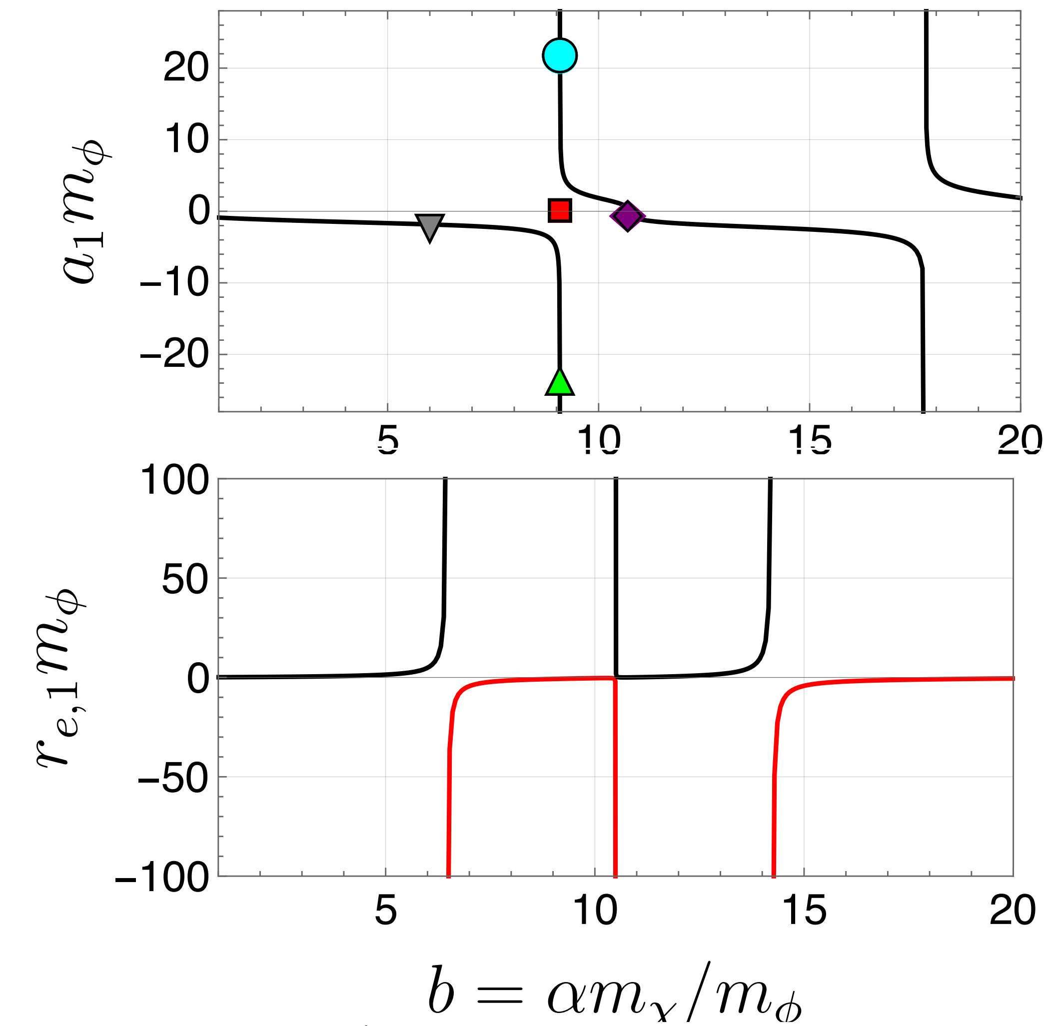
Omnès solution

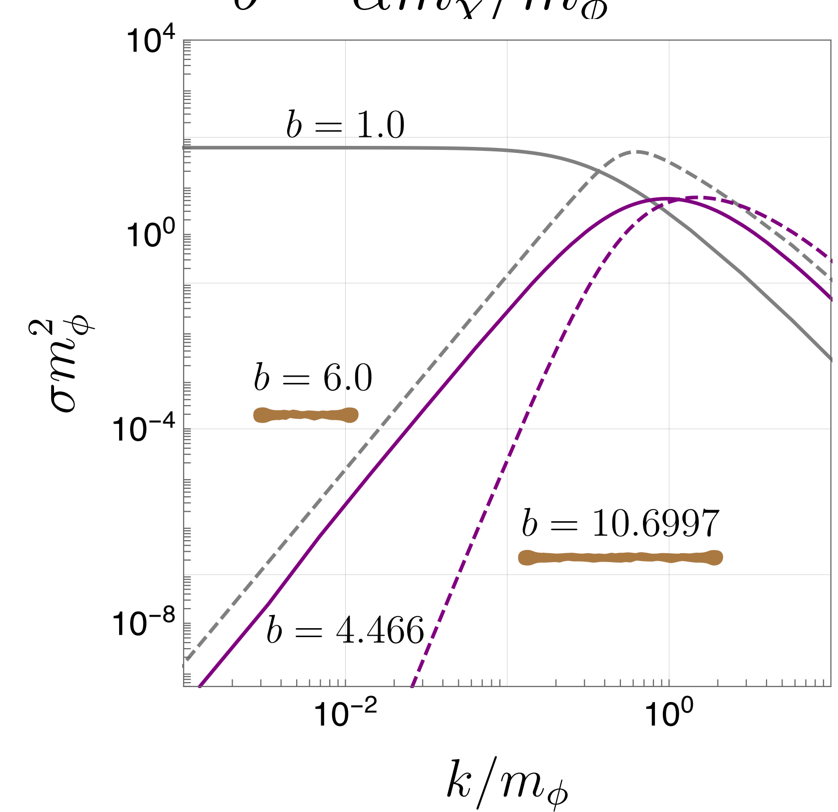
Yukawa potential

AK, Kuwahara and Patel, in preparation

- p-wave







Around resonances

Levison theorem

Weinberg, "Lectures on Quantum Mechanics"

- # of bound states is given by phase shift

$$\delta_\ell(k \to 0) - \delta_\ell(k \to \infty) = \left[\#b_\ell \left(+\frac{1}{2}\right)\right]\pi$$

