Dynamical Analysis in $4D$ Einstein-Gauss-Bonnet Gravity with Non-minimal Coupling

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based on arXiv:2210.06717 in collaboration with B. Bayarsaikhan, S. Khimphun, & P. Rithy

Contents:

Introduction

- General relativity (GR) has been tested at the precision 10^{-5} in the weak gravity regime, e.g., solar system, binary pulsars
- GW emission from binary pulsars is consistent with that in GR at the accuracy of $\mathcal{O}(0.1)\,\%$
- Still, the nature of gravity is not completely understood
  - cosmic accelerating expansion and/or dark matter
  - quantum effects of gravity, e.g., quantization of spacetime
  - big-bang and black hole singularities

GR could, in principle, deviate from what we know or be “modified”.

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GR could, in principle, deviate from what we know or be “modified”.

Introduction

- Thus, modified gravity theories are actively studied:
  - to better understand the nature of gravity
  - to understand the mysteries in the universe
    - inflation, dark energy, dark matter, etc
- By adding new gravitational degrees of freedom, one can modify GR:
  - Scalar-tensor theories: $L(g, \partial g, \partial^2 g, \cdots, \phi, \partial\phi, \partial^2\phi, \cdots)$
- "Healthy" higher-order theories must be degenerate:
  - Non-degenerate higher-order theories suffer from the so-called Ostrogradsky instability

Introduction

- One of the possible modifications of gravity is to consider the higher order of curvature corrections, e.g. $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, the Gauss-Bonnet (GB) term.

- GB gravity in $D > 4$ is well-investigated in the literature: suggested and motivated by string theory.

- In four dimensions ($4D$), the GB correction is known as a total derivative. Thus, the EoM derived from

$$S = \int d^D x \sqrt{-g}\,\alpha\mathcal{G},$$

vanishes identically

$$\frac{g^{\mu\nu}\delta S}{\sqrt{-g}\,\delta g^{\mu\nu}} = \frac{\alpha}{2}(D-4)\mathcal{G}.$$

- Common knowledge introduces the non-minimal coupling to GB term:

$$S = \int d^4 x \sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{\alpha}{2}\xi(\phi)\mathcal{G}\right) + S_{m,r}$$

PHYSICAL REVIEW LETTERS **124,** 081301 (2020)

Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime

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(Received 6 June 2019; revised manuscript received 26 September 2019; accepted 3 February 2020; published 26 February 2020)

In this Letter we present a general covariant modified theory of gravity in $D = 4$ spacetime dimensions which propagates only the massless graviton and bypasses Lovelock's theorem. The theory we present is formulated in $D > 4$ dimensions and its action consists of the Einstein-Hilbert term with a cosmological constant, and the Gauss-Bonnet term multiplied by a factor $1/(D-4)$. The four-dimensional theory is defined as the limit $D \to 4$. In this singular limit the Gauss-Bonnet invariant gives rise to nontrivial contributions to gravitational dynamics, while preserving the number of graviton degrees of freedom and being free from Ostrogradsky instability. We report several appealing new predictions of this theory, including the corrections to the dispersion relation of cosmological tensor and scalar modes, singularity resolution for spherically symmetric solutions, and others.

DOI: 10.1103/PhysRevLett.124.081301

$$S_{GB} = \int d^D x \sqrt{-g}\left[\frac{M_p^2}{2}R - \Lambda_0 + \frac{\alpha}{(D-4)}\mathcal{G}\right]$$

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

An intriguing idea in this paper:

- The theory is formulated in D dimensions,
- The GB term multiplied by a factor: $\dfrac{1}{(D-4)}$,
- As a result, the GB invariant yields finite nontrivial effects,
- It was also conjectured that the $D \to 4$ limit should have two dofs.

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$$\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}\right) + \frac{\alpha}{D-4}\mathcal{H}_{\mu\nu} = 0$$

$$\mathcal{H}_{\mu\nu} = 2\left[RR_{\mu\nu} - 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} + R_{\mu\alpha\beta\sigma}R_{\nu}{}^{\alpha\beta\sigma} - 2R_{\mu\alpha}R^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}\left(R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2\right)\right].$$

$$\mathcal{H}_{\mu\nu} = (D-4)\,\mathcal{Y}_{\mu\nu},$$

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In four dimensions, flat FLRW cosmology with ϕ:

$$3M_p^2H^2 + 6\alpha H^4 = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

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in contradiction with the common knowledge!

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THE EUROPEAN PHYSICAL JOURNAL C

Check for updates

Regular Article - Theoretical Physics

Is there a novel Einstein–Gauss–Bonnet theory in four dimensions?

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Abstract No! We show that the field equations of Einstein–Gauss–Bonnet theory defined in generic $D > 4$ dimensions split into two parts one of which always remains higher dimensional, and hence the theory does not have a non-trivial limit to $D = 4$. Therefore, the recently introduced four-dimensional, novel, Einstein–Gauss–Bonnet theory does not admit an *intrinsically* four-dimensional definition, in terms of metric only, as such it does not exist in four dimensions. The solutions (the spacetime, the metric) always remain $D > 4$ dimensional. As there is no canonical choice of 4 spacetime dimensions out of D dimensions for generic metrics, the theory is not well defined in four dimensions.

$$S_{GB} = \int d^Dx\sqrt{-g}\left[\frac{M_p^2}{2}R + \Lambda_0 + \frac{\alpha}{(D-4)}\mathcal{G}\right]$$

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$$\mathcal{H}_{\mu\nu} = (D-4)\,\mathcal{Y}_{\mu\nu},$$

$$\frac{\mathcal{H}_{\mu\nu}}{D-4} = 2\frac{\mathcal{L}_{\mu\nu}}{D-4} + \frac{2(D-3)}{(D-1)(D-2)}S_{\mu\nu},$$

$$\mathcal{L}_{\mu\nu} = C_{\mu\alpha\beta\gamma}C_\nu{}^{\alpha\beta\gamma} - \frac{1}{4}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}g_{\mu\nu} \Longrightarrow 0$$

: the Lanczos-Bach identity

$$S_{\mu\nu} = -\frac{2(D-1)}{(D-3)}C_{\mu\rho\nu\sigma}R^{\rho\sigma} - \frac{2(D-1)}{(D-2)}R_{\mu\rho}R_\nu^\rho + \frac{D}{(D-2)}R_{\mu\nu}R + \frac{(D-1)}{(D-2)}g_{\mu\nu}\left(R_{\rho\sigma}R^{\rho\sigma} - \frac{D+2}{4(D-1)}R^2\right).$$

$\nabla^\mu S_{\mu\nu} \neq 0$: the Bianchi identity is not satisfied!

