Higher-order initial state radiation in e^+e^- annihilation

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Outline

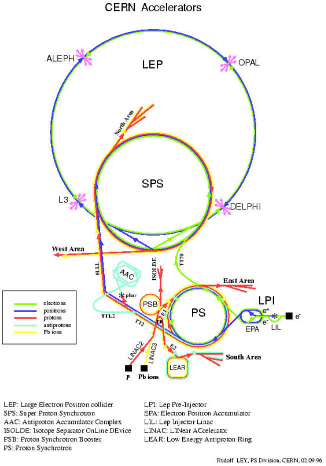
- Radiative corrections and future colliders
- Parton distribution functions approach
- Convolutions and regularization
- Evolution equation
- Factorization
- Parton distribution functions
- Higher order corrections
- Running coupling α
- Results
- New physics

Radiative corrections and future colliders

- Calculation of the radiative corrections $\longrightarrow$ accurate predictions of the high energy processes
- Tests of the Standard model
- Search for New Physics
- e^+e^- annihilation - for future lepton (electron-positron) colliders: ILC, CLIC, CEPC, FCC-ee

Electron-positron colliders

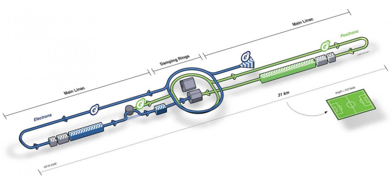
LEP - CERN, Switzerland



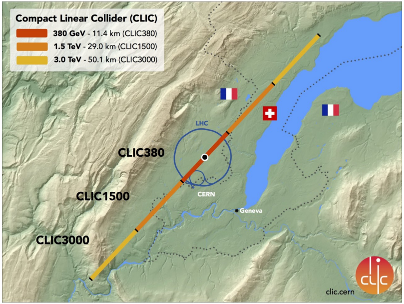
Max energy - 209 GeV
Worked until 2000
Results: discovery of Z and W-bosons, observing creation and decay of Z-boson, details of electroweak interaction, three generations of fermions
Closed to build the LHC

Future electron-positron colliders

ILC - International Linear Collider, Japan
250-500 GeV $\longrightarrow$ 1 TeV



CLIC - Compact Linear Collider, CERN
380 CeV $\longrightarrow$ 3 TeV



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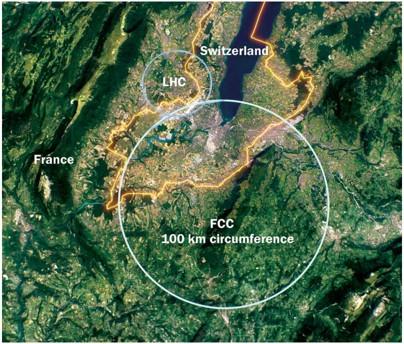
Future electron-positron colliders

FCC - Future Circular Collider, CERN
FCC-ee - 45-175 GeV



CEPC - Circular Electron Positron Collider, China
240 GeV

Electron-positron colliders

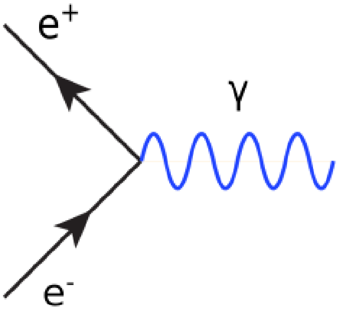
Opportunities:

- Higgs factory
- measurements the properties of the Z and W bosons
- generate top-quarks
- study strong interaction
- find new particles interacting with the SM particles

e^+e^- annihilation

$e^+ + e^- =$

- photon
- Z^0 boson
- pair of photons
- pions
- hadrons...



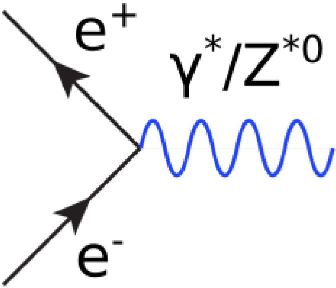
New physics

Dark matter candidate **heavy right-handed neutrino (RHN)**
ILC - 250-500 GeV $\longrightarrow$ 1 TeV
CLIC - 380 CeV $\longrightarrow$ 3 TeV

