# On Higher-dimentional Quantum Field Theories

IBS-PNU Joint Workshop
on Particle Physics and Cosmology
Paradise Pusan Dec. 7 2022

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# 4,3 dim quantum field theories

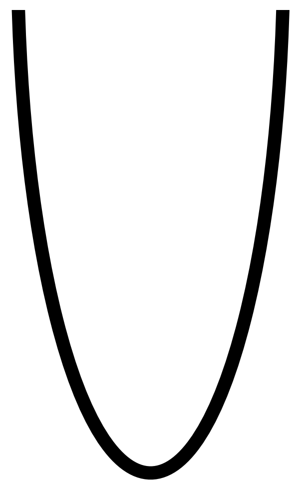
QFT: theory of identical particles with weak coupling vs. strong coupling

4d Yang-Mills: asymptotic freedom (UV free) and confinement (IR strong)

3d QFT with real scalar $\phi$ with $Z_2$ symmetry

$$\mathscr{L} = (\partial\phi)^2 + m^2\phi^2 + g^2\phi^4 + \lambda^2\phi^6 \text{ with } [\phi]^2 = M,\ [g^2] = M^2$$

for $m = \lambda = 0$ case, we get the perturbative expansion in $g_{\text{eff}}(p) = \dfrac{g^2}{p^2}$



In IR limit, there is a Wilson-Fisher fixed point $m_*^2 = 0,\ g_*^2 = \infty$.

It is the 3d Ising model conformal field theory.

conformal bootstrap method for $Z_2$ odd and even operators $\sigma, \epsilon \ \sim \phi, \phi^2$

$$\Delta_\sigma \approx 0.518154(15),\ \Delta_\epsilon = 1.41267(13)$$

# 3,2 dim quantum field theories

vortex-particle duality: U(1) complex scalar $\phi$ theory in WF fixed point = Maxwell theory with U(1) complex scalar theory in WF fixed point (Peskin)

$$\mathscr{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2 + \lambda|\phi|^4 \text{ vs } \mathscr{L} = -\frac{1}{e^2}F^2 + |D_\mu \phi'|^2 - m'^2|\phi'|^2 + \lambda'|\phi'|^4$$

monopole-fermion duality: U(1) CS theory with CS level $k = 1$ and a complex scalar at WF fixed point = a free massless Dirac fermion (Son, Karch and Tong, Seiberg and Witten)

Chern-Simons theories=symmetry protected topological phases

2d chiral conformal field theories= the boundary CFTs

classification of rational conformal field theories, moonshine of monster groups, modular tensor category, quantum computers, quantum codes....

# 5,6 dim quantum field theories

5d Yang-Mills field theory $\mathscr{L} = \frac{1}{4g^2}\mathrm{Tr}\ F^2$ with $[g^2] = M^{-1} = L$

We get the perturbative expansion in $g_{\rm eff}(p) = g^2|p|$ so the theory is strongly coupled at short distance.

In UV limit, one could expect a fixed point $g_*^2 = \infty$.

For the supersymmetric field theories, its existence as a conformal field theory is known.

Instantons provide $U(1)$ global symmetry with $j = *F \wedge F$.

For $\mathcal{N} = 1$ super Yang-Mills theories with gauge group $SU(2)_0$, $SU(2)_\pi$, where the discrete $\theta$ angle $0$, $\pi$ , the UV completion is 5d $\mathcal{N} = 1$ SCFs of global symmetry $SU(2)$, $U(1)$ respectively. They arise on the pq 5-brane webs in the type IIB string theory . Also they arise as the field theory limit of $R^{1\times 4} \cdot \mathrm{CY}_3$ in M-theory.

# 5,6 dim quantum field theories

For 5d $\mathcal{N} = 2$ super Yang-Mills qfts, their UV completions are 6d (2,0) superconformal theories of ADE types. It is selfdual tensor theory $H = dB,\ H = {}^*H, \psi, \phi_I$ on N M5 branes.....

the instantons provide the KK modes with $\dfrac{N}{R} = \dfrac{8\pi^2 N}{g^2}$ on a circle $x_6 \sim x_6 + 2\pi R.$

Two $\mathcal{N} = 2$ rank one theories of type $SU(2)_0,\ SU(2)_\pi$ have UV completion is 6d (2,0) superconformal theories of $A_1, A_2$ types on a circle. For the second case, outer morphism twisting of SU(3) type theories lead to the $Sp(1)_\pi = SU(2)_\pi$ theory. Note the relation between the products of the dual Coxeter number and the dimension of the Lie groups:

$$\frac{h^\vee_G d_G}{n_G} = h^\vee_{\tilde{H}} d_{\tilde{H}} \text{ with } n_G = 2,3,4. \quad \text{For SU(3) to Sp(1)=SU(2), } (3 \cdot 8)/4 = 2 \cdot 3$$

For $SU(2)$ theory with $N_f = 8$ fundamental hypermultiplets, its UV completion is 6-d $E_8$ string theory. It arises from a single M5 brane exploring a single Horava-Witten wall.

# Cardy limit on index function on $T^2 \times R^4_{\epsilon_{1,2}}$

6d (2,0) Theories $\mathrm{Re}(S) = 2\pi\sqrt{\sqrt{-\frac{2}{3}h_G d_G P(J_1+Q_R)(J_2+Q_R)} - Q_m^2}$

| $G$ | $h^\vee_G$ | $d_G$ | $n_G$ | $H$ | $H^\vee$ | $h^\vee_{H^\vee}$ | $d_{H^\vee}$ |
|---|---|---|---|---|---|---|---|
| $A_{2r-1}$ | $2r$ | $4r^2-1$ | 2 | $C_r$ | $B_r$ | $2r-1$ | $r(2r+1)$ |
| $A_{2r}$ | $2r+1$ | $4r(r+1)$ | 4 | $C'_r$ | $C'_r$ | $r+1$ | $r(2r+1)$ |
| $D_r$ | $2(r-1)$ | $r(2r-1)$ | 2 | $B_{r-1}$ | $C_{r-1}$ | $r$ | $(r-1)(2r-1)$ |
| $D_4$ | 6 | 28 | 3 | $G_2$ | $G_2$ | 4 | 14 |
| $E_6$ | 12 | 78 | 2 | $F_4$ | $F_4$ | 9 | 52 |