The theory must be defined in $D > 4$ dimensions!

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in contradiction with the common knowledge!

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THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Is there a novel Einstein–dimensions?

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A consistent theory of $D \to 4$ Einstein-Gauss-Bonnet gravity

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ABSTRACT

We investigate the $D \to 4$ limit of the D-dimensional Einstein-Gauss-Bonnet gravity, where the limit is taken with $\tilde{\alpha} = (D-4)\alpha$ kept fixed and α is the original Gauss-Bonnet coupling. Using the ADM decomposition in D dimensions, we clarify that the limit is rather subtle and ambiguous (if not ill-defined) and depends on the way how to regularize the Hamiltonian or/and the equations of motion. To find a consistent theory in 4 dimensions that is different from general relativity, the regularization needs to either break (a part of) the diffeomorphism invariance or lead to an extra degree of freedom, in agreement with the Lovelock theorem. We then propose a consistent theory of $D \to 4$ Einstein-Gauss-Bonnet gravity with two dynamical degrees of freedom by breaking the temporal diffeomorphism invariance and argue that, under a number of reasonable assumptions, the theory is unique up to a choice of a constraint that stems from a temporal gauge condition.

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Phys. Lett. B 809, 135717 (2020) & *Phys.Lett.B* 810 (2020) 135843

in contradiction with the common knowledge!

an extra degree of freedom!

Motivation

- In addition to **the scaling constant**, we consider an **extra "scalar" degree of freedom** that is non-minimally coupled to the GB term

- We investigate the **gravitational dynamics** of the $4D$ EGB gravity in the "late-time universe,"

- For this motivation, we perform the dynamical system analysis and study the fixed points of the system and their stability,

- We present the cosmic evolution of the universe and constrain the model with observational data to obtain the valid parameter space.

The setup and Equations of Motion (EoM)

- The action

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \frac{\alpha}{(D-4)} \xi^{(D-4)}(\phi) \mathcal{G} \right) + S_{m,r}$$

- In an arbitrary D dimensional flat FLRW universe with the metric,

$$ds^2 = -dt^2 + a(t)^2 \left(dx_1^2 + dx_2^2 + dx_3^2 + \ldots dx_{D-1}^2 \right),$$

we obtain the EoM

please, refer to arXiv:2210.06717 for $T_{\mu\nu}^{(\phi.GB,m,r)}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 \left(T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{GB} + T_{\mu\nu}^{m,r} \right),$$

$$\Box \phi - V_\phi - \frac{\alpha}{2(D-4)} (D-4) \xi^{(D-5)} \xi_\phi \mathcal{G} = 0, \qquad \text{where} \quad \xi_\phi \equiv \frac{d\xi}{d\phi}.$$

The setup and Equations of Motion (EoM)

- In the $D \to 4$ limit, the EoM read:

$$3H^2 = \kappa^2\left(\rho_m + \rho_r + \rho_\phi\right),$$

$$2\dot{H} + 3H^2 = -\kappa^2\left(\frac{\rho_r}{3} + P_\phi\right),$$

$$\ddot{\phi} + V_\phi + 3\dot{\phi}H + 12\alpha\frac{\xi_\phi}{\xi}H^2\left(\dot{H} + H^2\right) = 0,$$

where

with non-minimal coupling term

without non-minimal coupling

$$\rho_\phi \equiv \frac{1}{2}\dot{\phi}^2 + V + 12\alpha\frac{\dot{\xi}}{\xi}H^3 + 3\alpha H^4,$$

$$P_\phi \equiv \frac{\dot{\phi}^2}{2} - V - 8\alpha\frac{\dot{\xi}}{\xi}H(\dot{H} + H^2) - 3\alpha H^4 - 4\alpha H^2\left(\dot{H} - \frac{\dot{\xi}^2}{\xi^2} + \frac{\ddot{\xi}}{\xi}\right).$$

- The conservation equation: $\dot{\rho}_\phi + 3H(1 + \omega_\phi)\rho_\phi = 0$ with $\omega_\phi = P_\phi/\rho_\phi$.

Dynamic System Analysis (DSA)

- The DSA allows us to understand the cosmological dynamics of the system systematically.
- We define the dimensionless variables:

$$x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \quad \alpha_{GB} \equiv \sqrt{\alpha}\kappa H, \quad \mu \equiv -\frac{\sqrt{6}\xi_\phi}{\kappa\xi}, \quad \lambda \equiv -\frac{V_\phi}{\sqrt{6}\kappa V}, \quad \epsilon \equiv -\frac{\dot{H}}{H^2},$$

and rewrite the EoM as:

$$\frac{dx}{dN} = \frac{\kappa\ddot{\phi}}{\sqrt{6}H^2} - \frac{\kappa\dot{\phi}\dot{H}}{\sqrt{6}H^3} = x(\epsilon - \delta),$$

$$\frac{dy}{dN} = \frac{\kappa V_\phi\dot{\phi}}{2\sqrt{3}\sqrt{V}H^2} - \frac{\kappa\sqrt{V}\dot{H}}{\sqrt{3}H^3} = y(\epsilon - 3x\lambda),$$

$$\frac{d\alpha_{GB}}{dN} = \frac{\sqrt{\alpha}\kappa\dot{H}}{H} = -\alpha_{GB}\epsilon,$$

$$\frac{d\Omega_r}{dN} = -\frac{2\kappa^2\rho_r\dot{H}}{3H^4} + \frac{\kappa^2\dot{\rho}_r}{3H^3} = -4\Omega_r + 2\Omega_r\epsilon,$$

$$\frac{d\mu}{dN} = \frac{\sqrt{6}\xi_\phi^2\dot{\phi}}{\kappa\xi^2 H} - \frac{\sqrt{6}\xi_{\phi\phi}\dot{\phi}}{\kappa\xi H} = x\mu^2(1 - \Delta),$$

$$\frac{d\lambda}{dN} = \frac{V_\phi^2\dot{\phi}}{\sqrt{6}\kappa V^2 H} - \frac{V_{\phi\phi}\dot{\phi}}{\sqrt{6}\kappa V H} = 6x\lambda^2(1 - \Gamma),$$

where $\Omega_m \equiv \frac{\kappa^2\rho_m}{3H^2}, \quad \Omega_r \equiv \frac{\kappa^2\rho_r}{3H^2}, \quad \Omega_\phi \equiv x^2 + y^2 + \alpha_{GB}^2 - 4xz^2\mu$ and $1 = \Omega_m + \Omega_r + \Omega_\phi$,