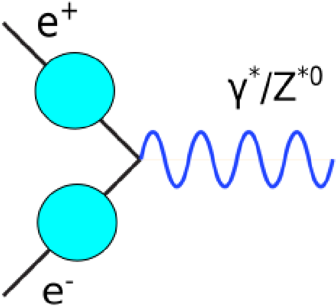
$$e^+e^- \longrightarrow b\bar{b} + \chi$$

e^+e^- annihilation to γ^*/Z^{0*}

$$e^+e^- \longrightarrow \gamma^*/Z^{0*}$$



Initial state radiation corrections



Parton distribution functions approach

- Based on perturbation theory
- Allows to calculate the most significant (logarithmic) corrections
- Expansion in powers of coupling constant and the large logarithm

Large logarithm

$$L = \frac{\mu^2}{\mu_0^2}$$

μ - factorization scale
μ_0 - renormalization scale (in QED $\mu_0 = m_e$)

Parton distribution functions approach

Parton distribution functions (PDFs)

A function $D_{ie}(x, s)$ describes the density of the distribution of the massles parton of type i (in QED - electron, positron or photon) in the initial massive electron . x is the energy of the parton relative to the total energy of the particle which emitted it. Structure functions correspond to transition of a massive particle to a massless, and fragmentation functions to transition from massless particle to a massive one, which can be observed.

Splitting functions

$P_{ij}(x)$ describes the probability density of a transition of a parton j to a parton i with the energy x.

Splitting functions and PDFs are independent of the process

Massless Wilson coefficients

$\sigma_{ij}(x)$ contain information of the process.

History

- Parton model of the hadron interactions was suggested by R. Feynman in 1969
- Then partons were associated with the quarks
- Parton distribution function approach was developed for QCD by
- They derived evolution equations of the parton distribution functions in QED (DGLAP equations)
- An analogous approach was developed for QED and the DGLAP equations were reduced to QED by by E.A. Kuraev and V.S.Fadin

Evolution equation

$$D^{(0)}_{ee}(x,\mu^2) = \delta(1-x) + \frac{\alpha}{2\pi} d^{(1)}_{ee}(x)$$

$$D^{(0)}_{\gamma e}(x,\mu^2) = \frac{\alpha}{2\pi} d^{(1)}_{\gamma e}(x)$$

$$D^{(0)}_{e\bar{e}}(x,\mu^2) = \frac{\alpha}{2\pi} d^{(1)}_{e\bar{e}}(x) = 0$$

$$d^{(1)}_{ee}(x) = \left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_0^2}{m_e^2} - 1 - 2\ln(1-x)\right)\right]_+ = \left[\frac{1+x^2}{1-x}(-1-2\ln(1-x))\right]_+$$

$$d^{(1)}_{\gamma e}(x) = -\frac{1+(1-x)^2}{x}(2\ln x + 1)$$

$$d^{(1)}_{e\gamma}(x) = 0$$

Evolution equation

$$D_{ba}(x,\mu^2,\mu_0^2) = \delta(1-x)\delta_{ba} + \sum_{i=e,\bar{e},\gamma} \int_{\mu_0^2}^{\mu^2} \frac{dt\,\alpha(t)}{2\pi t} \int_x^1 \frac{dy}{y} D_{ia}(y,t,\mu_0) P_{bi}\left(\frac{x}{y}\right)$$

Iterative solution

$$D_{ee}^{(k)} = D_{ee}^{(0)} + \frac{\alpha}{2\pi}\left(P_{ee}\otimes D_{ee}^{(k-1)} + P_{e\gamma}\otimes D_{\gamma e}^{(k-1)} + P_{e\bar{e}}\otimes D_{\bar{e}e}^{(k-1)}\right)$$

Evolution equation

$$D^{(\mathrm{I})}_{ee}(x,\mu) = D^{(0)}_{ee} + \frac{\alpha}{2\pi}\left(P_{ee}\otimes D^{(0)}_{ee} + P_{e\gamma}\otimes D^{(0)}_{\gamma e} + P_{e\bar{e}}\otimes D^{(0)}_{\bar{e}e}\right)$$

$$= \delta(1-x) + d^{(1)}_{ee}(x) + \frac{\alpha}{2\pi}\left(P^{(0)}_{ee} + \frac{\alpha}{2\pi}P^{(1)}_{ee}\right)\otimes\left(\delta(1-x) + d^{(1)}_{ee}(x)\right)$$

$$+\left(P^{(0)}_{eg} + \frac{\alpha}{2\pi}P^{(1)}_{eg}\right)\otimes d^{(1)}_{\gamma e}(x) = \delta(1-x) + \frac{\alpha}{2\pi}d^{(1)}_{ee}(x) + \frac{\alpha}{2\pi}LP^{(0)}_{ee}$$

