Introduction to Gravitational Waves: PTA Perspective

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Introduction to GW Astronomy

What are Gravitational Waves (GWs)?

- GWs are disturbances in the fabric of spacetime
- In Einstein's GR, GWs are propagating tidal interactions that travel with the speed of light
- General Relativity requires us to describe the local geometry of space-time

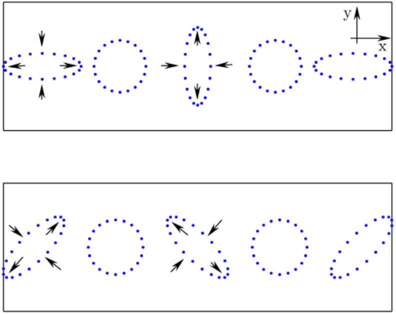
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Let i) $h_{\mu 0} = 0$ and ii) the curves $(x^{\mu}(t) = (ct, \vec{r}_0))$ with constant $\vec{r}_0$ be geodesics

- If so, $h_{\mu\nu}$ is a superposition of GWs in the TT gauge

How are GWs directly detected?

- GWs stretch and squeeze the space through which they propagate as they are essentially traveling tidal interactions



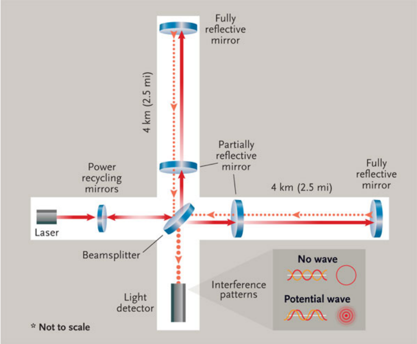


Figure: Basic diagram of the LIGO detectors. *Source: www.livingligo.org*

$h(t) = \frac{\delta L(t)}{L}$

Passing Gravitational wave $\rightarrow$ Strain produced in the detector arms

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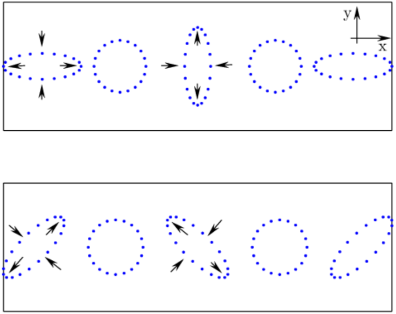

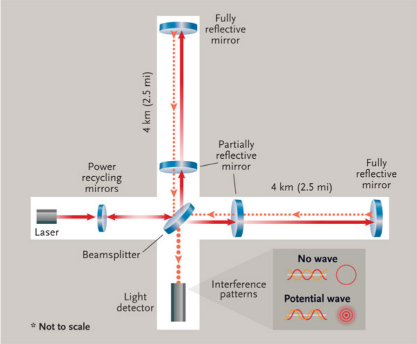

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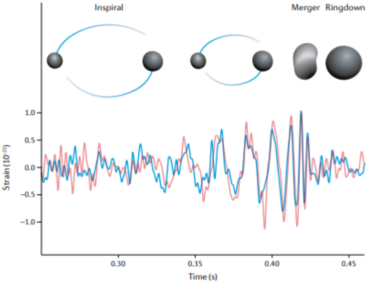
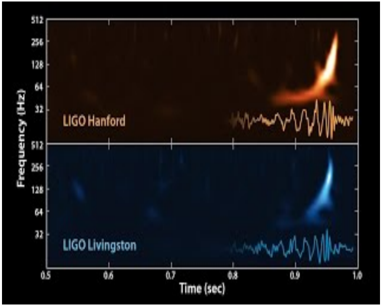
How are GWs directly detected?

GW Observatories measure GW amplitudes!!

For GW150914, $\mathcal{M} \sim 30\,M_\odot$, $R \sim 250 - 550\,Mpc$ and it lasted around 0.2 seconds in the LIGO's frequency window

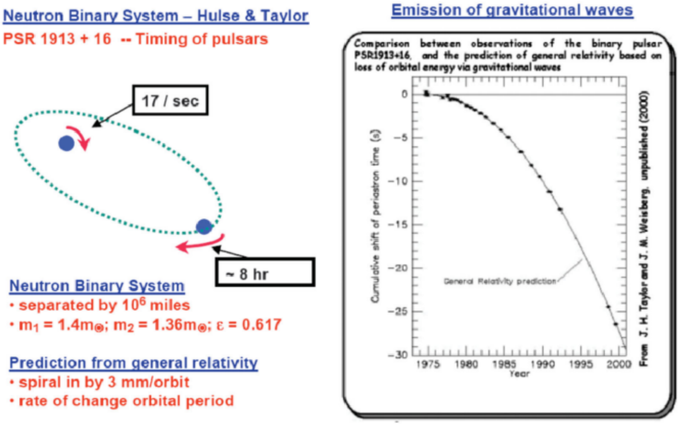




$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3} f^{-11/3} \dot{f}\right]^{3/5},$$

How are GWs indirectly detected?