For $B_6$=SO(7) from $A_5$=SU(6), we get (6*35)/2= 5*31

# Questions

BPS invariants on $S^1 \times R^4_{\epsilon_1,\epsilon_2}$ or $T^2 \times R^4_{\epsilon_1,\epsilon_2}$

Nekrasov partition function, elliptic genus of selfdual strings, Gopakumar-Vafa invariants, ADHM approach, blow-up formula, mirror symmetries

BPS invariant operators and states on $S^1 \times S^4$, or $S^1 \times S^5$

Cardy limit, black hole entropy, non-Lagrangian theories, ...6d theories

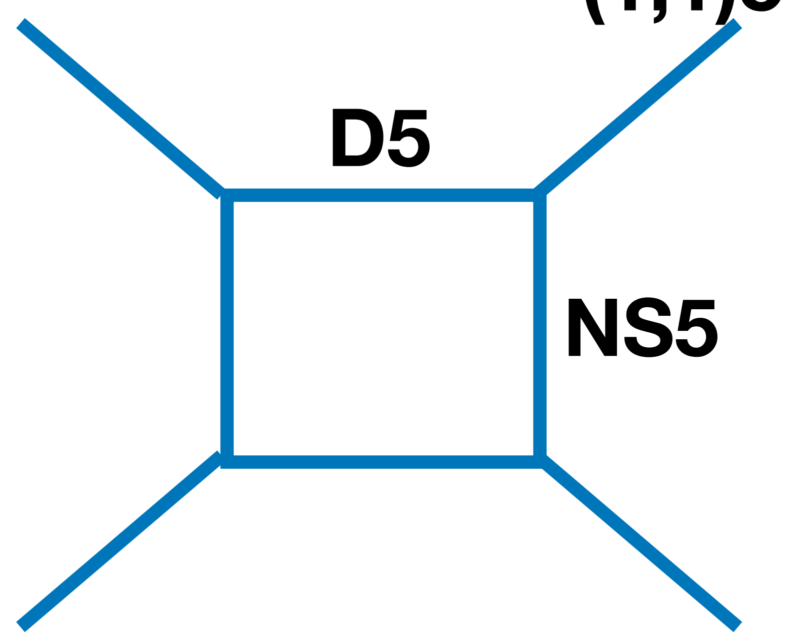
AdS-CFT correspondence to $AdS_6 \times \mathscr{M}_4$ or $AdS_7 \times \mathscr{M}_4$

Generalized symmetries

1-form and dual 2-form symmetries, 2-groups and 3-groups,...

NS-limit and integrable models, BPS-quivers on $R^{1,3}_{\epsilon_1} \times S^1$ or $R^{1,3}_{\epsilon_1} \times T^2$

# Brane Tiling and Bipartite Graphes




Algorithm for 5d SCFT arises from $R^{1+4} \times \mathrm{Toric\ CY}_3$

(1) Draw with pq 5-brane webs for Toric Calabi-Yau 3 manifold

(2) find the corresponding bi-partite graph on a torus

(3) write a BPS quiver theory for D0 branes of $R^{1+3} \times \mathrm{CY}_3 \times S^1_{\mathrm{M-circle}}$

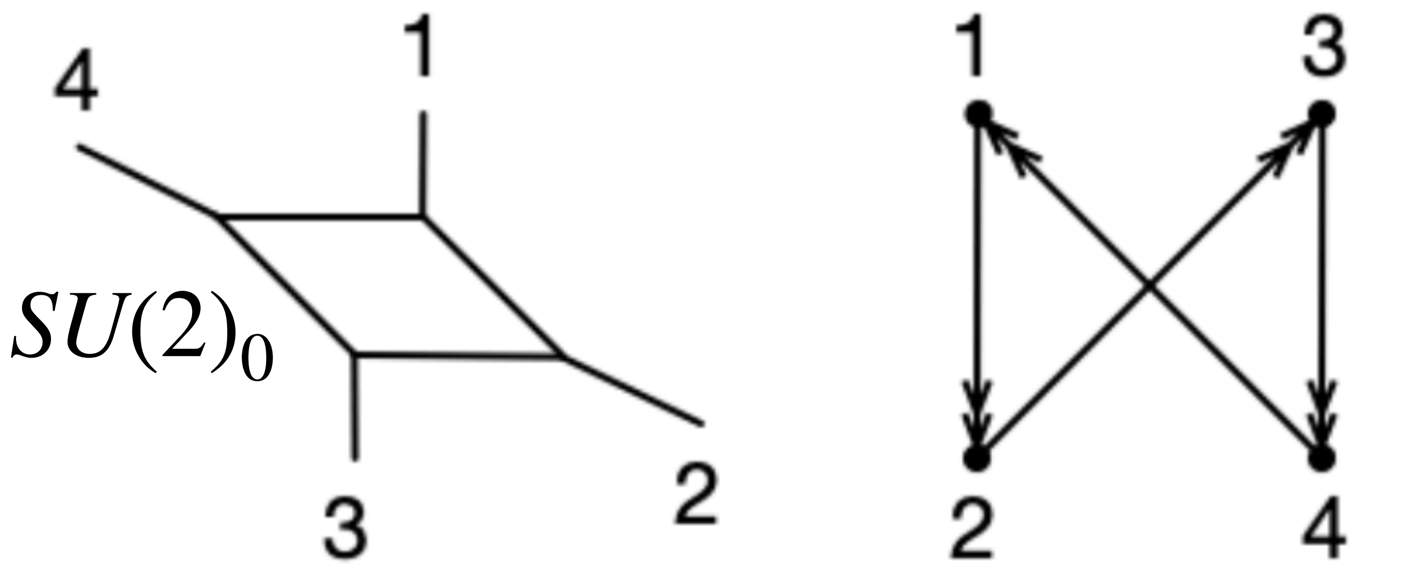
(4) write commuting Hamiltonians $H_a$ which commute each other