- excluding virtual states

- zero in our normalization
- only for s-wave resonances

- underlying idea
  - consider the system confined in a large sphere

$$R_{k\ell}(r) \to \frac{\sin(kr - \frac{1}{2}\ell\pi + \delta_\ell)}{r}$$

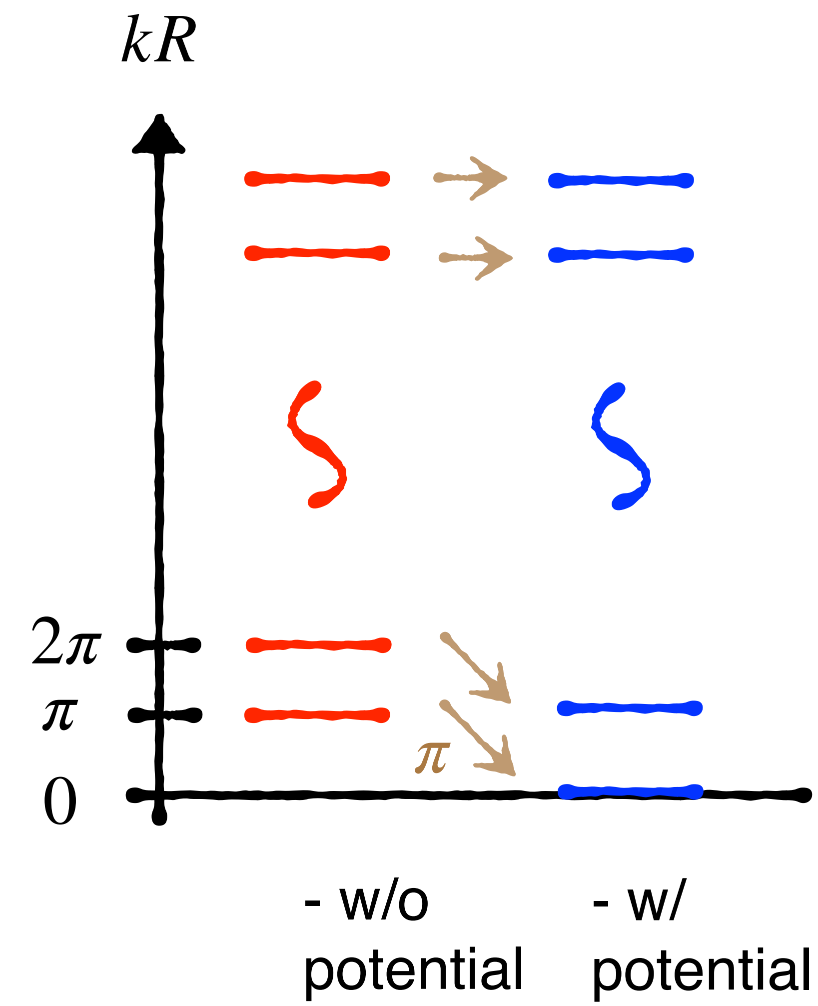
$$kR - \frac{1}{2}\ell\pi + \delta_\ell = n\pi$$

$n = 0, \pm 1, \pm 2\ldots$

$r \to \infty$

$k > 0$

  - scattering states are discretized (countable infinity)
  - decrease in # of scattering states = # of bound states
    - total number does not change