Rainer Weiss: Nobel Lecture: LIGO and the discovery of ...



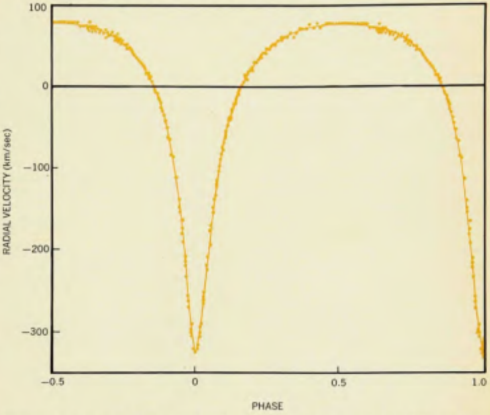
Pulsar in binary system may test fundamental theories

A pulsar in orbit about a second compact star would form a very convenient arrangement for studying gravitational interactions and learning the elusive details of the physics of neutron stars. So the announcement by Joseph Taylor and Russell Hulse of the University of Massachusetts, Amherst, that they have observed such a system has been greeted joyfully by general relativists and by astrophysicists who have been longing for a chance to test their models. Studying changes in the pulsar period as the rotating neutron star travels about its companion in a close, highly elliptical orbit should provide checks of time dilatation, precession and perihelion advance effects free of the complications that occur in the weaker gravitational fields of the solar system, perhaps settling the question of which theory of gravitation is the correct one. The changes in period may also yield more details of the structure of pulsars and perhaps even of the nature of the pulsed radiation.

Taylor and Hulse discovered the binary pulsar during a systematic search for new pulsars with the recently resurfaced radio dish at Arecibo, Puerto Rico. They used a multichannel receiver together with a small computer and achieved a sensitivity roughly twenty times better than that of previous pulsar surveys. The pulsar, PSR 1913+16, was first detected this past July and frustrated all attempts to measure its pulsation period (roughly 59 milliseconds) to within one microsecond: The period apparently changed by as much as 80×10^{-6} seconds from day to day and sometimes by as much as 8×10^{-6} seconds within five minutes, whereas the previously known maximum secular change for any pulsar was 10^{-5} sec per year. Then the astronomers realized that these apparent changes in period were actually a Doppler effect caused by the orbital motion of the pulsar about a companion, and by September they had plotted a velocity curve for the pulsar. To learn the orbital parameters, they measured the pulsar period directly during 200 separate five-minute intervals over ten days and found that it varied between 0.058697 sec and 0.059045 sec over a cycle of 0.3230 days (about eight hours).

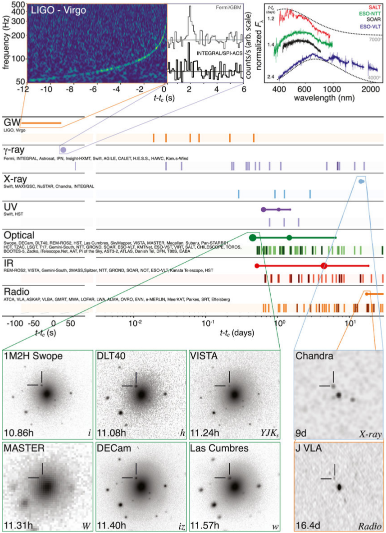
How did Taylor and Hulse realize that the companion must be compact—not a normal star? From their observations they determined the mass function $([M_2 \sin i]^3/[M_1 + M_2]^2$, of the