Model: $V(\phi) = V_0 e^{-\sqrt{6}\kappa\lambda\phi}$ & $\xi(\phi) = \xi_0 e^{-\kappa\mu\phi/\sqrt{6}} \Longrightarrow \Gamma \equiv \frac{V_{\phi\phi}V}{V_\phi^2} = 1, \quad \Delta \equiv \frac{\xi_{\phi\phi}\xi}{\xi_\phi^2} = 1.$

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~~$$\frac{d\mu}{dN} = \frac{\sqrt{6}\xi_\phi^2\dot{\phi}}{\kappa\xi^2 H} - \frac{\sqrt{6}\xi_{\phi\phi}\dot{\phi}}{\kappa\xi H} = x\mu^2(1-\Delta),$$~~

~~$$\frac{d\lambda}{dN} = \frac{V_\phi^2\dot{\phi}}{\sqrt{6}\kappa V^2 H} - \frac{V_{\phi\phi}\dot{\phi}}{\sqrt{6}\kappa V H} = 6x\lambda^2(1-\Gamma),$$~~

where $\Omega_m \equiv \frac{\kappa^2\rho_m}{3H^2}, \quad \Omega_r \equiv \frac{\kappa^2\rho_r}{3H^2}, \quad \Omega_\phi \equiv x^2 + y^2 + \alpha_{GB}^2 - 4xz^2\mu$ and $1 = \Omega_m + \Omega_r + \Omega_\phi$,

$$V(\phi) = V_0 e^{-\sqrt{6}\kappa\lambda\phi} \quad \& \quad \xi(\phi) = \xi_0 e^{-\kappa\mu\phi/\sqrt{6}}.$$

$$\omega_{eff} \equiv -1 - \frac{2\dot{H}}{3H^2}.$$

Represents the GR fixed points: *Quintessence* model (Phys. Rept. 775-777 (2018) 1-122)

Points	x	y	α_{GB}	Ω_r	Ω_m	Ω_ϕ	Existence	$\omega_{eff}=\omega_\phi$
$A_1^\pm$	$\frac{1}{2\lambda}$	$\pm\frac{1}{2\lambda}$	0	0	$1-\frac{1}{2\lambda^2}$	$\frac{1}{2\lambda^2}$	$\lambda^2 > 1/2$	0
$A_2^\pm$	$\frac{2}{3\lambda}$	$\pm\frac{\sqrt{2}}{3\lambda}$	0	$1-\frac{2}{3\lambda^2}$	0	$\frac{2}{3\lambda^2}$	$\lambda^2 > 2/3$	$1/3$
$A_3^\pm$	±1	0	0	0	0	1	$\forall\lambda$	1
A_4	0	0	0	0	1	0	$\forall\lambda$	0
A_5	0	0	0	1	0	0	$\forall\lambda$	$1/3$
$A_6^\pm$	λ	$\pm\sqrt{1-\lambda^2}$	0	0	0	1	$\lambda^2 < 1$	$-1+2\lambda^2$
$A_7^\pm$	0	$-\sqrt{\frac{2\mu}{2\mu-3\lambda}}$	$\pm\sqrt{\frac{3\lambda}{3\lambda-2\mu}}$	0	0	1	$(\mu\geq0,\lambda<0)$ or $(\mu\leq0,\lambda>0)$	-1
$A_8^\pm$	0	$\sqrt{\frac{2\mu}{2\mu-3\lambda}}$	$\pm\sqrt{\frac{3\lambda}{3\lambda-2\mu}}$	0	0	1	$(\mu\geq0,\lambda<0)$ or $(\mu\leq0,\lambda>0)$	-1
$A_9^\pm$	$\frac{3-\sqrt{9-80\mu^2}}{20\mu}$	0	$\pm\sqrt{\frac{6}{3+\sqrt{9-80\mu^2}}}$	0	0	1	$-\frac{3}{4\sqrt{5}}\leq\mu\leq\frac{3}{4\sqrt{5}}, \mu\neq0$	-1
$A_{10}^\pm$	$\frac{3+\sqrt{9-80\mu^2}}{20\mu}$	0	$\pm\sqrt{\frac{9+3\sqrt{9-80\mu^2}}{40\mu^2}}$	0	0	1	$-\frac{3}{4\sqrt{5}}\leq\mu\leq\frac{3}{4\sqrt{5}}, \mu\neq0$	-1

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H},\quad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H},\quad \alpha_{GB} = \sqrt{\alpha}\kappa H,\quad \Omega_r = \frac{\kappa^2\rho_r}{3H^2},\quad \Omega_m = \frac{\kappa^2\rho_m}{3H^2},\quad \Omega_\phi = 1-\Omega_m-\Omega_r.$$

The GB fixed points with $\alpha \neq 0$. The universe can be dominated by $\Omega_\phi = 1$ and the late-time acceleration with $\omega_{eff} = \omega_\phi = -1$

$$V(\phi) = V_0 e^{-\sqrt{6}\kappa\lambda\phi} \quad \& \quad \xi(\phi) = \xi_0 e^{-\kappa\mu\phi/\sqrt{6}}\,.$$

$$\Gamma \equiv \frac{V_{\phi\phi}V}{V_\phi^2} = 1, \quad \Delta \equiv \frac{\xi_{\phi\phi}\xi}{\xi_\phi^2} = 1.$$