(5) check the spectral curve of the integrable model with SW curve

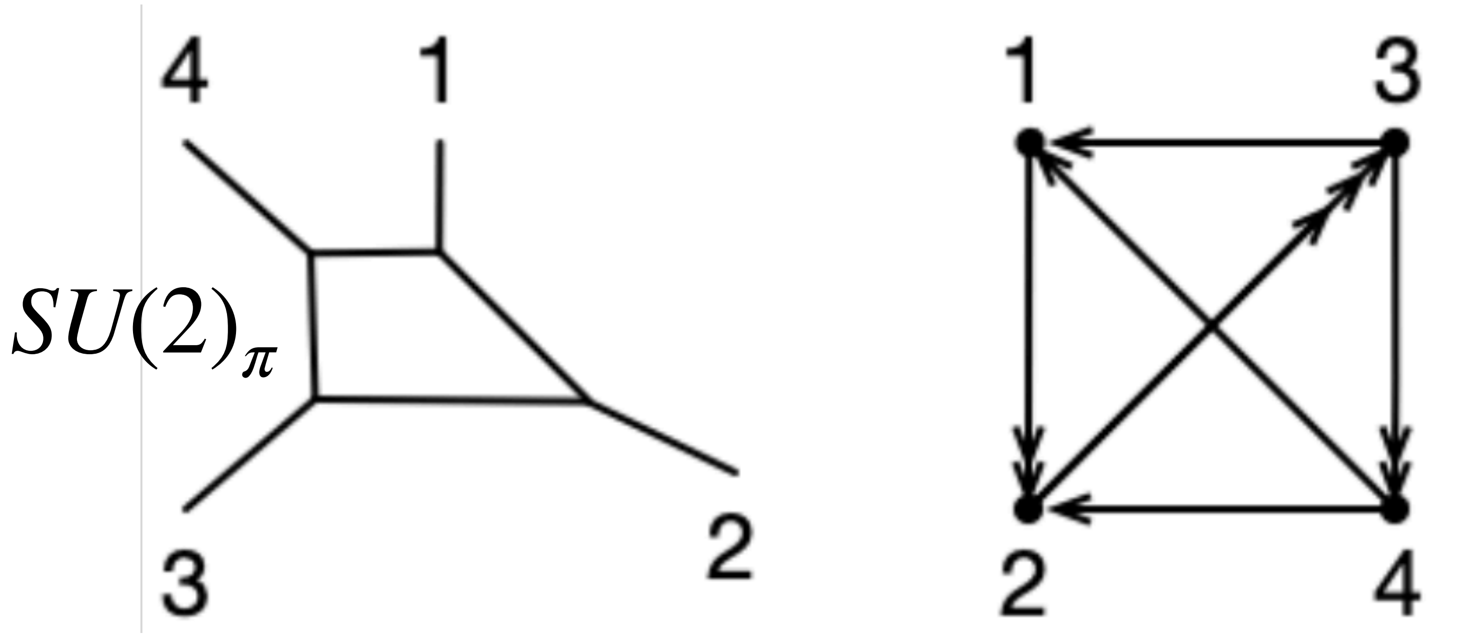
(6) calculate the codimension 2 defect partition function in the NS limit and find the commuting quantum Hamiltonians

(7) In NS limit, the SW curve becomes a quantum curve whose eigenfunction becomes the defect partition function and whose eigenvalue is the Wilson-loop.