continued on page 20



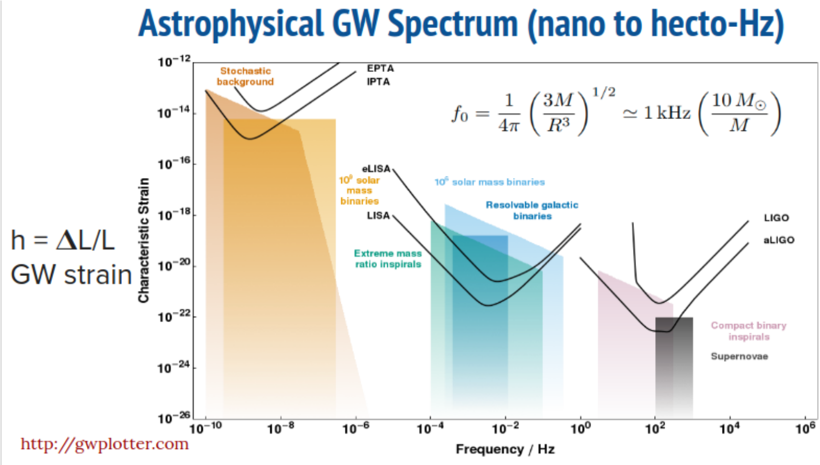
Velocity curve for binary pulsar shows Doppler effect. Points are experimental data over parts of ten different orbital periods; curve is a theoretical calculation based on orbital parameters.

GW event lasted around 100 seconds & from 40 Mpc (GW/EM170817)

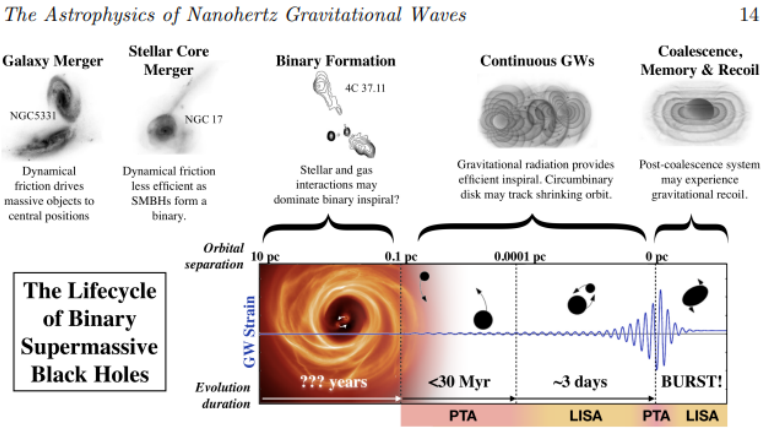


Multi-Band GW Astronomy

Is it possible to do Persient Multi-Messenger GW Astronomy in the coming years?



Pulsar Timing Arrays: I



Pulsar Timing Arrays for nano-hertz GW Astronomy

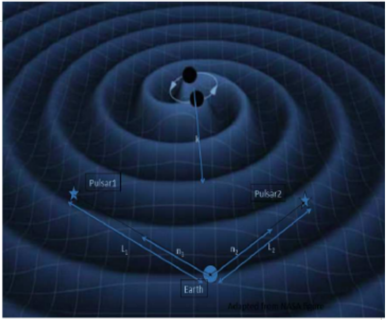
- SMBH binaries can provide nano hertz GWs with amplitudes ~10^{-15}

$$\omega = 2 \times 10^{-8}\,\mathrm{s}^{-1} \left(\frac{200M}{R_0}\right)^{3/2} \left(\frac{10^{10}\,M_\odot}{M}\right)$$

$$A \sim 5 \times 10^{-14} \left(\frac{200M}{R_0}\right) \left(\frac{M}{10^{10}\,M_\odot}\right) \left(\frac{10^{10}\,\text{lt-yr}}{r}\right).$$

Detweiler, S. (1979)

- We need highly accurate and stable celestial clocks
  Employ MSPs!!



Courtesy: Web

Theoretical Arguments Behind PTAs

General Relativity requires us to describe the local geometr
of space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Let i) $h_{\mu 0} = 0$ and ii) the curves $(x^{\mu}(t) = (ct, \vec{r}_0))$ with constant $\vec{r}_0$ be geodesics

If so, $h_{\mu\nu}$ is a superposition of GWs in the TT gauge

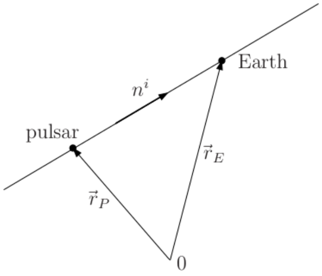
How Does a PTA Work?: I

- $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with the above 2 restrictions!
- $g_{\mu\nu} dx^\mu dx^\nu = -c^2\, dt^2 + \delta_{ij}\, dx^i\, dx^j + h_{ij}\, dx^i\, dx^j$
  It turns out that the above t coincides with the proper time!!
- We can identify frequencies with respect to the measured proper time intervals either at the pulsar or at the earth
- If pulsar emits pulses at a fixed frequency ν_p & they arrive at the Earth with a frequency $\nu_E(t_E)$ that depends on the time of arrival t_E
- Our Aim is to Determine $\nu_E(t_E)$