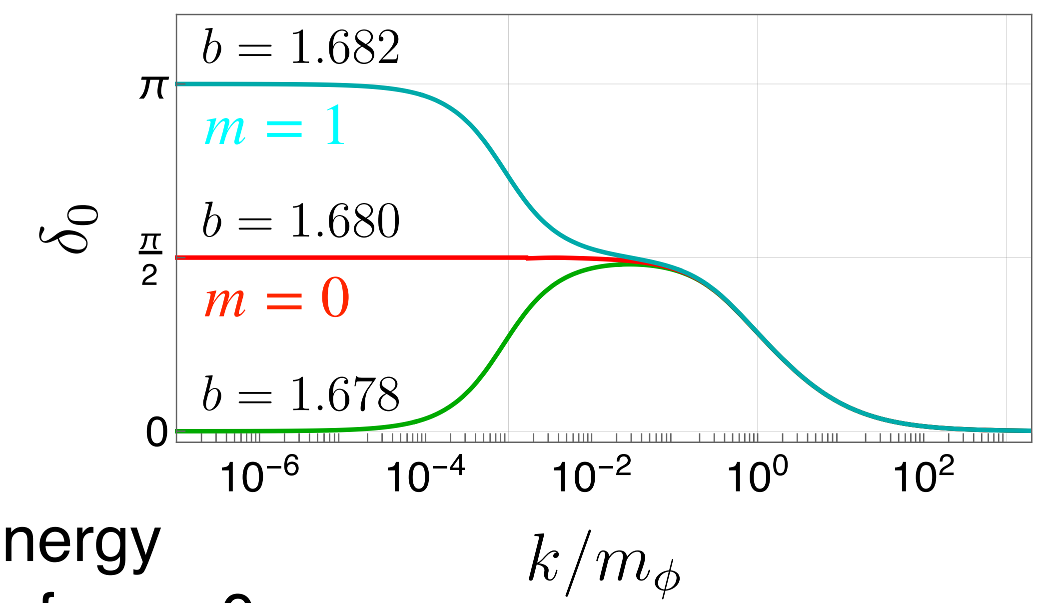
Around resonances

AK, Kuwahara and Patel, in preparation

Bound states

- s-wave

$$\delta_0(k \to 0) = \left[\#b_0 \left(+\frac{1}{2}\right)\right]\pi$$

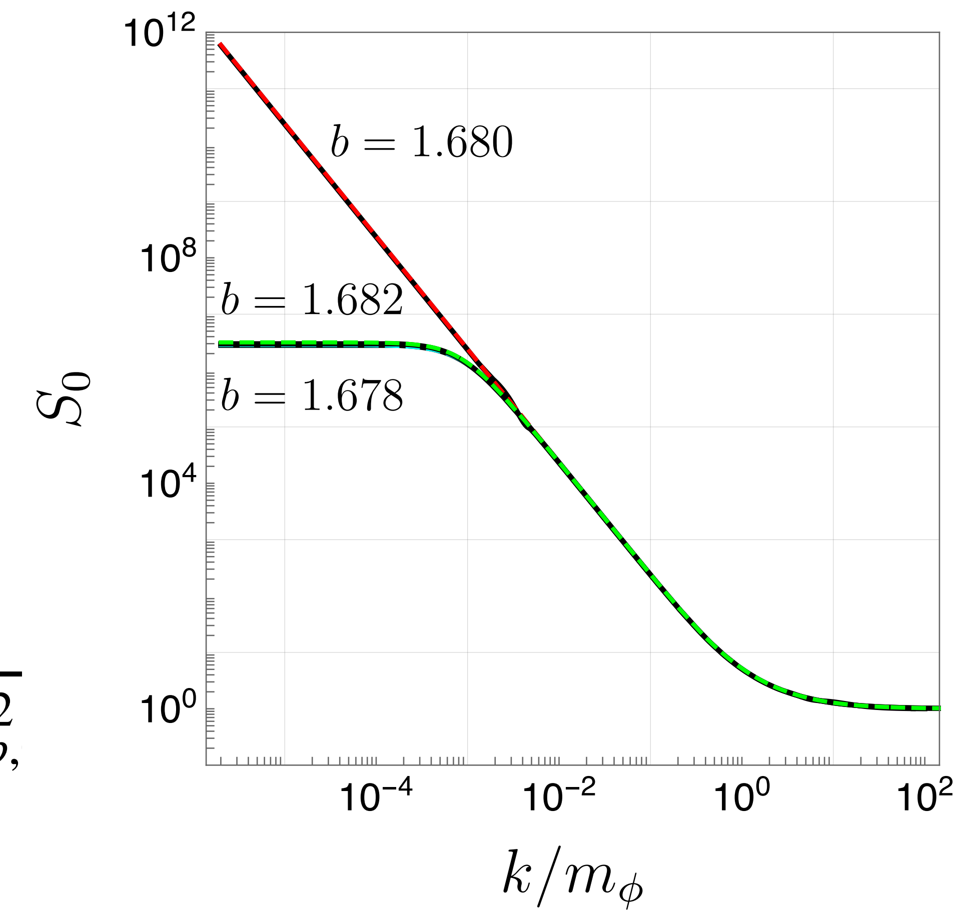


- resonances

$$k \to 0 \quad \delta_0 \to \left(\frac{1}{2} + m\right)\pi \quad \#b_0 = m$$

$$F_0(k^2) = \prod_{b_0=1}^{m} \frac{k^2}{k^2 + \kappa_{b,0}^2}$$

- only zero energy "virtual" state for m=0



- slightly below the 1st resonance

$k \to 0 \quad \delta_0 \to 0$ - no bound state $F_0(k^2) = 1$

- slightly above the 1st resonance

$$k \to 0 \quad \delta_0 \to \pi \quad \omega_0(k^2) \to -\ln(r_{e,0}^2 k^2)$$

- single bound state $F_0(k^2) = \dfrac{k^2}{k^2 + \kappa_{b,}^2}$

$\Gamma_0(k^2) \to \dfrac{1}{k^2 + \kappa_{b,0}^2}$ - saturates at low k

Around resonances

AK, Kuwahara and Patel, in preparation

Bound states

- p-wave

$$\delta_1(k \to 0) = \#b_1 \pi$$

- resonances

$$k \to 0 \quad k^3 \cot \delta_1 \to -\frac{1}{a_1^3} + \frac{1}{2r_{e,1}} k^2$$