Points	Eigenvalues	Stability
$A_1^\pm$	$\left\{-\frac{3}{2}, -1, -\frac{3}{4}\left(1+\sqrt{\frac{4}{\lambda^2}-7}\right), -\frac{3}{4}\left(1-\sqrt{\frac{4}{\lambda^2}-7}\right)\right\}$	stable for: $1/2 < \lambda^2 \leq 4/7$, stable-focus for: $\lambda^2 > 4/7$, saddle for: $\lambda^2 < 1/2$
$A_2^\pm$	$\left\{-2, +1, -\frac{1}{2}\left(1+\sqrt{\frac{32}{3\lambda^2}-15}\right), -\frac{1}{2}\left(1-\sqrt{\frac{32}{3\lambda^2}-15}\right)\right\}$	saddle.
$A_3^\pm$	$\{-3, +3, +2, 3\pm 3\lambda\}$	saddle.
A_4	$\{-\frac{3}{2}, -\frac{3}{2}, +\frac{3}{2}, -1\}$	saddle.
A_5	$\{-2, +2, -1, +1\}$	saddle.
$A_6^\pm$	$\{-3\lambda^2, 3\left(\lambda^2-1\right), 6\lambda^2-3, 6\lambda^2-4\}$	stable for: $0 < \lambda^2 < 1/2$; otherwise saddle.
$A_7^\pm$	$\left\{-4, -3, -\frac{3}{2}\left(1-\sqrt{1+\frac{32\lambda^2\mu^2-48\lambda^3\mu}{4\mu^2-\lambda^2(9-36\mu^2)}}\right), -\frac{3}{2}\left(1+\sqrt{1+\frac{32\lambda^2\mu^2-48\lambda^3\mu}{4\mu^2-\lambda^2(9-36\mu^2)}}\right)\right\}$	stable for: con_1, stable focus for: $\frac{3\lambda^2}{-8\lambda^3+2\sqrt{16\lambda^6+17\lambda^4+\lambda^2}} < \mu < \frac{3\lambda}{2\sqrt{1+9\lambda^2}}$, saddle for: $\mu > \frac{3}{2}\sqrt{\frac{\lambda^2}{1+9\lambda^2}}$ if $\lambda < 0$ or $\mu < -\frac{3}{2}\sqrt{\frac{\lambda^2}{1+9\lambda^2}}$ if $\lambda > 0$.
$A_8^\pm$	$\left\{-4, -3, -\frac{3}{2}\left(1-\sqrt{1+\frac{32\lambda^2\mu^2-48\lambda^3\mu}{4\mu^2-\lambda^2(9-36\mu^2)}}\right), -\frac{3}{2}\left(1+\sqrt{1+\frac{32\lambda^2\mu^2-48\lambda^3\mu}{4\mu^2-\lambda^2(9-36\mu^2)}}\right)\right\}$	
$A_9^\pm$	$\left\{-\frac{3\lambda}{20\mu}\left(3-\sqrt{9-80\mu^2}\right), -4, -3, -3\right\}$	stable for: con_2; otherwise saddle.
$A_{10}^\pm$	$\left\{-4, -3, -3, -\frac{3\lambda}{20\mu}\left(3+\sqrt{9-80\mu^2}\right)\right\}$	stable for: con_2; otherwise saddle.

$$\mathrm{con}_1 = \begin{cases} 0 < \mu \leq \frac{3\lambda^2}{-8\lambda^3+2\sqrt{16\lambda^6+17\lambda^4+\lambda^2}}, & \text{for } \lambda < 0 \\ -\frac{3\lambda^2}{8\lambda^3+2\sqrt{16\lambda^6+17\lambda^4+\lambda^2}} \leq \mu < 0, & \text{for } \lambda > 0 \end{cases},$$

$$\mathrm{con}_2 = \begin{cases} -\frac{3}{4\sqrt{5}} \leq \mu < 0\,, & \text{for } \lambda < 0 \\ 0 < \mu \leq \frac{3}{4\sqrt{5}}\,, & \text{for } \lambda > 0 \end{cases}.$$

$$V(\phi) = V_0 e^{-\sqrt{6}\kappa\lambda\phi} \quad \& \quad \xi(\phi) = \xi_0 e^{-\kappa\mu\phi/\sqrt{6}} .$$