How Does PTA a Work?: II

For a light ray from the pulsar to the Earth,

- $0 = -c^2\, dt^2 + \delta_{ij}\, dx^i\, dx^j + h_{ij}\, dx^i\, dx^j$
- $c^2\, (\frac{dt}{dl})^2 = 1 + h_{ij} \frac{dx^i}{dl} \frac{dx^j}{dl}$ with $dl^2 = \delta_{ij}\, dx^i\, dx^j$
- If there are NO GWs, a light ray will travel from the Pulsar to the Earth along a straightline
  We may define $\frac{dx^i}{dl} = n^i + \mathcal{O}(h)$



How Does a PTA Work?: III

- Therefore, $c\left(\frac{dt}{dl}\right) = \sqrt{1 + h_{ij}(x)n^i n^j + \mathcal{O}(h^2)}$
- $\rightarrow dl \sim c\left(1 - \frac{h_{ij}(x)n^i n^j}{2} + ..\right) dt$
- Integrate over the path of the light ray

$$\int_{t_P}^{t_E} \left(1 - \frac{h_{ij}(x)n^i n^j}{2}\right) dt = \frac{L}{c}$$

- Differentiate with respect to $t_P \rightarrow$

$$\frac{dt_P}{dt_E} = 1 + \frac{n^i n^j}{2}\left(h_{ij}(ct_P, \vec{r}_P) - h_{ij}(ct_E, \vec{r}_E)\right)$$

- Recall that we have identified the coordinate time with the proper time!!
  $\rightarrow$ We can identify 'time intervals' with 'frequencies'!!

How Does a PTA Work?: IV

- $\rightarrow$ the pulses, emitted at a constant frequency ν_P by the Pulsar, arrive with a time dependent frequency $\nu_{(t_E)}$ such that

$$\frac{\nu_{(t_E)} - \nu_P}{\nu_P} = \frac{dt_P}{dt_E} - 1 = \frac{n^i n^j}{2}\left(h_{ij}(ct_P, \vec{r}_P) - h_{ij}(ct_E, \vec{r}_E)\right)$$

- Propagating GWs induce frequency shifts in pulsar rotational fewquencies & it depends on the projection onto n^i of h_{ij} (both at the Pulsar and the Earth)
- If GWs are propagating along the x^3 direction, n^i should have non-vanishing components in the $x^1 - x^2$ plane !!
  Such frequency shifts influence pulse times-of-arrivals (TOAs)

How GWs affect TOAs?

- If a GW propagates along the positive z direction

$$z(t) = \frac{\nu_P - \nu(t)}{\nu_P} = \frac{\alpha^2 - \beta^2}{2(1+\gamma)} \Delta h_+(t) + \frac{\alpha\,\beta}{1+\gamma} \Delta h_\times(t)\,,$$

  α, β, and γ provide a pulsar's direction cosines w.r.t the x, y, and z axes (Detweiler 1979)
  Where $\Delta h_{+,\times} = h^P_{+,\times} - h^E_{+,\times}$

- The observed/measured TOAs are compared with their predictions, based on an appropriate pulsar timing model
  The resulting differences are usually referred to as the 'Pulsar Timing Residuals'

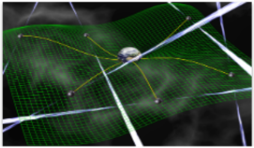
- Observable timing residuals, induced by the GWs, are given by the integral over time of the above Equation

$$R(t) = \int_0^t \frac{\nu_0 - \nu(t')}{\nu_0} \delta t'$$

Pulsar Timing Array’s GW Sources

- Observe a bunch of MSPs at different locations in the sky using the ***best radio telescopes*** of our times.

- Search for the correlated timing residual for detecting nHz GWs.



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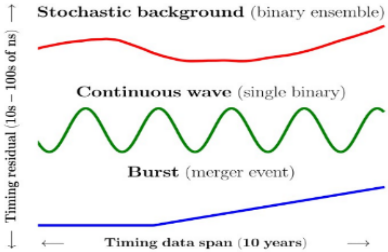

Astrophysical Stochastic GW Background

Stochastic GWB from massive BH binaries ?: I

Cosmological population of MBH binaries is expected to provide a diffusive GW background for PTAs !!
We need to compute # of sources in a frequency interval $\Delta f = 1/T_{\rm obs}$

- $$\Delta N = \left(\frac{dN}{df}\right)\Delta f \;\; = \frac{dN}{dt}\left(\frac{df}{dt}\right)^{-1}\Delta f \tag{1}$$
- $$\Delta N \propto \frac{dN}{dt}\left({\mathcal{M}_c}^{-5/3} f_{\rm GW}^{-11/3}\right)\Delta f \tag{2}$$
- There are some 10^{11} galaxies in our Universe and each galaxy is likely to experience one merger with another galaxy in Hubble time (10^{10} year)
- $$\rightarrow \frac{dN}{dt} \sim 10 \text{ mergers/year}$$