$$a_1 \to \infty$$

$$k \to 0 \quad \delta_1 \to m\pi \qquad \#b = m$$

$$\omega_1(k^2) \to -m \ln(r_{e,1}^2 k^2)$$

for m=1

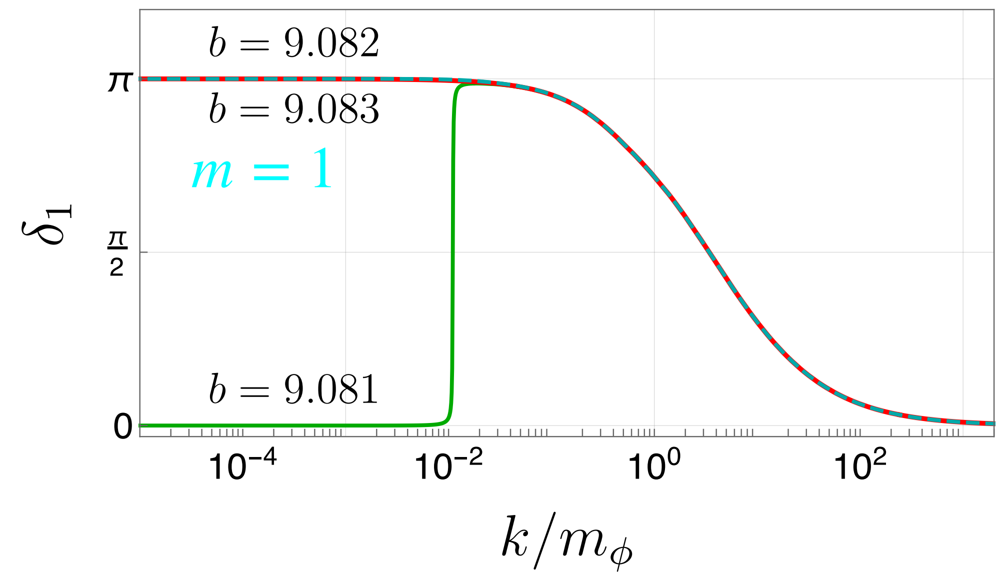
$$\Gamma_1(k^2) \to \frac{1}{k^2}$$

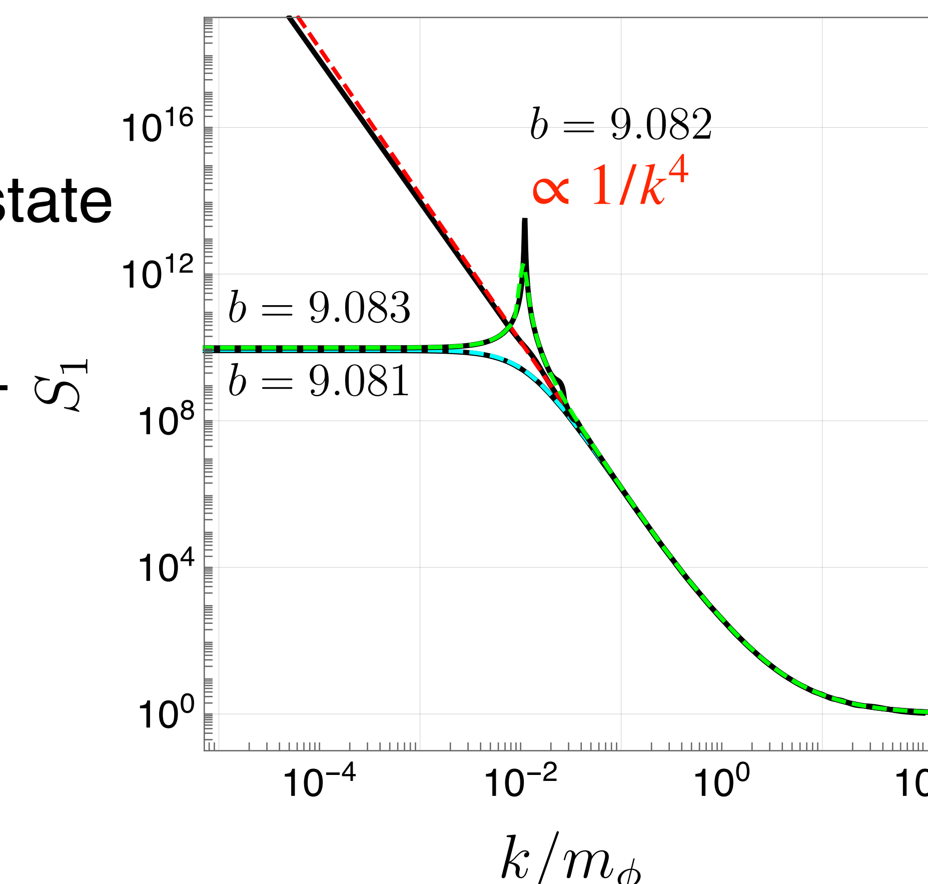
- including zero energy bound state

$$F_1(k^2) = \prod_{b_1=1}^{m-1} \frac{k^2}{k^2 + \kappa_{b,1}^2}$$

- slightly below/above the 1st resonance

  - similar to s-wave





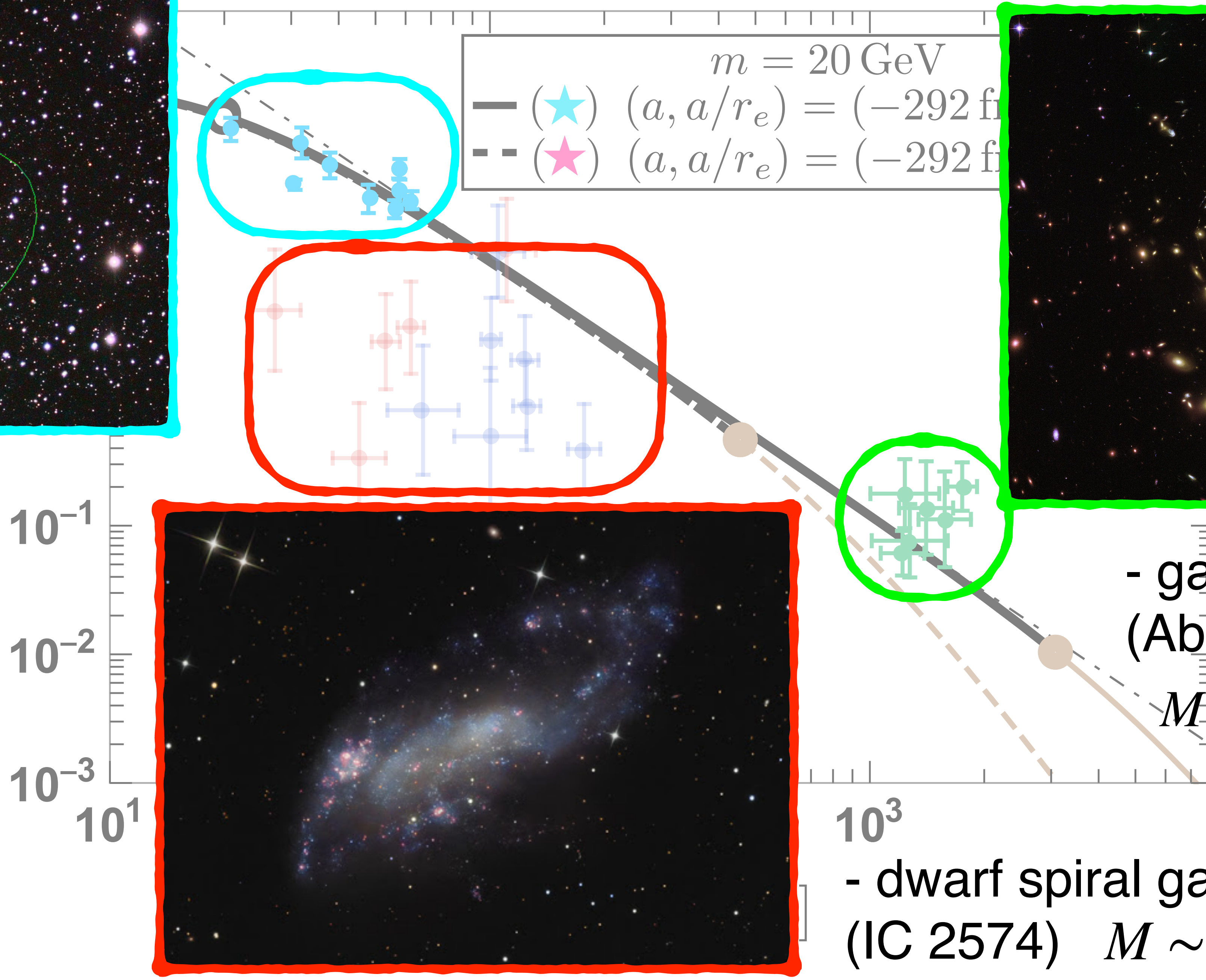
Data points

Overview

- cores in various-size halos

- MW satellite (Draco)

$M_{\rm infall} \sim 10^9\, M_\odot$



- galaxy cluster (Abell 2744)

$M \sim 10^{14}\, M_\odot$

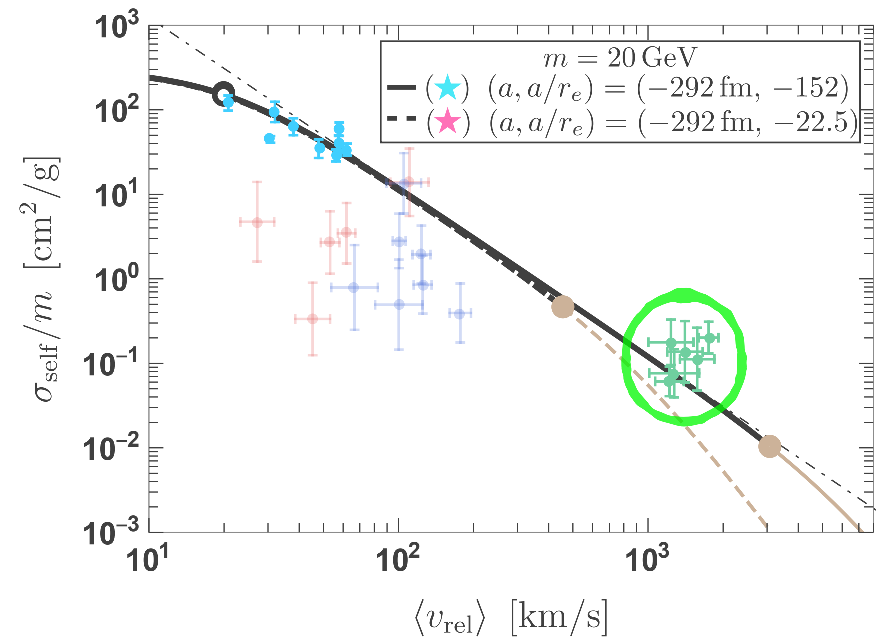
- dwarf spiral galaxy (IC 2574) $M \sim 10^{11}\, M_\odot$