$$D^{(\mathrm{I})}_{\gamma e}(x,\mu) = D^{(0)}_{\gamma e} + \frac{\alpha}{2\pi}(P_{\gamma\gamma}\otimes D^{(0)}_{\gamma e} + P_{\gamma e}\otimes D^{(0)}_{ee} + P_{\gamma\bar{e}}\otimes D^{(0)}_{\bar{e}e}) = d^{(1)}_{\gamma e}(x)$$

$$+\frac{\alpha}{2\pi}\left(P^{(0)}_{\gamma\gamma} + \frac{\alpha}{2\pi}P^{(1)}_{\gamma\gamma}\right)\otimes d^{(1)}_{\gamma e}(x) + \frac{\alpha}{2\pi}\left(P^{(0)}_{\gamma e} + \frac{\alpha}{2\pi}P^{(1)}_{\gamma e}\right)\otimes\left(\delta(1-x)\right.$$

$$\left.+d^{(1)}_{ee}(x)\right) = \frac{\alpha}{2\pi}d^{(1)}_{\gamma e}(x) + \frac{\alpha}{2\pi}LP^{(0)}_{\gamma e}$$

$$D^{(\mathrm{I})}_{e\bar{e}}(x,\mu) = D^{(0)}_{e\bar{e}} + \frac{\alpha}{2\pi}(P^{(0)}_{ee}\otimes D^{(0)}_{e\bar{e}} + P^{(0)}_{e\bar{e}}\otimes D^{(0)}_{ee} + P^{(0)}_{e\gamma}\otimes D^{(0)}_{e\gamma}) = 0$$

Running coupling α

$$\alpha(\mu^2) = \frac{\alpha(\mu_0)}{1+\overline{\Pi}(\mu,\mu_0,\alpha(0))} = \alpha(0)\left\{1+\frac{\alpha(0)}{2\pi}\left(-\frac{10}{9}+\frac{2}{3}L\right)\right.$$
$$\left.+\left(\frac{\alpha(0)}{2\pi}\right)^2\left(-\frac{1085}{324}+4\zeta_3-\frac{13}{27}L+\frac{4}{9}L^2\right)+\mathcal{O}\left(\alpha^3(0)\right)\right\}$$

P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn, C. Sturm, Nucl. Phys. B 867 (2013), 182-202

Gorishnii, S. G., Kataev, A. L., Larin, S. A., Phys. Lett. B 273 (1991), 141–144

$$\alpha(\mu^2) = \alpha(0)\left\{1+\frac{\alpha(0)}{2\pi}\left(\frac{2}{3}L\right)\right\}$$

Cross-section

$$
\begin{aligned}
d\sigma^{\mathrm{NLO}}_{ab\to cd} &= \sum_{i,j,k,l} \int\limits_{\bar{z}_1}^{1} dz_1 \int\limits_{\bar{z}_2}^{1} dz_2 D^{\mathrm{str}}_{ia}(z_1, \frac{\mu_0^2}{\mu^2}) D^{\mathrm{str}}_{jb}(z_2, \frac{\mu_0^2}{\mu^2}) \\
&\times \left(d\sigma^{\mathrm{Born}}_{ij\to kl}(z_1, z_2) + d\bar{\sigma}^{(1)}_{ij\to kl}(z_1, z_2) + \mathcal{O}(\alpha^2 L^0) \right) \\
&\times \int\limits_{\bar{y}_1}^{1} \frac{dy_1}{Y_1} \int\limits_{\bar{y}_2}^{1} \frac{dy_2}{Y_2} D^{\mathrm{frg}}_{ck}(\frac{y_1}{Y_1}, \frac{\mu_0^2}{\mu^2}) D^{\mathrm{frg}}_{dl}(\frac{y_2}{Y_2}, \frac{\mu_0^2}{\mu^2}) + \mathcal{O}\left(\frac{\mu_0^2}{\mu^2}\right)
\end{aligned}
$$

$$
d\sigma^{\mathrm{NLO}}_{ab\to cd} = d\sigma^{(0)}_{ab\to cd} \left\{ 1 + \sum_{k=1}^{\infty} \left(\frac{\alpha}{2\pi}\right)^k \sum_{l=k-1}^{k} c_{k,l} L^l + \mathcal{O}(\alpha^k L^{k-2}) \right\}
$$