Stochastic GWB from massive BH binaries ?: I

- $\rightarrow$ a rough estimate for the number of binary BH sources in a frequency interval $\Delta f = 1/T_{\rm obs}$
-

$$\Delta N \sim 3 \times 10^{12} \left(\frac{\mathcal{M}}{10^9\, M_\odot}\right)^{-5/3} \left(\frac{f_{\rm GW}}{10^{-8}\, Hz}\right)^{-11/3} \left(\frac{T_{\rm obs}}{10\, yr}\right)^{-1} \times \left(\frac{dN/dt}{10{\rm merger/yr}}\right)$$

- This is clearly $>> 1$

  This ensures a diffuse GW background in the PTA GW frequency window from merging massive BHs in the universe

- It is NOT very difficult to show that its characteristic strain spectrum is given by $h_c(f) = A_{1yr} \times (f/{\rm yr}^{-1})^{-2/3}$

$h_c(f)$ derivation : I

This derivation for $h_c(f)$ is essentially due to Phinney

- Let $\mathcal{E}_{gw}$ be the total present day energy density in GWs
- Let $\Omega_{gw}(f)$ be the RATIO between the present-day energy density per logarithmic frequency interval, in GWs AND the rest-mass energy density $\rho_c c^2$ that would be required to close the universe
  $\rho_c = 3H_0^2/(8\pi G)$
- $\mathcal{E}_{gw} \equiv \int_0^\infty \rho_c c^2 \Omega_{gw}(f)\, df/f$........(1)
- Let $N(z)dz$ be the number of events in unit comoving volume between redshifts z and $z + dz$
- $\mathcal{E}_{gw} \equiv \int_0^\infty N(z) \frac{1}{1+z}\, dE_{gw}\, dz$
  $\mathcal{E}_{gw}$ must be equal to sum of the energy densities radiated at each redshift, divided by $(1 + z)$
- $\mathcal{E}_{gw} \equiv \int_0^\infty \int_0^\infty N(z) \frac{1}{1+z} \frac{dE_{gw}}{df_r} f_r \frac{df_r}{f_r}\, dz$ $\qquad f = f_r/(1 + z)$

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$h_c(f)$ derivation : II

- $\mathcal{E}_{gw} = \int_0^\infty \int_0^\infty N(z) \frac{1}{1+z} f_r \frac{dE_{gw}}{df_r} \, dz \, \frac{df}{f}$
- We have already argued that
  $\mathcal{E}_{gw} \equiv \int_0^\infty \rho_c c^2 \Omega_{gw}(f) \, df/f$........(1)
- Equating the above two equations $\rightarrow$

$$\rho_c c^2 \Omega_{gw}(f) = \int_0^\infty N(z) \frac{1}{1+z} \left(f_r \frac{dE_{gw}}{df_r} \right) \Bigg|_{f_r = f(1+z)} dz \; .$$

- The energy density in GWs per log frequency interval is equal to the comoving number density of event remnants, times the (redshifted) energy each event produced per log frequency interval.

$h_c(f)$ derivation : III

- It is fairly easy to compute $\frac{dE_{gw}}{df_r}$ associated with a massive BH binary inspiraling along circular orbits
- $$\frac{dE_{gw}}{df_r} = \frac{\pi}{3}\frac{1}{G}\frac{(G\mathcal{M})^{5/3}}{(\pi f_r)^{1/3}} \quad \text{for } f_{\min} < f_r < f_{\max} \ , \qquad (3)$$
- This leads to $\Omega_{gw}(f) = \frac{8\pi^{5/3}}{9}\frac{1}{c^2 H_0^2}(G\mathcal{M})^{5/3} f^{2/3} N_0 \langle (1+z)^{-1/3} \rangle$ $N_0 = \int_0^\infty N(z)\, dz$ is the present-day comoving number density of merged remnants
- It is convenient to introduce h_c: the characteristic amplitude of the gravitational wave spectrum over a logarithmic frequency interval as $\mathcal{E}_{gw} \equiv \int_0^\infty \rho_c c^2 \Omega_{gw}(f)\, df/f \equiv \int_0^\infty \frac{\pi}{4}\frac{c^2}{G} f^2 h_c^2(f) \frac{df}{f}$,
- For massive BH binaries, $h_c^2(f) = \frac{4}{3\pi^{1/3}}\frac{1}{c^2}\frac{(G\mathcal{M})^{5/3}}{f^{4/3}} N_0 \langle (1+z)^{-1/3} \rangle$ $\qquad h_c \propto f^{-2/3}$
- We require another observational evidence to detect nHz SGWB

On the Hellings & Downs Curve: I

PTAs search for correlations among timing residuals!!