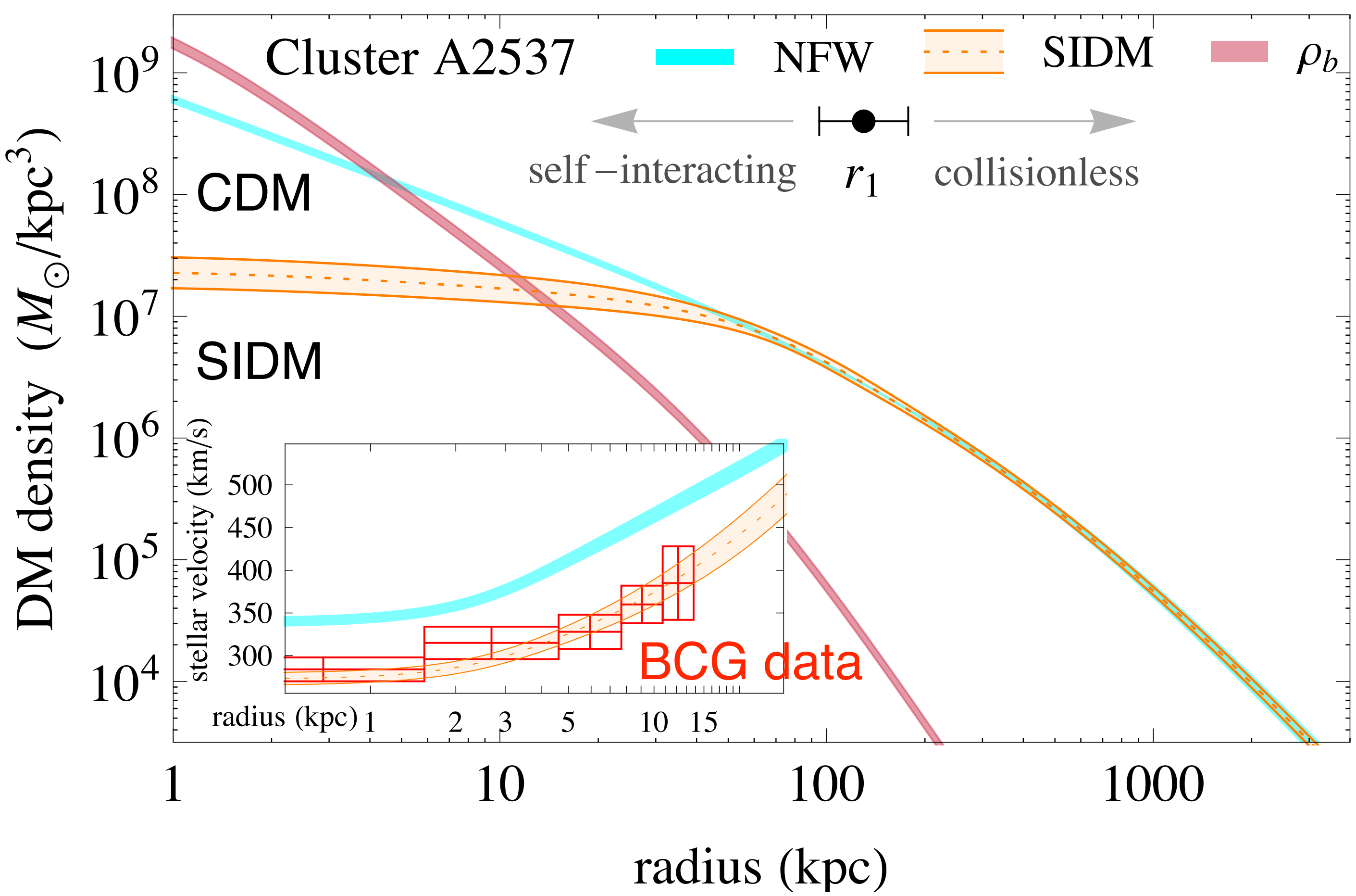
Data points

Galaxy clusters

- mass distribution in the outer region is determined by strong/weak gravitational lensing

- stellar kinematics in the central region (brightest cluster galaxies) prefer cored SIDM profile