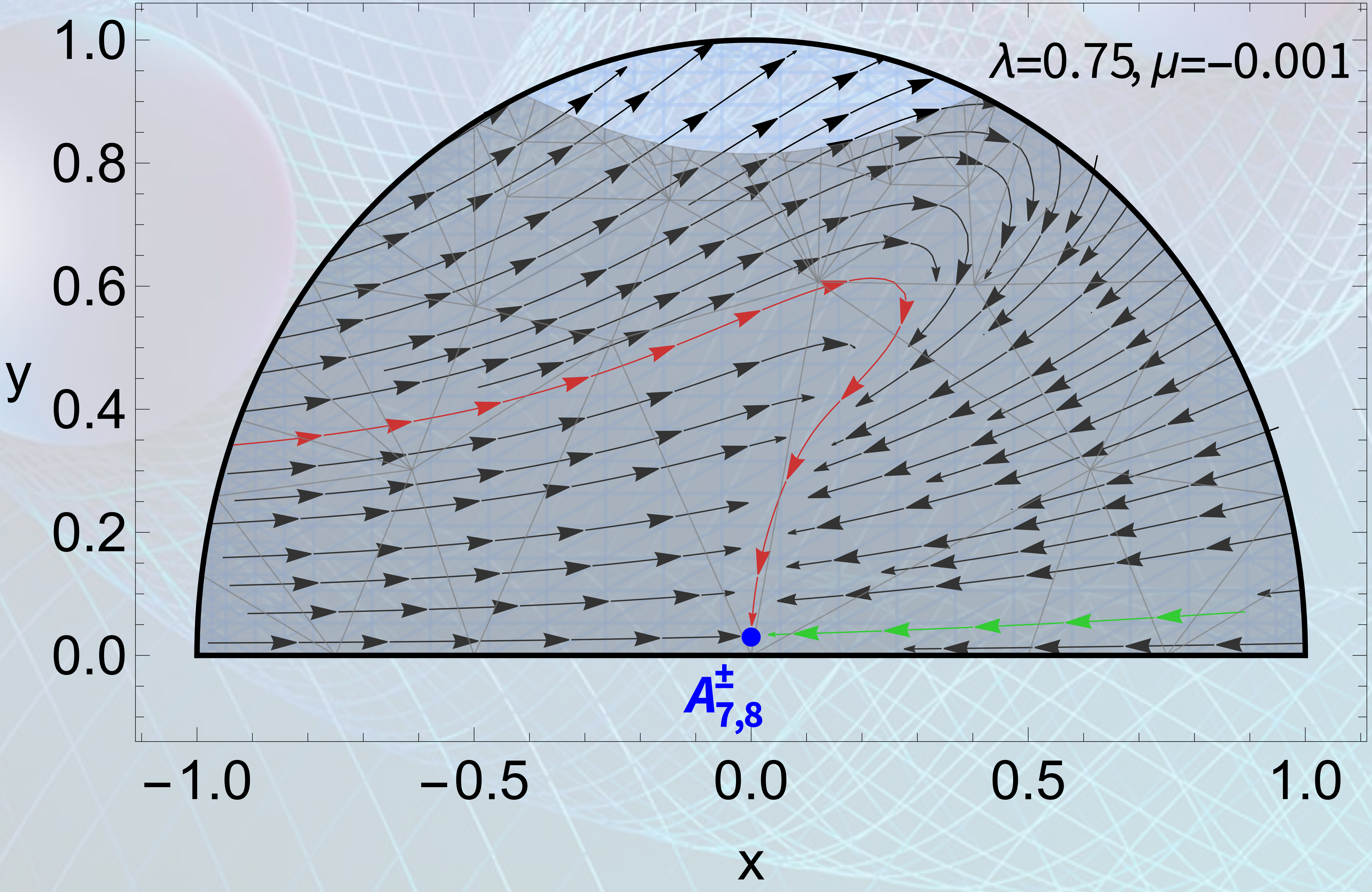
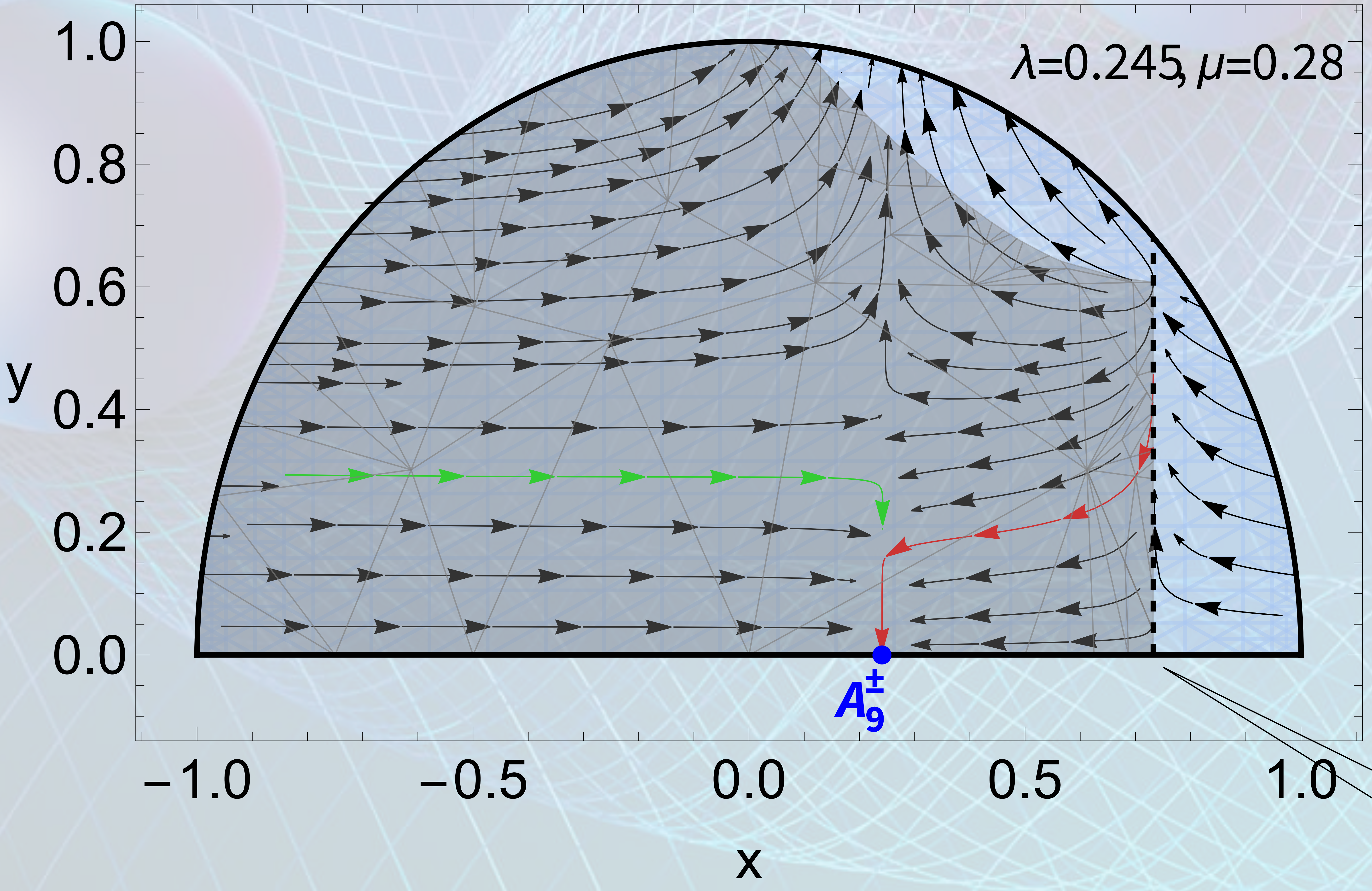


Figure 1. Phase space portraits of the autonomous system on a x vs. y plane. The acceleration of the universe is possible within the gray-shaded region. The green and red lines are particular attractor trajectories toward the fixed points $A^{\pm}_{7,8}$, $A^{\pm}_{9}$, and $A^{\pm}_{10}$ when their stability conditions are satisfied.

$$V(\phi) = V_0 e^{-\sqrt{6}\kappa\lambda\phi} \quad \& \quad \xi(\phi) = \xi_0 e^{-\kappa\mu\phi/\sqrt{6}}.$$



$$x \equiv x_{div} = \frac{-1 + 2\alpha_{GB}^2 - 4\alpha_{GB}^4\mu^2}{4\alpha_{GB}^2\mu},$$

