⟨QCD vacuum structure, strong CP problem & Axion⟩

Refs) Peccei 0607268

Kim-Carosi 0807.3125

Scherer 0210398

QCD at low energy

"vector-like"

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \underbrace{\bar{q}_L \not{D} q_L + \bar{q}_R \not{D} q_R}_{\to\ \bar{q}\not{D}q}$$

$$+ \underbrace{\bar{q}_L M q_R + \bar{q}_R M^\dagger q_L}_{\to\ \bar{q}Mq}$$

$$M = \text{Diag.}\left[m_u, m_d, m_s, m_c, m_b, m_t\right]$$

2MeV 4.7 93 1.3GeV 4.2 173.

Below $\Lambda_h \sim 1\,\text{GeV} \gg m_{u,d,s}$, $\alpha_s = \frac{g_3^2}{4\pi} \gtrsim 1$.

quarks and gluons are confined to form color-neutral hadrons.

mesons & baryons

In the limit of $m_{u,d,s} \to 0$, $\mathcal{L}_{QCD}$ has a chiral symmetry $U(3)_L \times U(3)_R$

$$q_{L,R} \to e^{i T^a \alpha_A^{L,R}} q_{L,R}, \quad a = \underset{\uparrow \atop \mathbb{1}}{0}, \underbrace{1, \cdots 8}_{SU(3)}$$

equivalently $U(3)_V \times U(3)_A$

$$q \to e^{i T^a \alpha_a^V} q \qquad \alpha_a^V = \alpha_a^L = \alpha_a^R$$

$$q \to e^{i \gamma_5 T^a \alpha_a^A} q \qquad \alpha_a^A = \alpha_a^L = -\alpha_a^R$$

The <u>chiral</u> symmetry $U(3)_A$ is broken <u>spontaneously</u> by quark condensate

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle \neq 0$$

(explicitly by $m_{u,d,s} \neq 0$)

$$U(3)_A = \underset{\substack{\uparrow \\ \text{octet}}}{SU(3)_A} \times \underset{\substack{\uparrow \\ \eta'}}{U(1)_A}$$

Goldstone bosons
||
mesons

$U(3)_V = SU(3)_V \times U(1)_V$ "good" symmetry

$SU(3)_V$ ↑ Isospin ⊕ Strangeness

$U(1)_V$ ↑ Baryon member

meson octet

K^0 K^+

η', π^- π^0, η^0 π^+

K^- $\bar{K}^0$

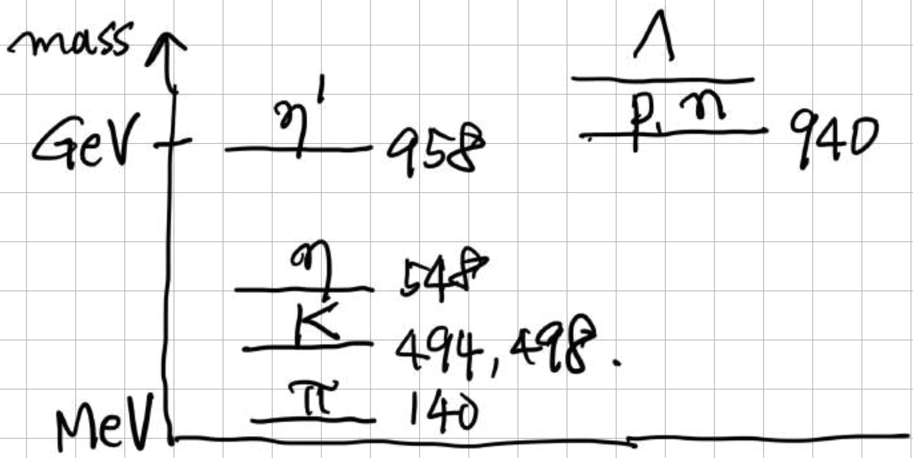
S, I_3

baryon octet

n P

Σ^- Σ^0, Λ^0 Σ^+

Ξ^- Ξ^0



"Why $m_{\eta'} \gg m_{\pi, K, \eta}$?"

$\simeq$ 1 GeV

Weinberg

Effective field theory of QCD

chiral structure $SU(3)_L \times SU(3)_R$.