$$\sigma_{\rm self}/m \sim 0.1\,{\rm cm^2/g}$$

$$\langle v_{\rm rel}\rangle \sim 10^3\,{\rm km/s}$$

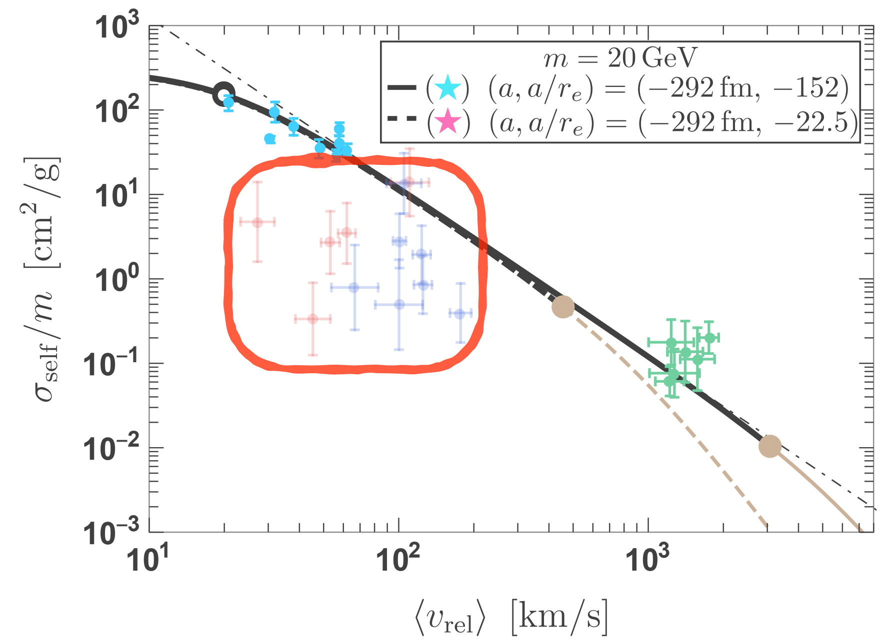
Kaplinghat, Tulin and Yu, PRL, 2016

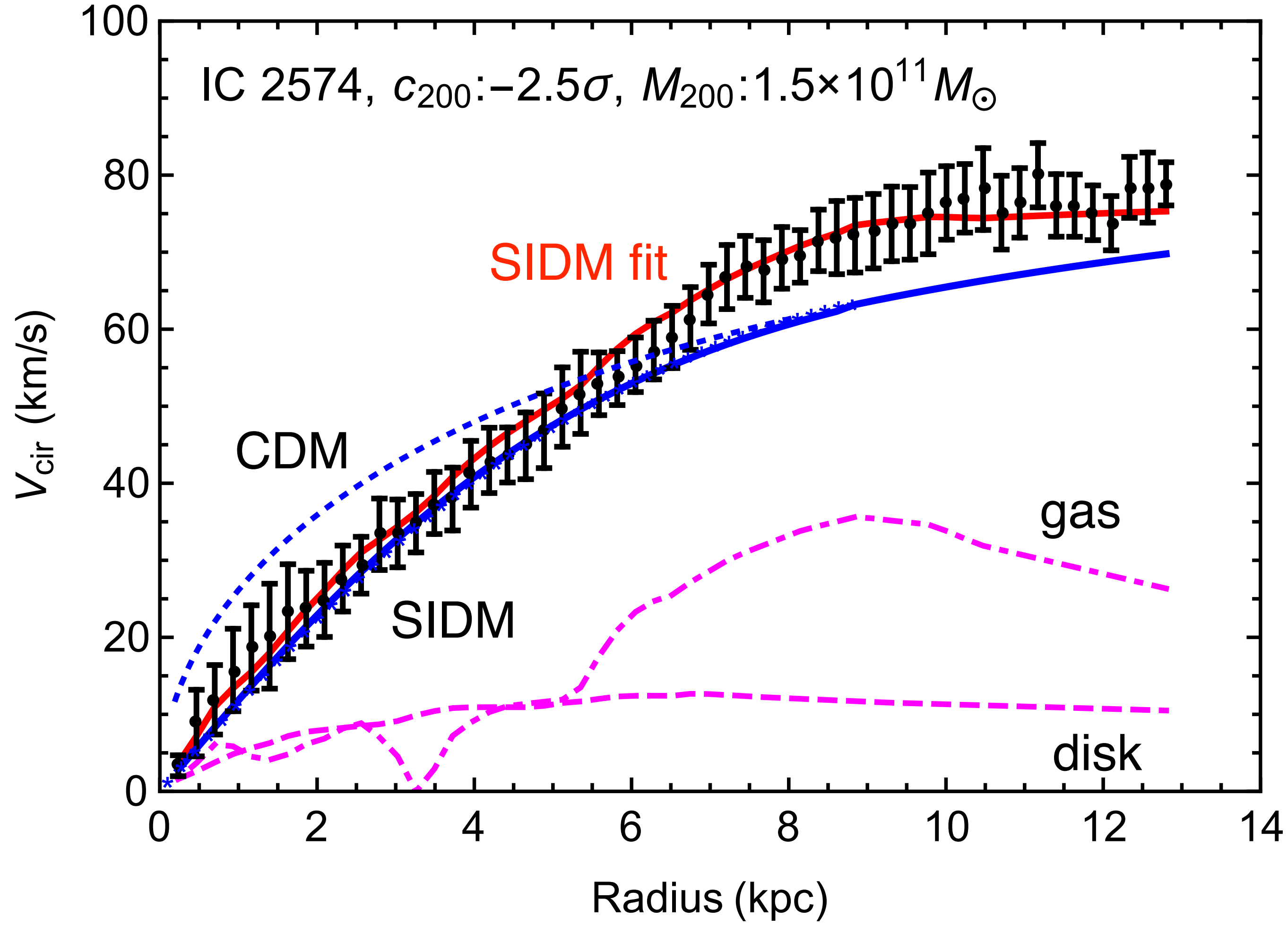
Data points

Dwarf spiral galaxies

- mass distribution is broadly determined by rotation curves

- rotation velocity in central region (of some galaxies) prefer cored SIDM profile





$$\sigma_{\rm self}/m \sim 1\,\mathrm{cm}^2/\mathrm{g}$$

$$\langle v_{\rm rel}\rangle \sim 10^2\,\mathrm{km/s}$$

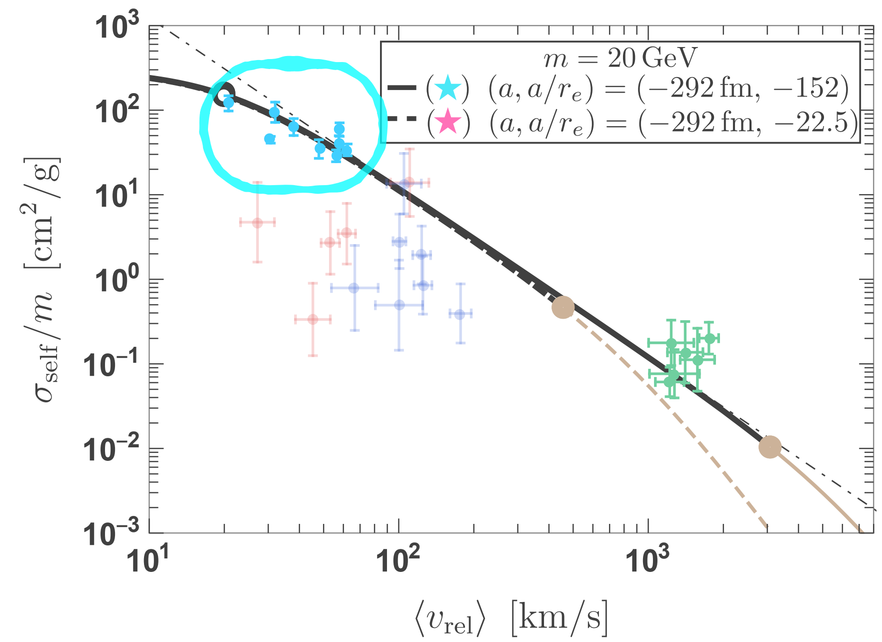
AK, Kaplinghat, Pace and Yu, PRL, 2017

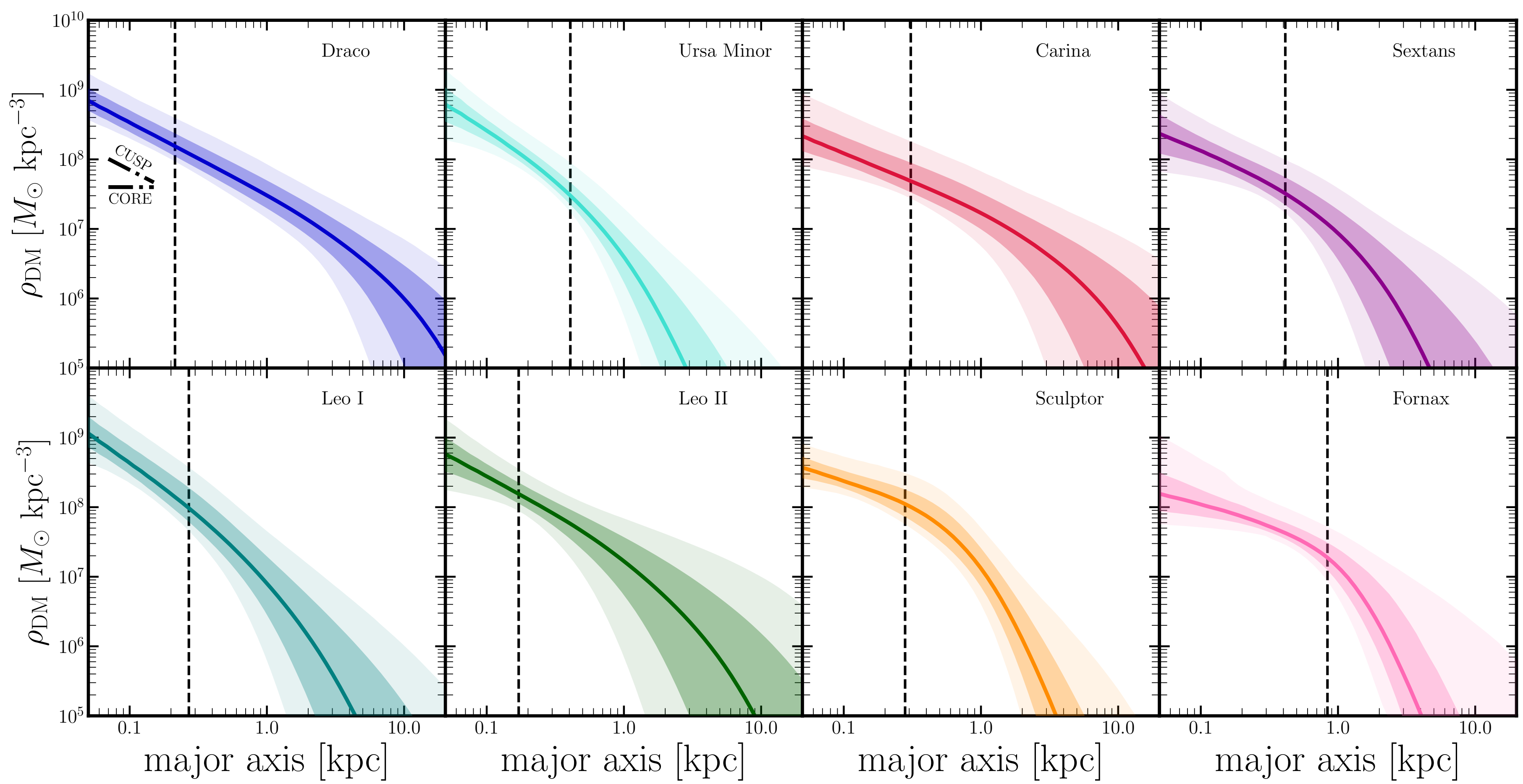
Data points

MW satellites

- mass distribution is determined by stellar kinematics

- stellar kinematics in the central region (of some satellites) prefer cuspy CDM profile