Leading logarithmic approximation: $\alpha^k L^k$
Next-to-leading logarithmic approximation: $\alpha^k L^{k-1}$

PDF evolution equation in QED

Convolution and regularization:

$$(f \otimes g)(x) \equiv \int_0^1 dz \int_0^1 dy\ f(z)g(y)\delta(x - yz) = \int_x^1 \frac{dz}{z} f(z)g(\frac{x}{z})$$

$$f(x) = \lim_{\Delta \to 0} \Big(f_\Theta(x)\Theta(1 - x - \Delta) + f_\Delta \delta(1 - x) \Big)$$

$$\int_z^1 dx[f(x)]_+ g(x) = \int_0^1 dx f(x) \Big[g(x)\Theta(x - z) - g(1) \Big]$$

$$f_\Delta = - \int_0^{1-\Delta} f_\Theta(z)dz$$

$$\Big(f \otimes g\Big)_\Theta (z) = \lim_{\Delta \to 0} \Big\{ \int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x} f_\Theta(x) g_\Theta \left(\frac{z}{x}\right) + f_\Delta g_\Theta(z) + f_\Theta(z) g_\Delta \Big\}$$

Splitting functions

$$P_{ee}^{(0)}(x) = \left[\frac{1+x^2}{1-x}\right]_+, \qquad P_{e\gamma}^{(0)}(x) = x^2 + (1-x)^2, \qquad P_{e\bar{e}}^{(0)}(x) = 0,$$

$$P_{\gamma e}^{(0)}(x) = \frac{1+(1-x)^2}{x}, \qquad P_{\gamma\gamma}^{(0)}(x) = \frac{\beta_0}{2}\delta(1-x),$$

$$P_{ee}^{(1)NS}(x) = C_f^2\left(\left(-2\ln x\ln(1-x) - \frac{3}{2}\ln x\right)\frac{1+x^2}{1-x} - \left(\frac{3}{2}+\frac{7}{2}x\right)\ln x - \right.$$

$$\left. -\frac{1}{2}(1+x)\ln^2 x - 5(1-x)\right) + C_f T_f\left(-\frac{2}{3}\ln x - \frac{10}{9}\frac{1+x^2}{1-x} - \frac{4}{3}(1-x)\right) +$$

$$+\delta(1-x)\left(C_f^2\left(\frac{3}{8} - \frac{\pi}{2} + 6\zeta_3\right) - C_f T_f\left(\frac{1}{6} + \frac{2\pi^2}{9}\right)\right)$$

Cross-section

$$\frac{d\sigma_{\bar{e}e}}{ds'} = \sigma^{(0)}\Big[D_{\bar{e}\bar{e}} \otimes D_{ee} \otimes \tilde{\sigma}_{\bar{e}e} + D_{\gamma e} \otimes D_{ee} \otimes \tilde{\sigma}_{e\gamma}$$
$$+D_{ee} \otimes D_{e\bar{e}} \otimes \tilde{\sigma}_{ee} + D_{\gamma e} \otimes D_{\bar{e}\bar{e}} \otimes \tilde{\sigma}_{\bar{e}\gamma} + D_{\gamma e} \otimes D_{\gamma\bar{e}} \otimes \tilde{\sigma}_{\gamma\gamma}$$
$$+D_{\gamma e} \otimes D_{e\bar{e}} \otimes \tilde{\sigma}_{e\gamma} + D_{\bar{e}e} \otimes D_{\bar{e}\bar{e}} \otimes \tilde{\sigma}_{\bar{e}\bar{e}}$$
$$+D_{\bar{e}e} \otimes D_{\gamma\bar{e}} \otimes \tilde{\sigma}_{\bar{e}\gamma} + D_{\bar{e}e} \otimes D_{e\bar{e}} \otimes \tilde{\sigma}_{\bar{e}e}\Big]$$