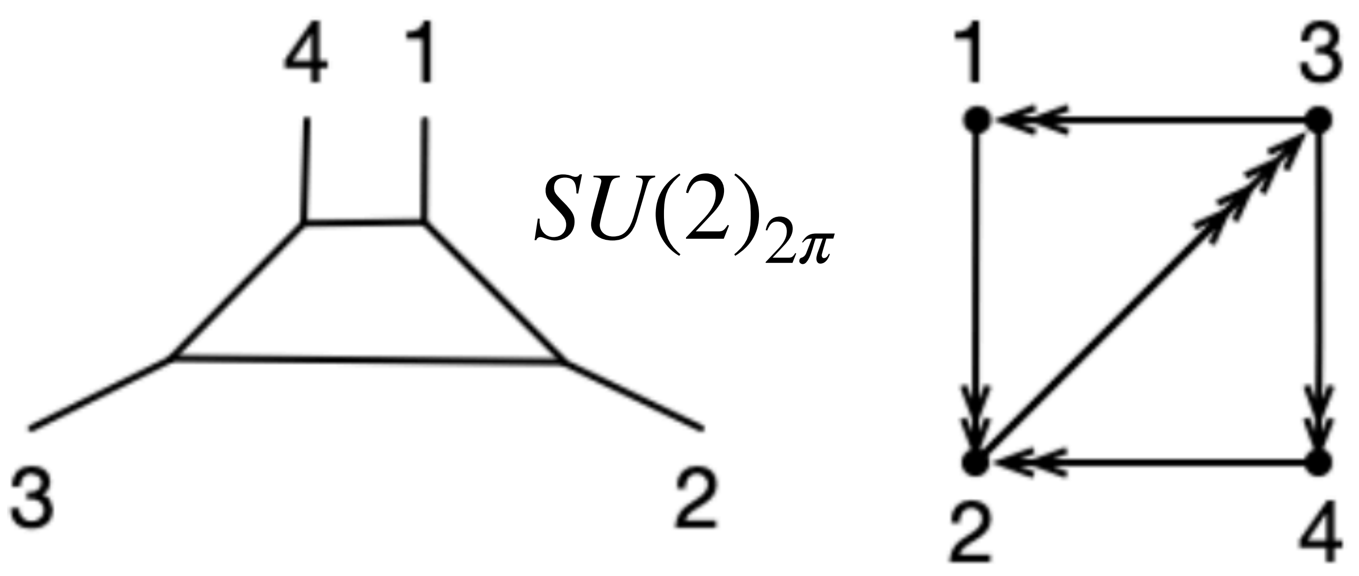
# rank 1



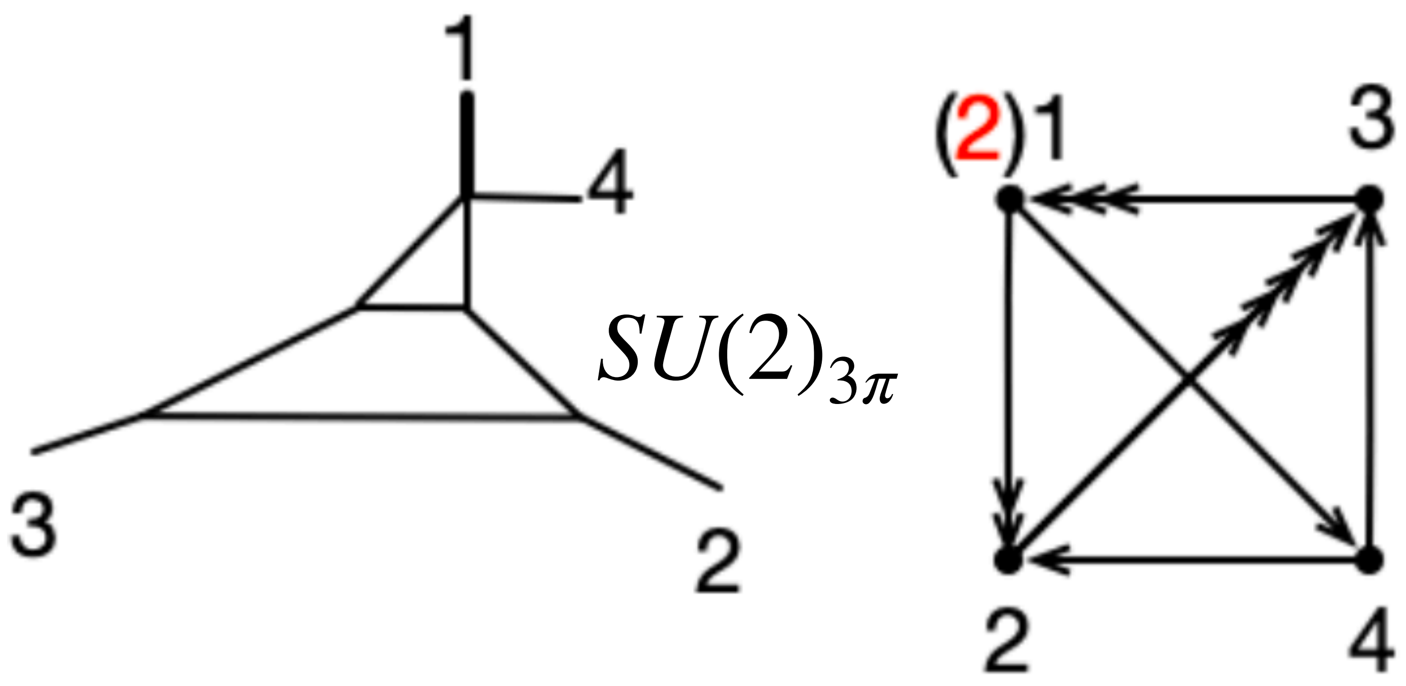
4
1
$SU(2)_0$
3
2
1
3
2
4

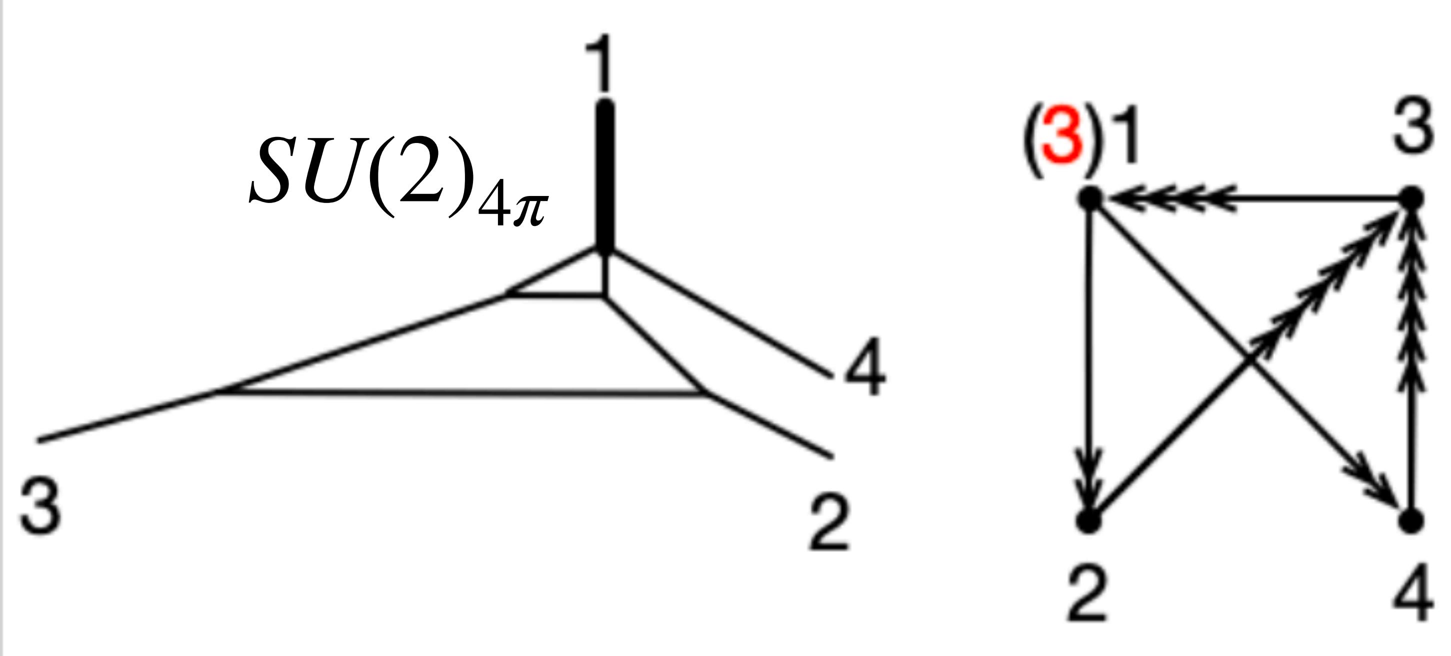
4
1
$SU(2)_\pi$
3
2
1
3
2
4

4
1
$SU(2)_{2\pi}$
3
2
1
3
2
4

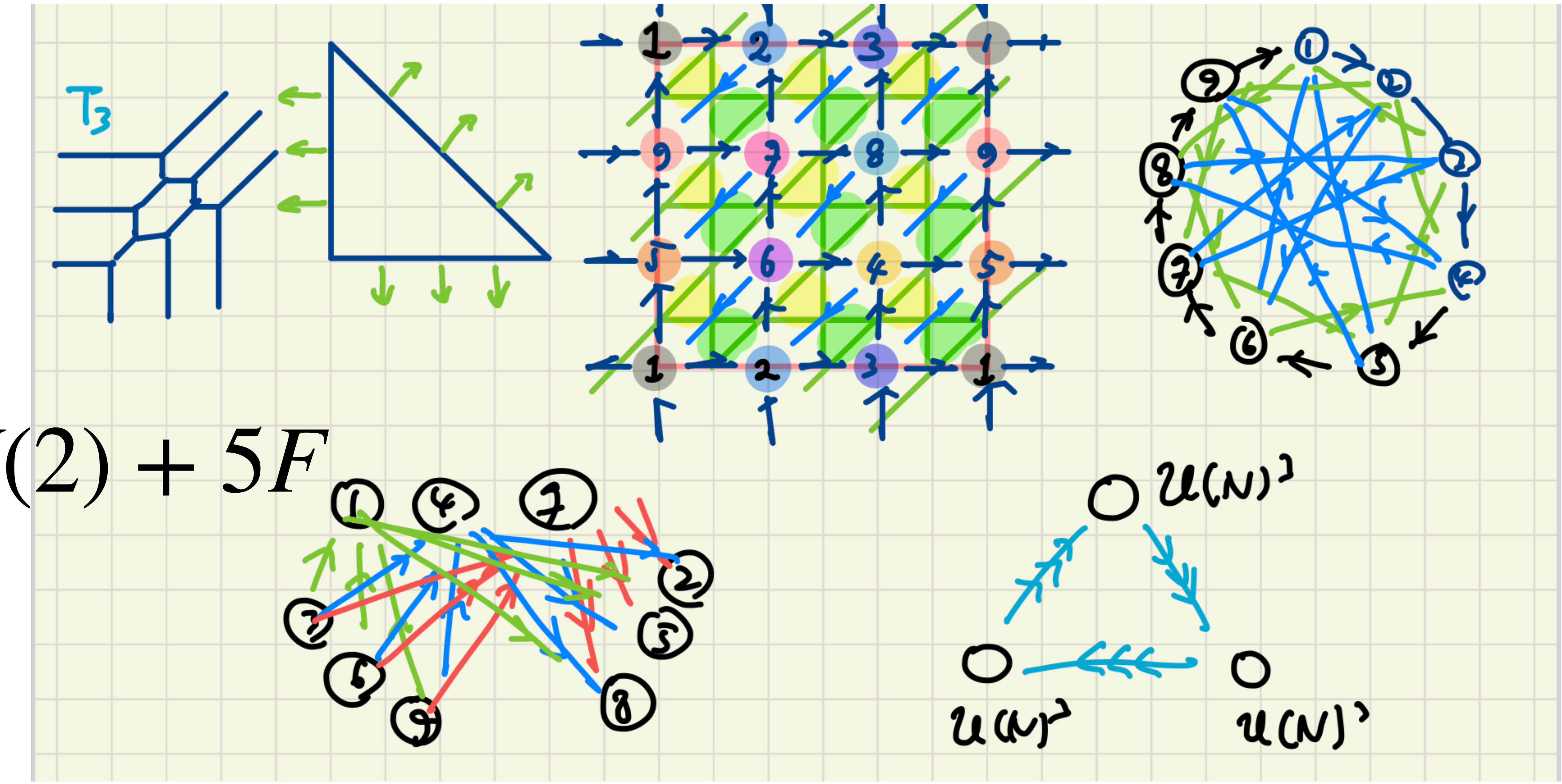



1
4
$SU(2)_{3\pi}$
3
2
(2)1
3
2
4

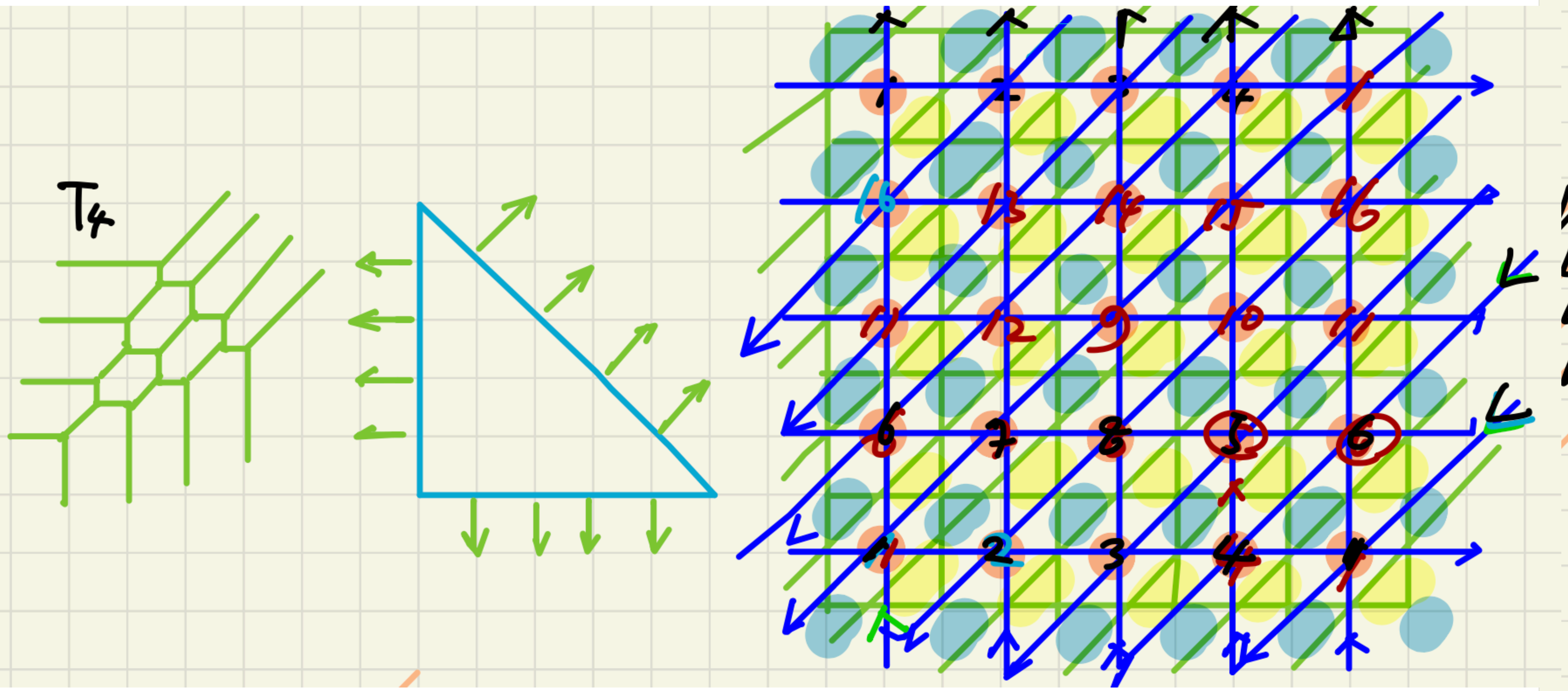
1
$SU(2)_{4\pi}$
4
3
2
(3)1
3
2
4

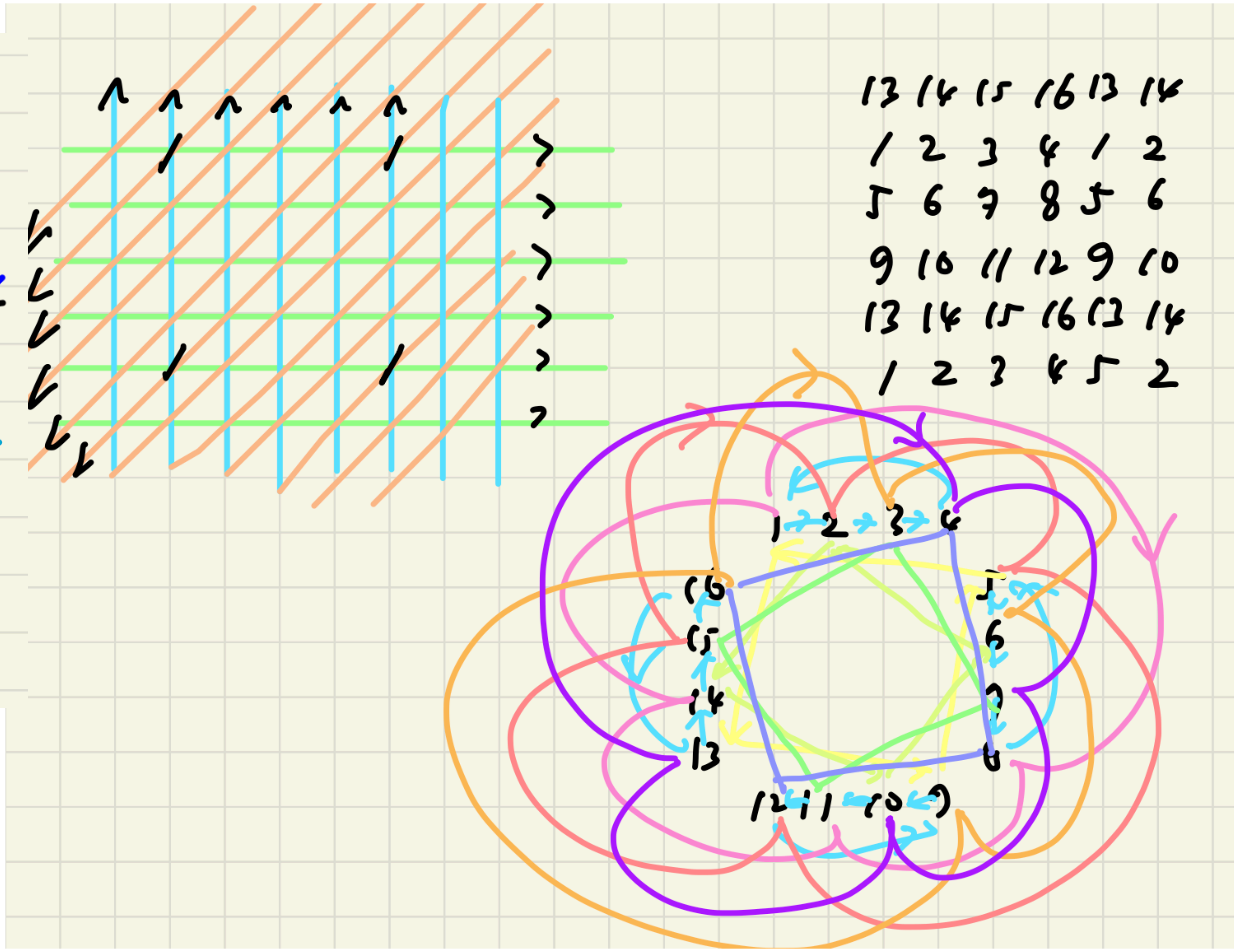

$SU(2) + 5F$



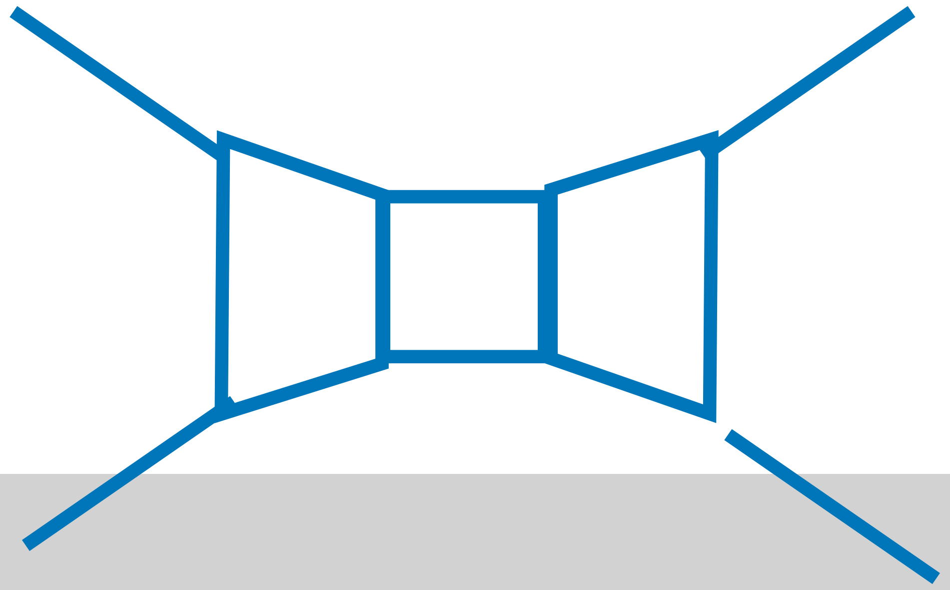
$T_3$

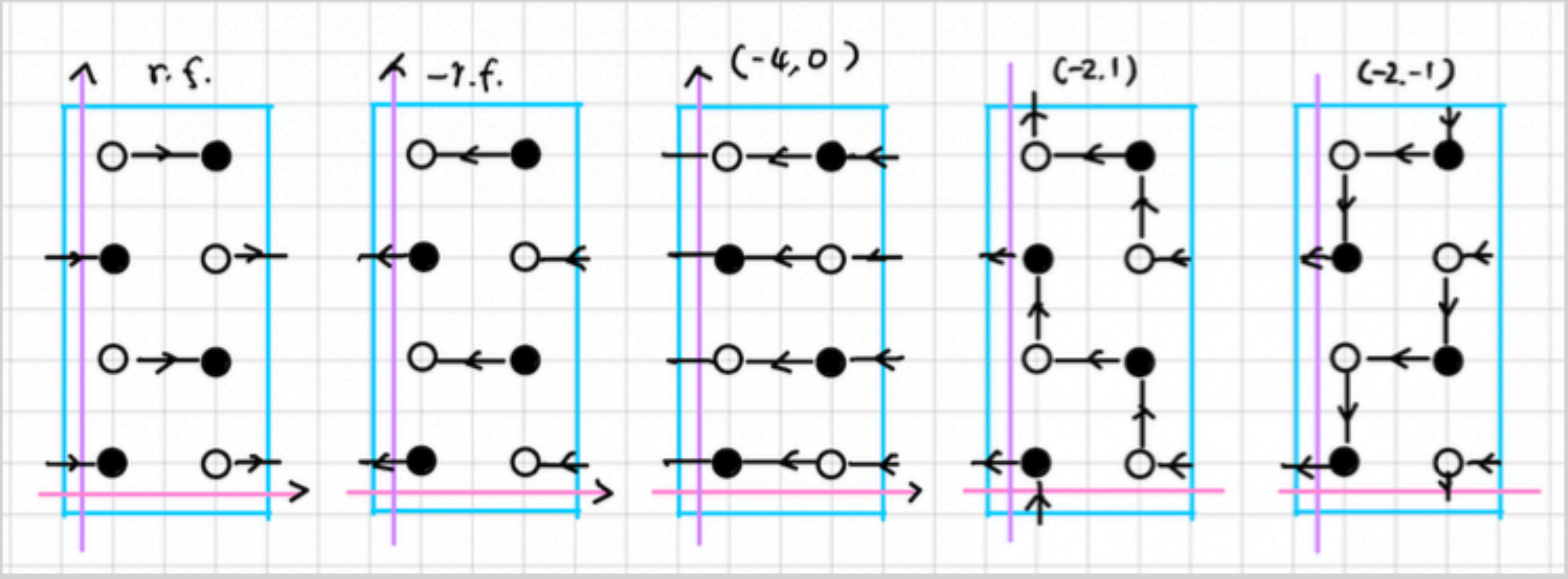
# rank 3

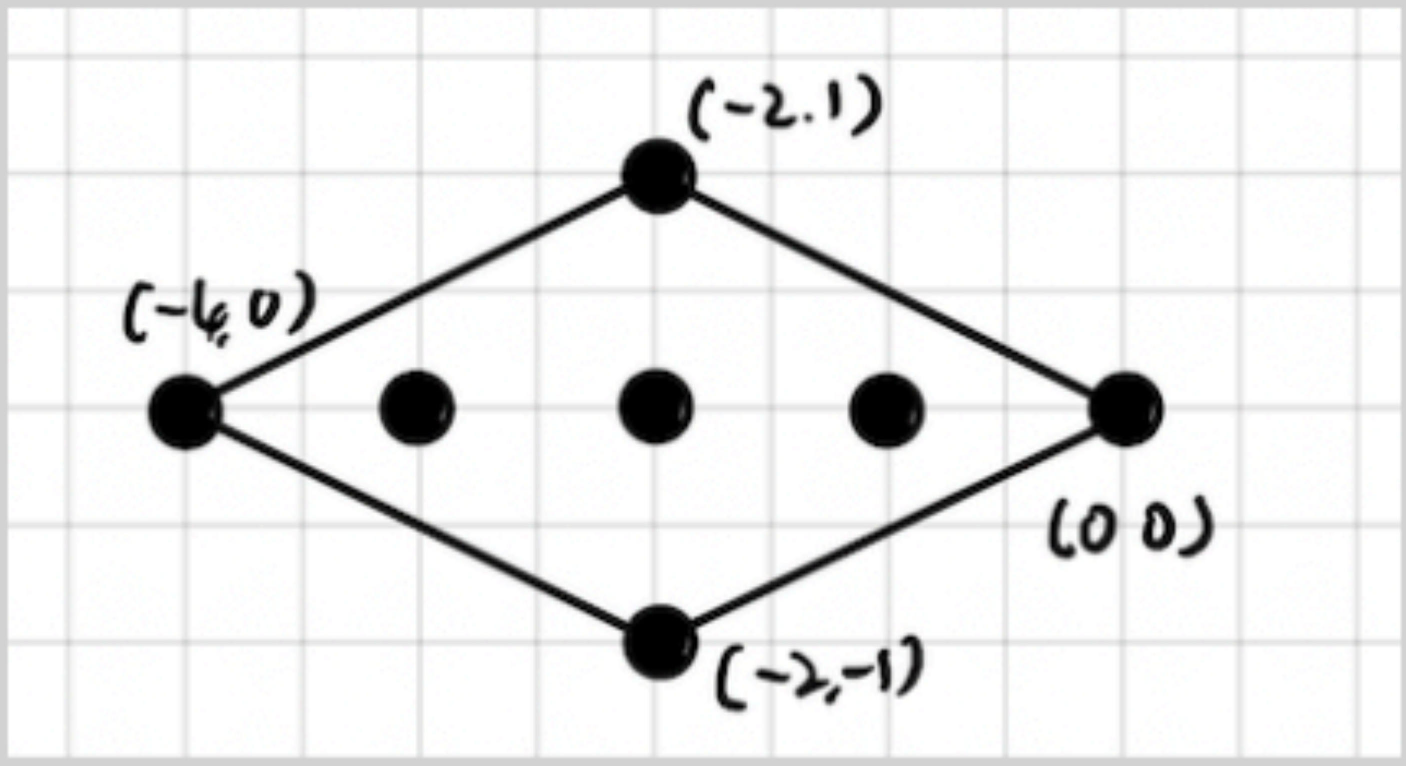