Hayashi, Chiba and Ishiyama, ApJ, 2020

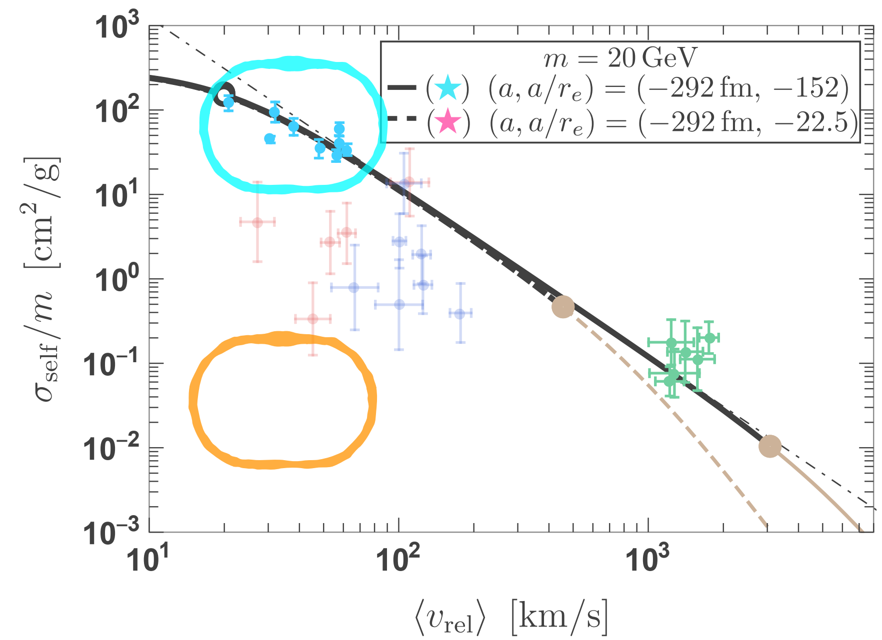
Data points

MW satellites

- one possibility is to take as a tiny cross section as $\sigma_{\rm self}/m \simeq 0.01\,{\rm cm^2/g}$

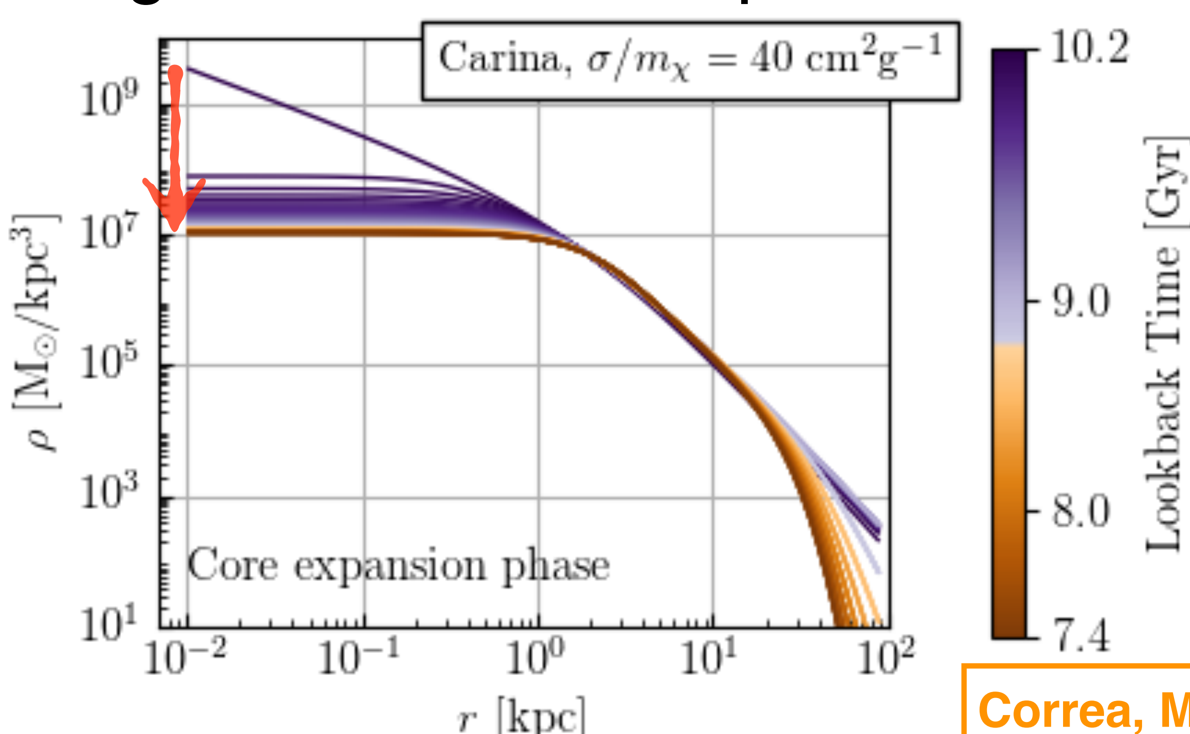
$\langle v_{\rm rel}\rangle \sim 30\,{\rm km/s}$

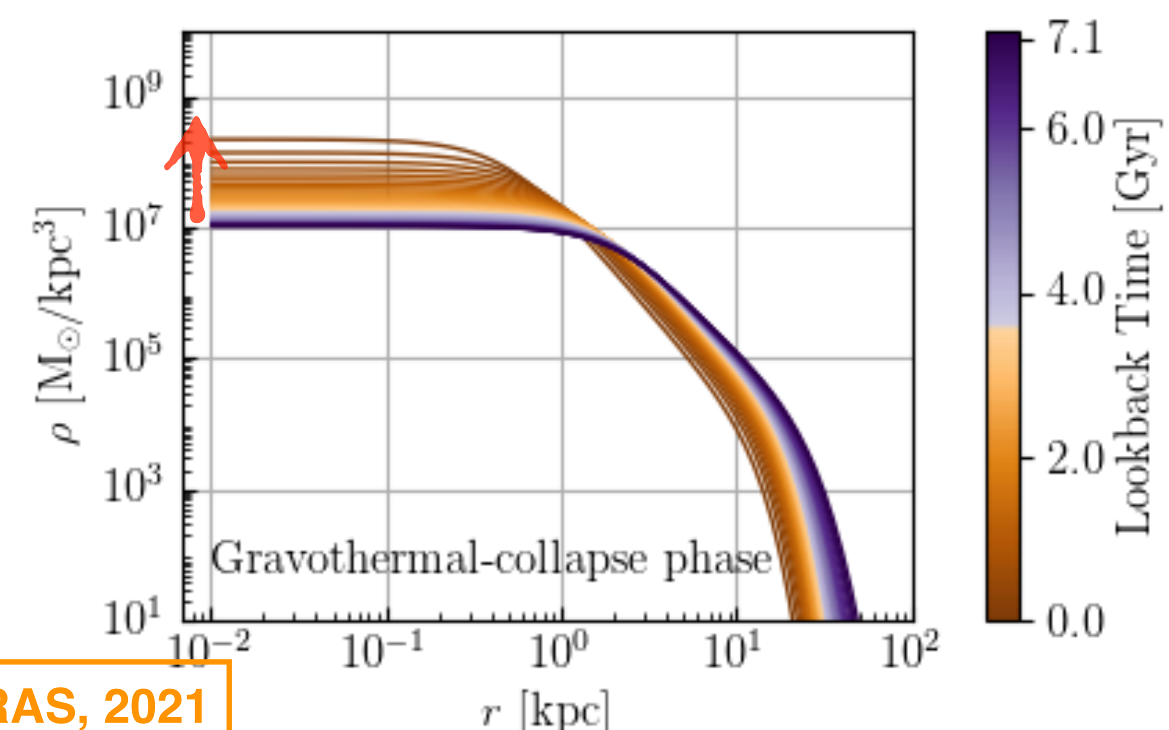
  - resonance? Chu, Garcia-Cely and Murayama, PRL, 2019