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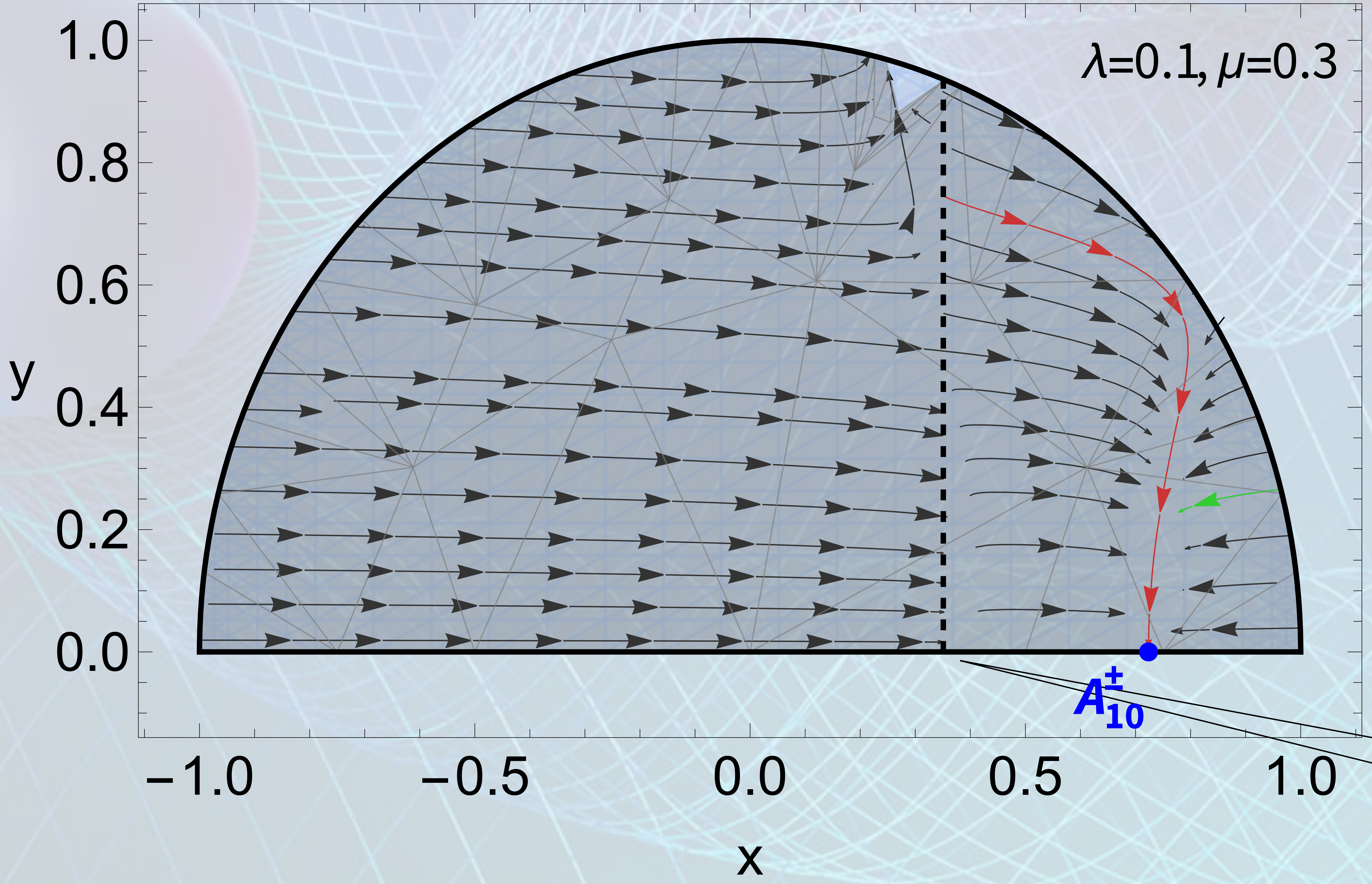

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Figure 1. Phase space portraits of the autonomous system on a x vs. y plane. The acceleration of the universe is possible within the gray-shaded region. The green and red lines are particular attractor trajectories toward the fixed points $A_{7,8}^{\pm}$, $A_9^{\pm}$, and $A_{10}^{\pm}$ when their stability conditions are satisfied.