- We write GW induced frequency shift for a Pulsar i as

$$\delta\nu_i/\nu_i = \alpha_i\, h(t) + n_i(t)$$

- Cross-correlate such frequency variations from two Pulsars (i, j)

  $$c_{ij}(\tau) = \alpha_i\, \alpha_j < h^2 > + \alpha_i < h\, n_j > + \alpha_j < h\, n_i > + < n_i n_j >$$

  where $< h^2 >= \int_{T-\tau}^{T+\tau} h(t)h(t+\tau)dt$, τ being time lag between the 2 Pulsars

- For isotropic GWB, $< h^2 >$ is direction independent &

$$\alpha_{ij} = \frac{1}{4\,\pi} \int \alpha_i\, \alpha_j d\Omega$$

- For sufficiently large data span, the noise from the two pulsars n_i & n_j and the gravitational wave strain $h(t)$ are uncorrelated to each other (the last 3 terms $\rightarrow 0$)

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On the Hellings & Downs Curve: II

- The cross-correlation of GW induced rotaional frequency variations between 2 Pulsars leads to

$$c_{ij}(\tau) = \alpha_{ij} < h^2 > + \delta c_{ij} \,, \tag{4}$$

  where δc_{ij} is an estimation error due to ‘finite‘ T and due to ‘finite‘ # of SMBH binaries in SGWB!!

- The average of the angular factors

$$\alpha_{ij} = \frac{1-\cos\theta_{ij}}{2} \ln\left(\frac{1-\cos\theta_{ij}}{2}\right) - \frac{1}{6}\frac{1-\cos\theta_{ij}}{2} + \frac{1}{3} \tag{5}$$

  where θ_{ij} gives the angle between two pulsars.

- The plot of the above correlation function with angle is commonly referred to as the Hellings and Downs curve

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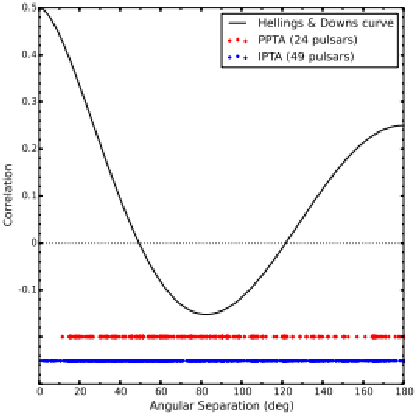
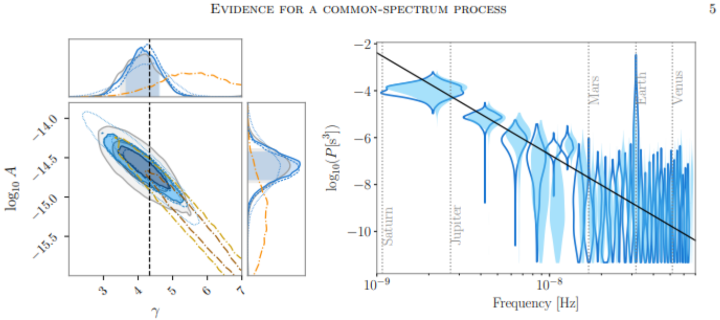
On the Hellings & Downs Curve: III



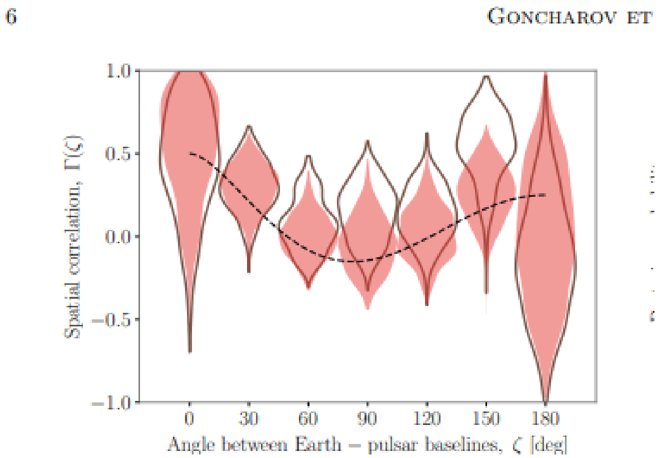
(https://arxiv.org/abs/1707.01615)