Goldstone bosons

$$\phi = \sqrt{2} \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta^0}{\sqrt{6}} \end{bmatrix}$$

$$\Sigma = e^{i\frac{\phi}{F}}$$

Invariants

$$\Sigma \longrightarrow L \Sigma R^{\dagger} \qquad \mathrm{Tr}\left(\partial_\mu \Sigma \, \partial^\mu \Sigma^{\dagger}\right)$$

Quark masses in perturbation

$$M \longrightarrow L M R^{\dagger}$$

$$q_{L,R} \longrightarrow L_{,R}\, q_{L,R} \qquad \bar{q}_L M q_R + \bar{q}_R M^{\dagger} q_L$$

$$\mathrm{Tr}\left(M \Sigma^{\dagger} + \Sigma M^{\dagger}\right)$$

Model Lagrangian

$$\mathcal{L}_{eff} = \frac{F^2}{4}\left[\mathrm{Tr}\left(\partial_\mu \Sigma \, \partial^\mu \Sigma^{\dagger}\right) + 2B\, \mathrm{Tr}\left(M \Sigma^{\dagger} + \Sigma M^{\dagger}\right)\right]$$

$$\downarrow$$

$$\partial_\mu \pi^+ \partial_\mu \pi^- + \frac{1}{2} \partial_\mu \pi^0 \partial_\mu \pi^0 + \cdots$$

Mass relations : (숙제)

(i) $m_\pi^2 = B(m_u+m_d)$

$m_{K^\pm}^2 = B(m_u+m_s)$

$m_{K^0}^2 = B(m_d+m_s)$

$m_\eta^2 = \frac{B}{3}(m_u+m_d+m_s)$

Gellman-Okubo

$2m_{K^\pm}^2 + 2m_{K^0}^2 = 3m_\eta^2 + m_\pi^2$

(Check!)

(ii) $$\frac{m_u}{m_d} = \frac{m_{K^+}^2 - m_{K^0}^2 + m_\pi^2}{m_{K^0}^2 - m_{K^+}^2 + m_\pi^2}$$

$$\frac{m_s}{m_d} = \frac{m_{K^0}^2 + m_{K^+}^2 - m_\pi^2}{m_{K^0}^2 - m_{K^+}^2 + m_\pi^2}$$

(iii) Isospin breaking by e.m.

$$Q = e\,\mathrm{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$

$$\mathcal{L}_{em} = C\,\mathrm{Tr}\left[Q\Sigma Q\Sigma^\dagger\right]$$

$$\begin{cases} m_{\pi^\pm}^2 = B(m_u+m_d) + \dfrac{2Ce^2}{F^2} \\ m_{K^\pm}^2 = B(m_u+m_s) + \quad '' \end{cases}$$

Dashen formula : $m_{\pi^\pm}^2 - m_{\pi^0}^2 = m_{K^\pm}^2 - m_{K^0}^2$

(Check!)

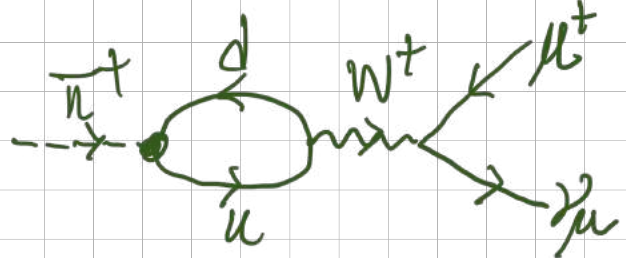
- $\underline{\pi \text{ decay constant} : F \approx f_\pi}$

$$\langle 0 | j^a_\mu(x) | \pi^b \rangle = i\, p_\mu\, \delta^{ab} \frac{f_\pi}{2} e^{-ip\cdot x}$$

$$\Leftrightarrow j^a_\mu(x) = \frac{f_\pi}{2} \partial_\mu \pi^a(x)$$

$$\mathcal{L}_W = \frac{g_2}{\sqrt{2}} W^-_\mu \left(\bar{l}_L \gamma^\mu \nu_L + V_{ud}\, \bar{d}_L \gamma^\mu u_L \right) + h.c.$$

$$\Leftrightarrow \frac{f_\pi}{2} \partial^\mu \pi^+$$



$$\Gamma_{\pi^+ \to \mu^+ \nu_\mu} = \frac{G_F^2}{4} |V_{ud}|^2 f_\pi^2 m_\pi^2 m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)$$

$$\Rightarrow f_\pi = 92.4 \text{ MeV} \qquad (f_K = 1.22 f_\pi)$$

- Quark condensation scale Λ_{QCD}.