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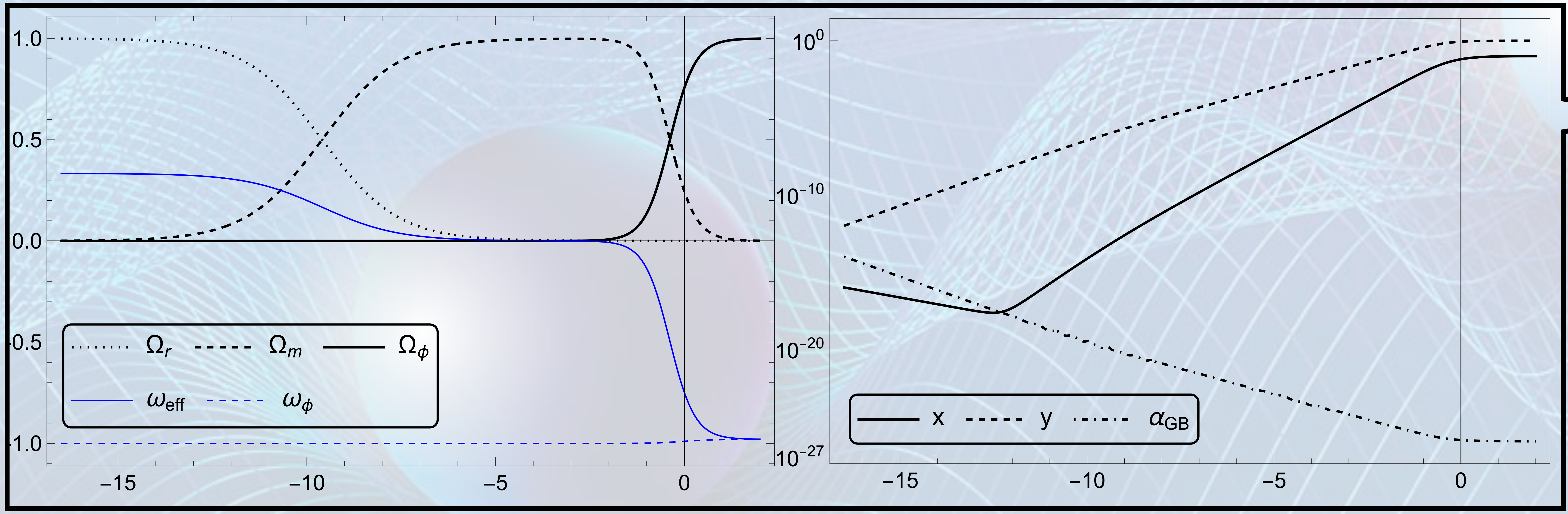
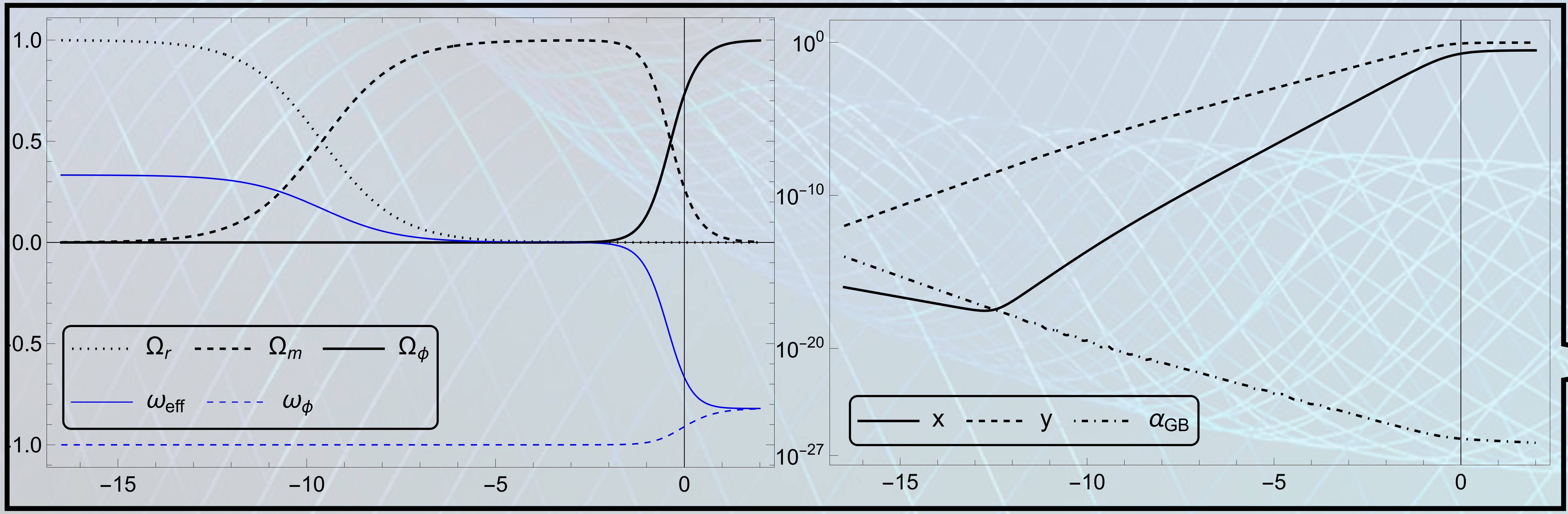


$\lambda = 0.1,$

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \quad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \quad \alpha_{GB} = \sqrt{\alpha}\kappa H,$$

$$\Omega_r = \frac{\kappa^2\rho_r}{3H^2}, \quad \Omega_m = \frac{\kappa^2\rho_m}{3H^2}, \quad \Omega_\phi = 1 - \Omega_m - \Omega_r.$$