On the Hellings & Downs Curve: IV



(https://arxiv.org/abs/2107.12112)

On the Hellings & Downs Curve: V



(https://arxiv.org/abs/2107.12112)

InPTA is part of the 3P+ efforts to substantiate these results

Introduction to GWs:

- Aim is to provide an introduction to Gravitational Waves (GWs) without using General Relativity
- We use arguments from Special Theory & Electrodynamics
- Rough estimates for GW amplitudes & Luminosity..
- Bernard F. Schutz, Gravitational waves on the back of an envelope; American Journal of Physics 52, 412 (1984); *https://doi.org/10.1119/1.13627*

Theoretical Introduction to GWs

GWs: I

General Relativity (GR) defines GWs as ripples in the curvature of space-time that propagate with the speed of light !

It is possible to **COMPUTE** most of the crucial effects of GWs using Newtonian Gravitational Theory, Classical Electrodynamics & some elements of Special Relativity

- Newtonian gravity involves a scalar potential $\phi_{\rm N}({\rm x}, t)$ such that $\nabla\phi_{\rm N} = 4\,\pi\,G\,\rho$, such that

$$\phi_{\rm N}({\rm x}, t) = -G \int \frac{\rho({\rm y}, t)}{r}\, d^3y\,, \quad r \equiv |{\rm x} - {\rm y}| \tag{6}$$

  A change in $\phi_{\rm N}({\rm x}, t)$ due to a change in $\rho({\rm y}, t)$ propagate instantaneously

- Special Relativity demands that no information should be able to propagate faster than c: the speed of light

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GWs: II

To make $\phi_{\rm N}$ consistent with Special Relativity, we modify it

- $$\phi_{\rm R}(\mathrm{x}, t) = -G \int \frac{\rho(\mathrm{y}, t - \frac{r}{c})}{r} \, d^3 y \qquad (7)$$

  Dominant effects of GWs can be deducted from such a retarded gravitational potential

  This simple insertion will help us to define GWs in a non-rigorous way
- $\phi_{\rm R}$ satifies the scalar wave equation
  $\Box\phi_{\rm R} \equiv \left(\nabla^2 - \frac{\partial^2}{c^2\,\partial^2 t}\right)\phi_{\rm R} = 4\,\pi\,G\,\rho$
- Take the spatial gradient of $\phi_{\rm R}$

$$\nabla\phi_{\rm R} = G \int \left(\frac{\rho}{r} - \frac{\partial\rho}{c\,\partial t}\right)\frac{\mathrm{x}-\mathrm{y}}{r^2}\, d^3 y \qquad (8)$$

GWs: II

To make $\phi_{\rm N}$ consistent with Special Relativity, we modify it

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- If $|\mathsf{x}| \gg |\mathsf{y}_{\rm mx}|$, we have $r \sim |\mathsf{x}|$ & can negelct $1/r$ term in the previous Eq.

$$\mathsf{n} \cdot \boldsymbol{\nabla}\phi_{\rm R} \sim \frac{\partial \phi_{\rm R}}{c\,\partial t}\,, \quad \mathsf{n} = \mathsf{x}/|\mathsf{x}| \tag{9}$$

$$\phi_{\rm R}/\lambda \sim 1/c \times \phi_R/T$$

- If we are far away from a GW source, the typical length scale over which $\phi_{\rm R}$ varies is **c** $\times$ the typical time scale over which $\phi_{\rm R}$ changes

- This is true for a wave traveling at speed c
  This is our Gravitational Wave

- Recall that $\phi_R \sim v^2$. Therefore, the amplitude of GW should be $\sim$

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An estimate for the dominate contribution to h: I

Consider a region $|\mathrm{x}| \gg |\mathrm{y}| b$ & we have far-zone expansion

-
$$1/r \equiv |\mathrm{x}-\mathrm{y}|^{-1} \sim \frac{1}{|\mathrm{x}|} + \mathrm{y}\cdot\mathrm{n}\,|\mathrm{x}|^{-2} \tag{11}$$

  This leads to (if we neglect $\mathcal{O}(|\mathrm{x}|^{-2})$ terms)

$$\phi_{\mathrm{R}} = -\frac{G}{|\mathrm{x}|} \int \rho(\mathrm{y}, t_r)\, d^3y\,, \quad t_r = t - \frac{r}{c} \tag{12}$$