Table: Orders of different contributions

a \ b	$\bar{e}$	γ	e
e	$D_{ee}D_{\bar{e}\bar{e}}\sigma_{e\bar{e}}$ LO (1)	$D_{\gamma e}D_{ee}\sigma_{e\gamma}$ NLO $(\alpha^2 L)$	$D_{ee}D_{e\bar{e}}\sigma_{ee}$ NNLO $(\alpha^4 L^2)$
γ	$D_{\gamma e}D_{\bar{e}\bar{e}}\sigma_{\bar{e}\gamma}$ NLO $(\alpha^2 L)$	$D_{\gamma e}D_{\gamma\bar{e}}\sigma_{\gamma\gamma}$ NNLO $(\alpha^4 L^2)$	$D_{\gamma e}D_{e\bar{e}}\sigma_{e\gamma}$ NLO $(\alpha^4 L^3)$
$\bar{e}$	$D_{\bar{e}e}D_{\bar{e}\bar{e}}\sigma_{\bar{e}\bar{e}}$ NNLO $(\alpha^4 L^2)$	$D_{\bar{e}e}D_{\gamma\bar{e}}\sigma_{\bar{e}\gamma}$ NLO $(\alpha^4 L^3)$	$D_{\bar{e}e}D_{e\bar{e}}\sigma_{\bar{e}e}$ LO $(\alpha^4 L^4)$

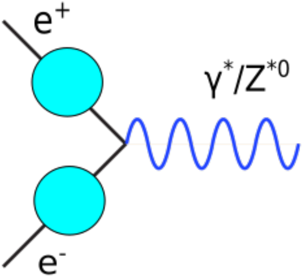
Factorization

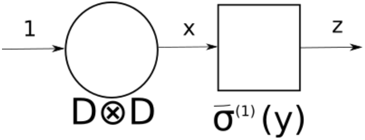
Large logarithm

$$L = \frac{\mu^2}{\mu_0^2} \text{ or } L = \frac{\mu^2}{\mu_0^2} + \ln z$$

μ - factorization scale
μ_0 - renormalization scale (in QED $\mu_0 = m_e$)

Factorization





$$\tilde{\sigma}^{(1)}_{\bar{e}e}(y) = \frac{\alpha}{\pi}\frac{1+y^2}{1-y}\ln\left(\frac{sx}{m_e^2}-1\right)$$

Results

Blumlein scheme (*Blumlein et al., Nucl.Phys.B 855 (2012), 508-569, Berends, Van Neerven, Burgers, Nucl.Phys.B 297 (1988), 429,*):

$$\bar{\sigma}^{(1)}_{ee}(x) = \frac{\alpha}{\pi}\frac{1+x^2}{1-x}\left(2\ln(1-x) - \ln x\right)$$

Our scheme

$$\tilde{\sigma}^{(1)}_{\bar{e}e}(x) = \frac{\alpha}{\pi}\frac{1+x^2}{1-x}\left(2\ln(1-x) + \ln z - \ln x\right)$$

Results

Conditions for $\bar{\sigma}^{(1)}_{e\bar{e}}(x)$

$$\sigma^{(1)}_{e\bar{e}}(x) = \bar{\sigma}^{(1)}_{e\bar{e}}(x) + 2\frac{\alpha}{2\pi}\Big[P^{(0)}_{ee}(x)$$
$$+d^{(1)}_{ee}(x)\Big] \otimes \sigma^{(0)}_{e\bar{e}}(x)L + \mathcal{O}\left(\frac{m_e^2}{\mu^2}\right)$$

Results

$$
\begin{aligned}
c_{32} = {} & \sigma^{(0)}_{e\gamma} \otimes \Big(2P^{(0)}_{\gamma e} + P^{(0)}_{\gamma e} \otimes P^{(0)}_{e\bar{e}} + P^{(0)}_{\gamma e} \otimes P^{(0)}_{\gamma\gamma} + 3P^{(0)}_{ee} \otimes P^{(0)}_{\gamma e} \Big) \\
& + \frac{4}{9}\, \sigma^{(1)}_{e\bar{e}} + \sigma^{(1)}_{e\bar{e}} \otimes \left(P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} + 2P^{(0)}_{ee} + 2P^{(0)}_{ee} \otimes P^{(0)}_{ee} \right) \\
& + \sigma^{(0)}_{e\bar{e}} \otimes \Big(P^{(0)}_{e\bar{e}} \otimes P^{(1)}_{e\bar{e}} + P^{(0)}_{\gamma e} \otimes P^{(1)}_{e\gamma} + P^{(0)}_{e\gamma} \otimes P^{(1)}_{\gamma e} + \frac{2}{3} P^{(0)}_{e\gamma} \otimes d^{(1)}_{\gamma e} \\
& + P^{(0)}_{e\gamma} \otimes d^{(1)}_{\gamma e} \otimes P^{(0)}_{\gamma\gamma} - \frac{20}{9} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} + \frac{4}{3} P^{(1)}_{ee} + 2d^{(1)}_{ee} \otimes P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \\
& - \frac{13}{27}\, P^{(0)}_{ee} + 3P^{(0)}_{ee} \otimes P^{(0)}_{e\gamma} \otimes d^{(1)}_{\gamma e} + 4P^{(0)}_{ee} \otimes P^{(1)}_{ee} + \frac{4}{3} P^{(0)}_{ee} \otimes d^{(1)}_{ee} \\
& - \frac{40}{9}\, P^{(0)}_{ee} \otimes P^{(0)}_{ee} + 4P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes d^{(1)}_{ee} \Big);
\end{aligned}
$$