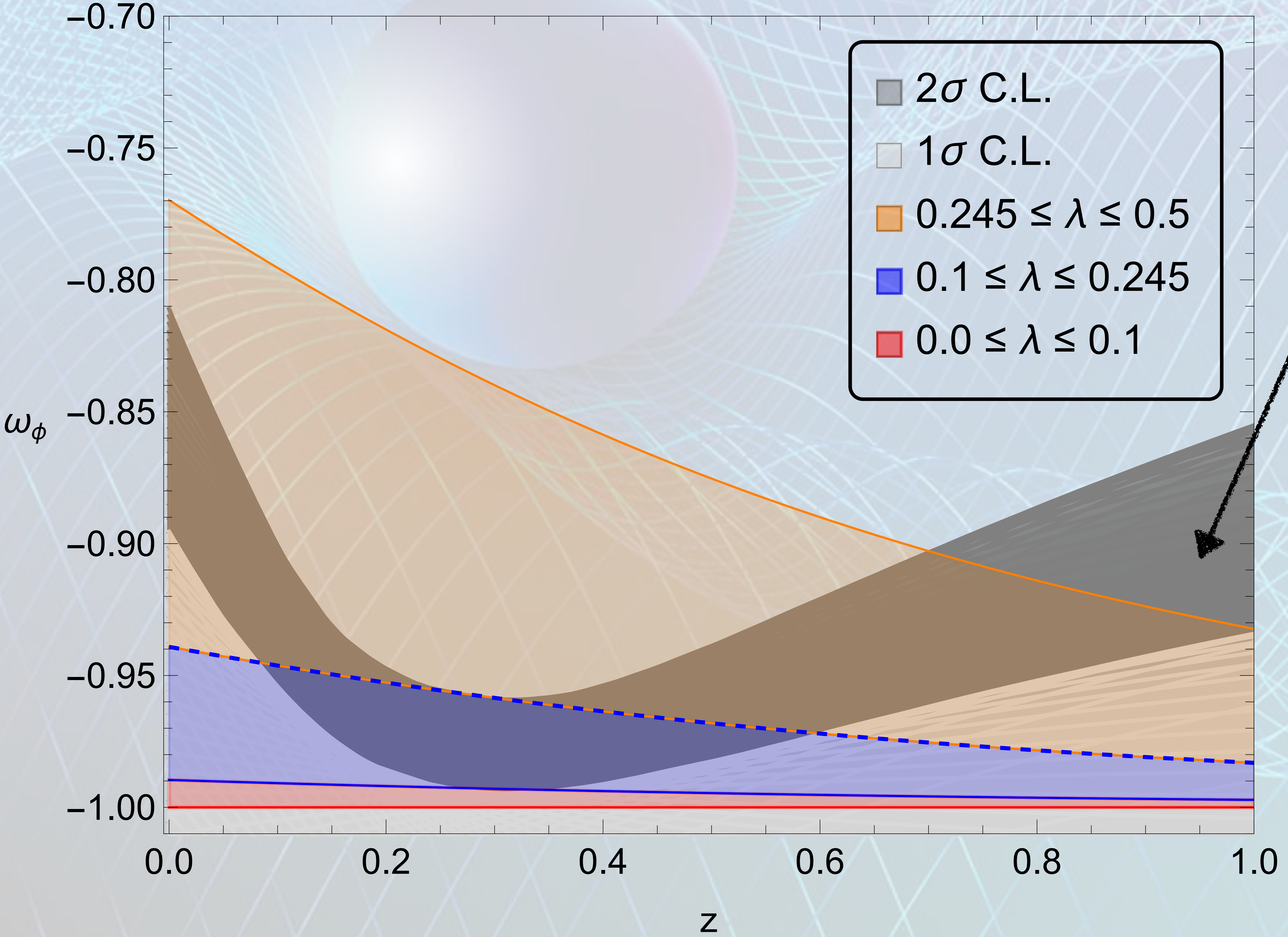
$\lambda = 0.3,$

$-\ln(1+z)$

$$V(\phi) = V_0 e^{-\sqrt{6}\kappa\lambda\phi} \quad \& \quad \xi(\phi) = \xi_0 e^{-\kappa\mu\phi/\sqrt{6}}.$$

$$w_\phi = \frac{1}{3(x^2+y^2+\alpha_{GB}^2-4x\alpha_{GB}^2\mu)(1+4\alpha_{GB}^4\mu^2-2\alpha_{GB}^2+4x\alpha_{GB}^2\mu)}\Big[(-3y^2+12y^2\alpha_{GB}^2\lambda\mu) + x^2\left(3+4\alpha_{GB}^2\mu^2(1-\Delta)\right) - 4x\alpha_{GB}^2\mu(4+\Omega_r) + \alpha_{GB}\left(3+2\Omega_r-4\alpha_{GB}^2\mu^2(1+\Omega_r)\right)\Big],$$

CPL: $\omega(z) = \omega_0 + \frac{z}{1+z}\omega_a$



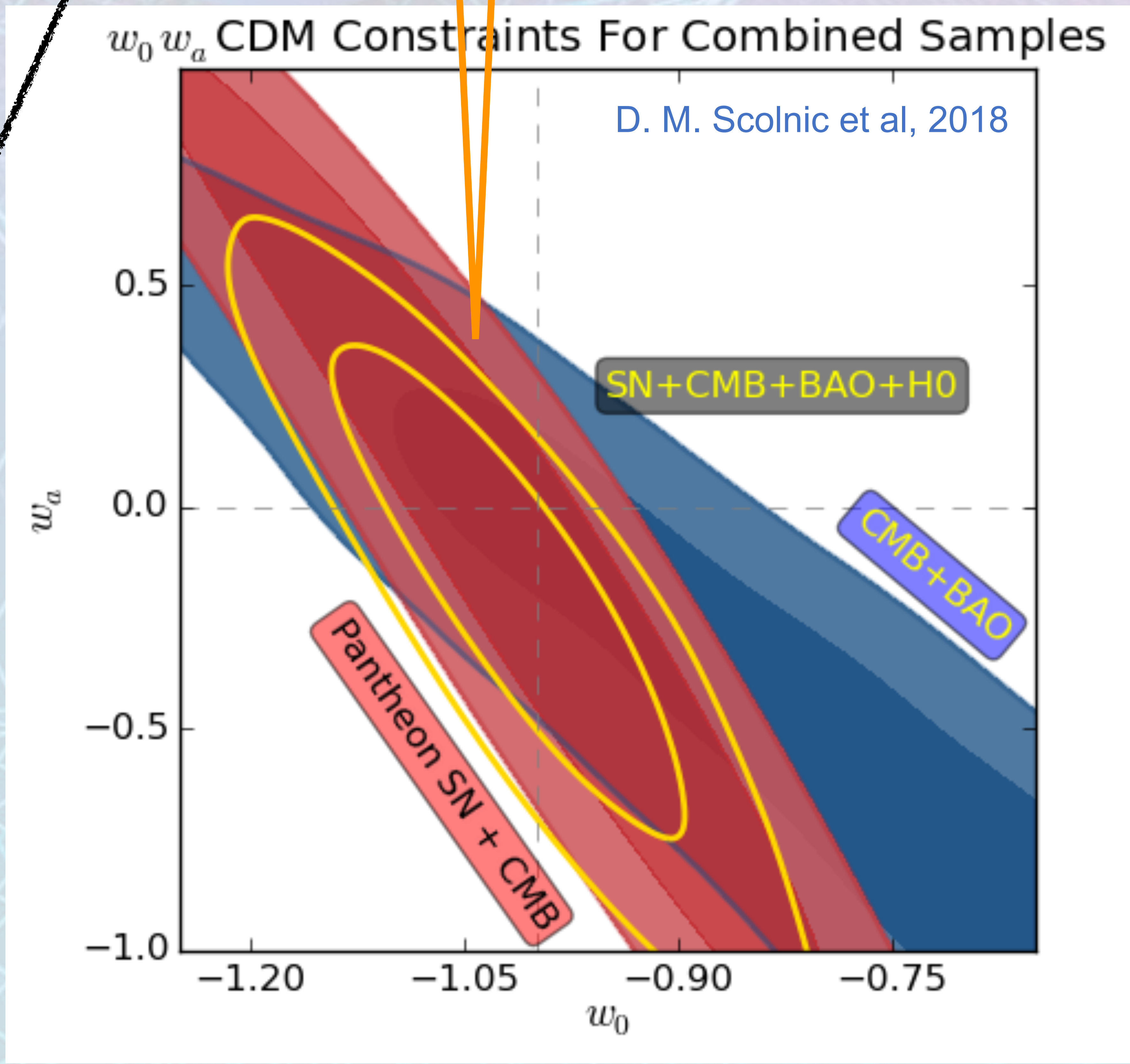


Figure 21. Confidence contours at 68% and 95% (including systematic uncertainty for SNe) for the w and w_a cosmological parameters for the w_0w_aCDM model.

Conclusion and outlook

- We considered $4D$ EGB gravity with both the scaling constant and the non-minimal coupling between the scalar field and GB term to obtain well-defined EoMs,

- There exist stable fixed points for the $4D$ EGB gravity for a preferred potential and coupling function,

- No missing cosmological history,

- We obtained the parameter space for non-minimal coupling function and potential in light of the observational data.

- As a further extension, we need to work on the perturbation analysis.

THANK YOU FOR YOUR KIND ATTENTION!