- another possibility is to take as a large cross section as $\sigma_{\rm self}/m \sim 40\,{\rm cm^2/g}$ $\langle v_{\rm rel}\rangle \sim 30\,{\rm km/s}$

- gravothermal collapse



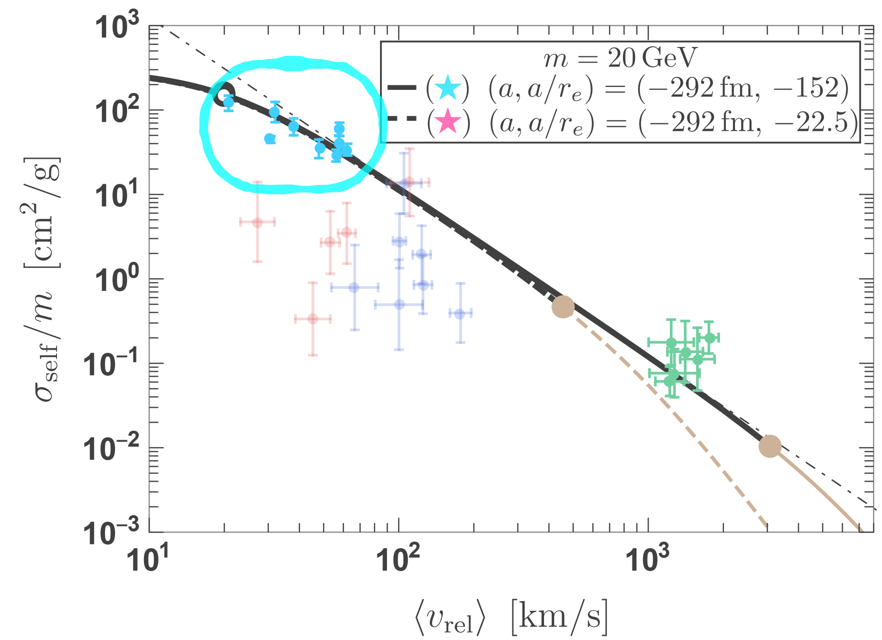


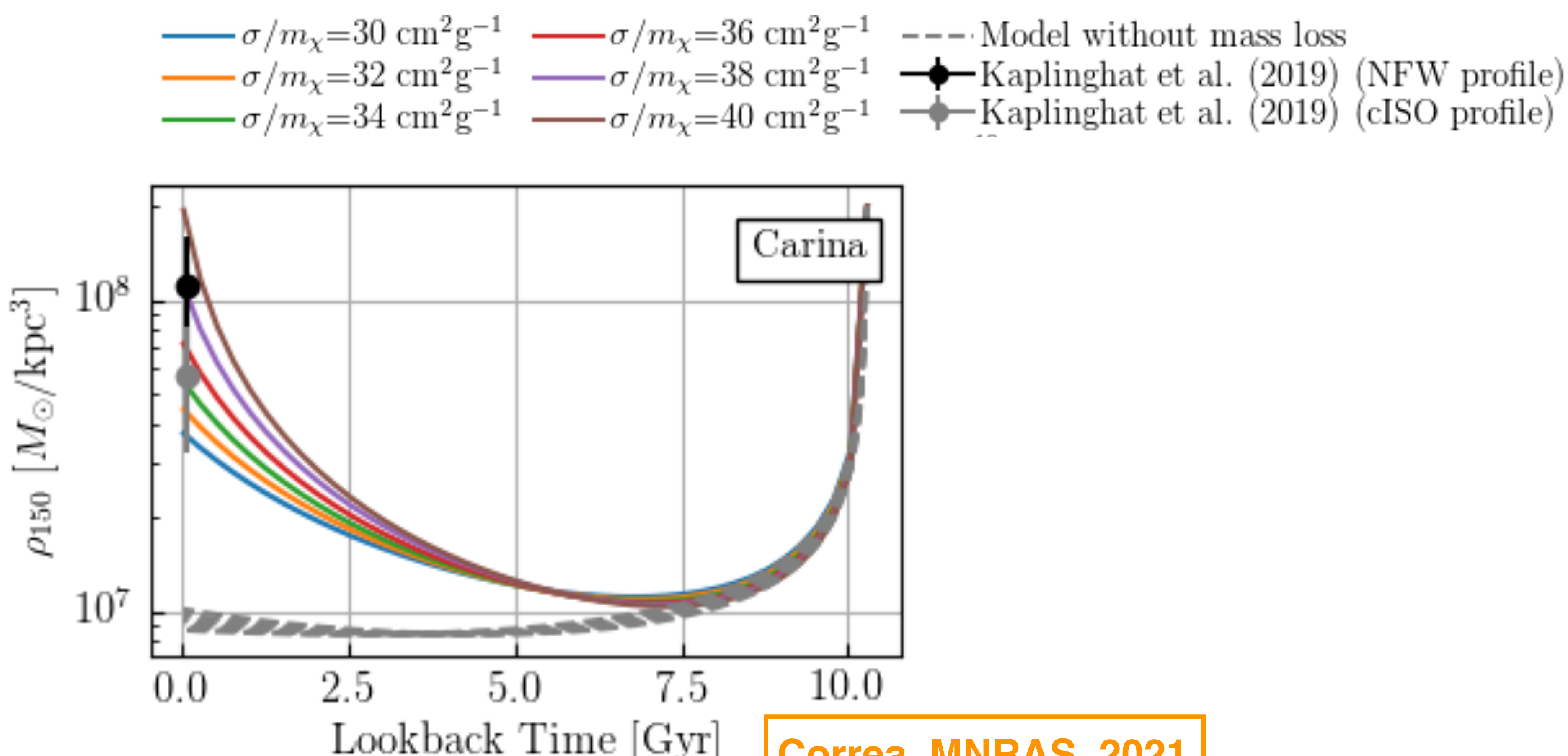
Correa, MNRAS, 2021

Data points

MW satellites

- gravothermal collapse
  - core shrinks and central density gets higher
  - central density at present is very sensitive to the cross section





Correa, MNRAS, 2021