# $SU(4)_0$






$$\{c_i, c_{i+1}\} = -c_i c_{i+1},\ \{c_i, d_{i+1}\} = -c_i d_{i+1},\ \{c_i, d_i\} = c_i d_i.$$

$$\begin{aligned}
H_{(0,4)} &= \mathcal{H}_d = d_1 d_2 d_3 d_4, \\
H_{(0,3)} &= \mathcal{H}_d \mathcal{H}_-, \\
H_{(0,2)} &= d_1 d_3 + d_2 d_4 + \sum_{i=1}^{4} d_i (d_{i+1} + c_{i+1} + c_{i+2}) + c_1 c_3 + c_2 c_4, \\
H_{(0,1)} &= d_1 + d_2 + d_3 + d_4 + c_1 + c_2 + c_3 + c_4 \\
H_{(0,0)} &= 1,
\end{aligned}$$

$$\mathcal{H}_+ = \sum_i (d_j + c_j),\quad \mathcal{H}_- = \sum_j \frac{d_j + c_j}{d_j d_{j+1}}$$

**SW curve:** $t + t^{-1} = P(w) = w^2 H_{(0,2)} + wH_+ + H_0 + w^{-1}H_- + w^{-2}H_{(0,-2)}$

# $SU(4)_0 + 8F$

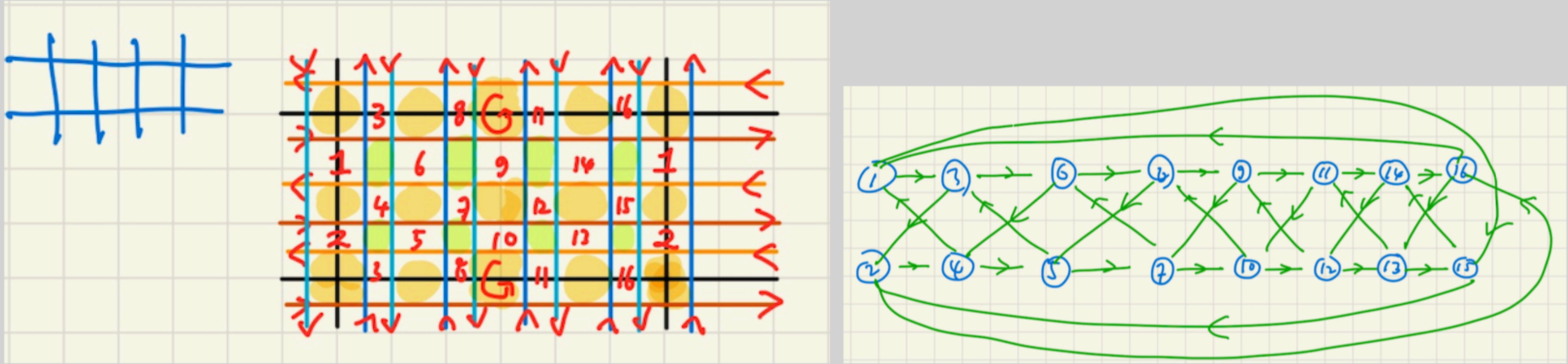