Results

$$
\begin{aligned}
c_{44} = {} & \frac{11}{27} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} + \frac{1}{3} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \otimes P^{(0)}_{e\bar{e}} + \frac{1}{3} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \otimes P^{(0)}_{\gamma\gamma} \\
& + \frac{1}{12} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \otimes P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{e\bar{e}} + \frac{1}{12} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \otimes P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{\gamma\gamma} \\
& + \frac{2}{3} P^{(0)}_{e\gamma} \otimes P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \otimes P^{(0)}_{\gamma e} + \frac{4}{27} P^{(0)}_{ee} + \frac{5}{3} P^{(0)}_{ee} \otimes P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \\
& + \frac{1}{2} P^{(0)}_{ee} P^{(0)}_{e\gamma} P^{(0)}_{\gamma e} P^{(0)}_{e\bar{e}} \\
& + \frac{1}{2} P^{(0)}_{ee} \otimes P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \otimes P^{(0)}_{\gamma\gamma} + \frac{22}{27} P^{(0)2}_{ee} + \frac{17}{12} P^{(0)}_{ee} \otimes P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \\
& + \frac{4}{3} P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{ee} + \frac{2}{3} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{ee}
\end{aligned}
$$

Summary

- Calculation of radiation correction is necessary to predict and describe high-energy processes
- Electron-positron annihilation process can be used to test the SM and to search for new physics
- Electron and positron can annihilate to a dark matter particle at future collider energies
- We calculated ISR correction to the process $e^+e^- \longrightarrow \gamma^*/Z^{0*}$
- Our main point is a different factorization and expressions of the Wilson coefficients

Thank you for your attention!

Splitting functions

$$P_{ee}^{(0)}(x) = \left[\frac{1+x^2}{1-x}\right]_+, \qquad P_{e\gamma}^{(0)}(x) = x^2 + (1-x)^2, \qquad P_{e\bar{e}}^{(0)}(x) = 0,$$

$$P_{\gamma e}^{(0)}(x) = \frac{1+(1-x)^2}{x}, \qquad P_{\gamma\gamma}^{(0)}(x) = \frac{\beta_0}{2}\delta(1-x),$$

$$P_{e\gamma}^{(1)}(x) = C_f T_f \left(4 - 9x - (1-4x)\ln x - (1-2x)\ln^2 x + 4\ln(1-x) + \right.$$
$$\left. + \left(2(\ln(1-x) - \ln x)^2 - 4(\ln(1-x) - \ln x) - 4\zeta(2) + 10\right) P_{e\gamma}^{(0)}(x)\right),$$

$$P_{\gamma e}^{(1)}(x) = C_f^2 \left(-\frac{5}{2} - \frac{7}{2}x + \left(2 + \frac{7}{2}x\right)\ln x - \left(1 - \frac{1}{2}x\right)\ln x^2 - 2x\ln(1-x)\right.$$
$$\left. -(3\ln(1-x) + \ln^2(1-x))P_{\gamma e}^{(0)}(x)\right) + C_f\left(-\frac{4x}{3} - \left(\frac{20}{9} + \frac{4\ln(1-x)}{3}\right)P_{\gamma e}^{(0)}(x)\right),$$

$$P_{ee}^{(1)NS}(x) = C_f^2\left(\left(-2\ln x\ln(1-x) - \frac{3}{2}\ln x\right)\frac{1+x^2}{1-x} - \left(\frac{3}{2} + \frac{7}{2}x\right)\ln x - \right.$$
$$\left. -\frac{1}{2}(1+x)\ln^2 x - 5(1-x)\right) + C_f T_f\left(-\frac{2}{3}\ln x - \frac{10}{9}\frac{1+x^2}{1-x} - \frac{4}{3}(1-x)\right) +$$
$$+\delta(1-x)\left(C_f^2\left(\frac{3}{8} - \frac{\pi}{2} + 6\zeta_3\right) - C_f T_f\left(\frac{1}{6} + \frac{2\pi^2}{9}\right)\right)$$