- Let $t_0 = t - \frac{|\mathrm{x}|}{c}$ & this leads to $t_r \sim t_0 - \mathrm{y}\cdot\mathrm{n}/c$
- Expand $\rho(t_r)$ about t_0

$$\phi_R = \frac{-G}{|\mathrm{x}|}\left\{\int \left[\rho(t_0) - \frac{\dot{\rho}}{c}\,\mathrm{n}\cdot\mathrm{y} + \frac{\ddot{\rho}}{2\,c^2}\,(\mathrm{n}\cdot\mathrm{y})^2 + \ldots\right] d^3y\right\} \tag{13}$$

An estimate for the dominate contribution to h: II

- First term

$$\int \rho(t_0) d^3 y \equiv M \tag{14}$$

- Let $\mathrm{v} = \frac{d\mathrm{y}}{dt}$; the second term involves

$$n_i \int \dot{\rho} y_i \, d^3 y = n_i \int \rho \, v_i \, d^3 y = \mathrm{n} \cdot \mathrm{P} \tag{15}$$

  where P is the conserved momentum of the source

- Third term contains

$$\int \ddot{\rho} \, y_i \, y_j \, d^3 \, y = \ddot{I}_{ij} \tag{16}$$

  where $I_{ij}(t) \equiv \int \rho(t) \, y_i \, y_i \, d^3 y$ is the quadrupole moment of gravitating source & $\ddot{I}_{ij} = \int \rho \, v_i \, v_j \, d^3 y$

- Retarded potential becomes

$$\phi_R \sim -\frac{G\,M}{|\mathrm{x}|} + \frac{G\,\mathrm{n} \cdot \mathrm{P}}{c\,|\mathrm{x}|} - \frac{G}{2\,c^2} \frac{\ddot{I}_{ij}\,n_i\,n_j}{|\mathrm{x}|}, \tag{17}$$

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- This leads to

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- Note that h depends only on the components of I_{ij} along n, the direction of propagation of the wave & this is due to the fact that we are dealing with **scalar waves**
- In GR, GWs are *ripples in the curvature of space-time* & space-time & its disturbances are described by tensors
- $I_{ij} \rightarrow$ transverse components of trace-free tensor $\mathcal{I}_{ij} = I_{ij} - \frac{\delta_{ij}}{3}\,I_{kk}$
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$$h_{ij}^{TT} = \frac{2\,G}{c^4\,r}\,\ddot{\mathcal{I}}_{ij}(t - r/c) \tag{19}$$

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- In classical ED, the dominant order multipole radiation fron a charge distribution is the dipole radiation. The vector potential A_i in the wave-zone

$$A_j = \frac{1}{c\, r}\, \dot{d}_j(t_r) \tag{20}$$

The $1/r$ EM fields E & B depend only on the components of $\dot{\mathrm{d}}$ transverse to n; $d_j^{\mathrm{T}} \equiv P_{jk}\, d_k$, $P_{jk} = \delta_{jk} - n_j\, n_k$

- The Larmor formula provides the expression for EM luminosity

$$\mathcal{L}_{\mathrm{EM}} = \frac{2}{3\, c^3}\, \ddot{d}_j\, \ddot{d}_j \,, \;\; d_j = e\, y_i \tag{21}$$

- For gravitating systems, linear & angular momenta provide electric & magnetic *type* dipole moments & they are conserved $\boldsymbol{\mu} = \frac{1}{c} \sum_a \mathrm{y}^a \times \mathrm{d}^a = \frac{1}{c} \sum_a \mathrm{y}^a \times m_a \mathrm{v}^a = \frac{1}{c} \sum_a \mathrm{L}^a$

An estimate for GW luminosity :II

$\mathcal{L}_{\rm GW} \propto \overset{(3)}{I_{ij}}\,\overset{(3)}{I_{ij}}$ & dimensional consideration require us to have G/c^5

- Explicit calculations in GR provides

$$\mathcal{L}_{GW} = \frac{G}{5\,c^5}\,\mathcal{I}_{ij}^{(3)}\,\mathcal{I}_{ij}^{(3)} \tag{22}$$

- $^{(3)}I_{ij} \sim M\,R^2/T^3 \sim M\,V^3/R$

$$\mathcal{L}_{GW} \sim \frac{G}{c^5}\,(M/R)^2\;V^6 \sim L_0\,(r_{Sch}/R)^2\,(V/c)^6\;, \tag{23}$$

where $L_0 = \frac{c^5}{G} \sim 3.6 \times 10^{52}\,\mathrm{J}/s$ & $r_{\rm Sch} = G\,M/c^2$

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Summary

- We provided i) Introduction to GW Astronomy
- How PTAs should detect GWs
- Why we expected SGWB from an ensmeble of spiarling in SMBH binaries
- Rough estimates for GW amplitudes & Luminosity..