FIGURE 7. 2F-SU(2)-SU(2)-SU(2)-2F

# Relativistic Toda

$$
\begin{aligned}
H_1 &= \sum_{n=1}^{N} \left[1 + q^{-1/2} e^{\frac{2\pi}{w_2}(x_n - x_{n+1})}\right] e^{w_1 p_n}, \\
H_{N-1} &= H_N \sum_{n=1}^{N} \left[1 + q^{-1/2} e^{\frac{2\pi}{w_2}(x_n - x_{n+1})}\right] e^{-w_1 p_{n+1}}, \\
H_N &= \prod_{n=1}^{N} e^{w_1 p_n}.
\end{aligned}
$$

$$x_{n+1} = x_1 + m_0$$

$$[x_i, p_j] = i\hbar$$

# Higgsing & Non-Toric

For cases, we made some progress recently in the integrable models in terms of the defect partition function which can be obtained by blow-up formula for non-toric cases.

For BPS quiver from brane -tiling, one needs a deeper understanding and extension of the bipatite graphs. This would leads to the nontrivial BPS quivers on torus, which would open new area to explore.

# Conclusion

The compactification of 6d (2,0) theories to 4-dim, the so-called class S theories,  or 3-dim, are very rich topics, which taught us many new things about 3,4 theories.

The exploration of the relation between the 5,6 SCFTs and 6d little string theories would enrich us in the view of the defect partition function and also the integrable model.The relation between the integrable model and BPS quiver is highly interesting.

They have been providing a new perspectives of non-perturbative physics, even including 3,4 dimensional physics.

The  BPS-quivers and integrable models related to 5,6 dimensional QFTs is a rich subject just started to be explored by many people and we will find many new things there.

Our ambition in understanding the structure of Nature is endless. While we are a part of Nature, we are free to do the imagination of the possibilities.

Quantum Field Theories contain  gravity and string theory already as the part of its structures. We are just at the beginning of exploring this connection.