$$\langle \bar{q} q \rangle = \langle \bar{u} u \rangle = \langle \bar{d} d \rangle = \langle \bar{s} s \rangle = \Lambda^3_{QCD}$$

flavor-blind QCD

$$\mathcal{L}_{mass} \Rightarrow m_q \langle \bar{q} q \rangle \Leftrightarrow \frac{F^2}{4} 4B\, m_q$$

$$\therefore \langle \bar{q} q \rangle = F^2 B = f_\pi^2 \frac{m_\pi^2}{m_u + m_d} = \Lambda^3_{QCD}$$

Note 1) $\Lambda_{QCD} = \left(f_\pi^2 \frac{m_\pi^2}{m_u + m_d} \right)^{\frac{1}{3}} \approx 280\,\text{MeV}$

2) $\Lambda_h \approx 4\pi f_\pi \approx 1.2\,\text{GeV}$

UV cutoff of the effective theory

$$\frac{E_{max}^2}{(4\pi)^2 f_\pi^2} \lesssim 1, \quad E_{max} = \Lambda_h$$

3) $m_{\eta'} \sim \Lambda_h \gg m_{\pi, K, \eta}$

$\uparrow$ scale of SSB by $\langle \bar{q} q \rangle$

※: η' is <u>not</u> a GB;

remains <u>massive</u> even in the chiral limit $m_{u,d,s} \to 0$.

- <u>π-N-N coupling</u>

$$\mathcal{L}_{\pi NN} = -g_A \frac{\partial_\mu \pi^a}{f_\pi} \bar{N} \gamma^\mu \gamma_5 \frac{\sigma^a}{2} N, \quad N = \begin{pmatrix} p \\ n \end{pmatrix}$$

$g_A \approx 1.27$

$$\simeq g_{\pi NN} \pi^a \bar{N} \gamma_5 \frac{\sigma^a}{2} N$$

$$g_{\pi NN} \simeq g_A \frac{m_N}{f_\pi} \approx 13.2$$

$U(1)_A$ Problem

"Is η' a GB?"

Consider $U(2)_L \times U(2)_R = U(2)_V \times U(2)_A$

$$\begin{pmatrix} u \\ d \end{pmatrix}_{L,R} \rightarrow e^{i\sigma^a \alpha^a_{L,R}} \begin{pmatrix} u \\ d \end{pmatrix}_{L,R} \qquad a = 0, 1, 2, 3$$

$$\uparrow$$
$$\mathbb{1}$$

$$U(2)_V : \alpha^a_L = \alpha^a_R = \alpha^a_V$$

$$U(2)_A : \alpha^a_L = -\alpha^a_R = \alpha^a_A .$$

$U(2)_A$ is broken spontaneously by $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \neq 0$.

Goldstone Boson matrix

$$\Sigma = e^{i\frac{\phi}{f_\pi}} . \qquad \phi = \sigma^a \phi_a = \begin{pmatrix} \eta' + \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & \eta' - \pi^0 \end{pmatrix}$$

$$\mathcal{L}^{mass}_{eff} = \frac{B}{2} f_\pi^2 \, \mathrm{Tr}\left(M \Sigma^\dagger + \Sigma M^\dagger \right)$$

$$= -B(m_u + m_d) \left[\pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 + \frac{1}{2}\eta'\eta' + \frac{m_u - m_d}{m_u + m_d} \pi^0 \eta' \right]$$

Mass relations

$$2 m_{\pi^\pm}^2 = m_{\pi^0}^2 + m_{\eta'}^2 \quad : \quad m_{\eta'} = 958 \text{ MeV}$$

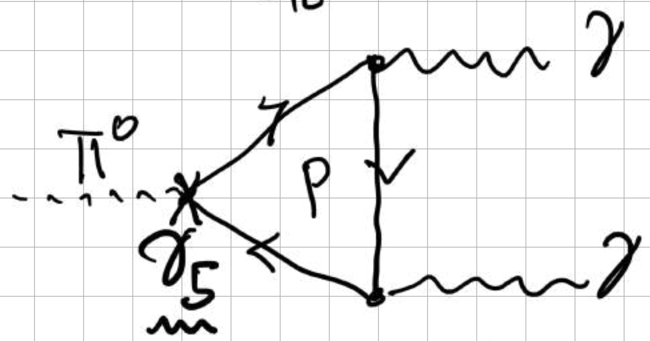
$$m_{\pi^\pm, 0} \simeq 140 \text{ MeV}$$

No ! ?

Chiral Anomaly   ABJ 1969

$\pi^0 \to \gamma\gamma$

$$\mathcal{L}_{\pi NN} = -g_A \frac{\partial_\mu \pi^0}{f_\pi} \bar{N}\gamma^\mu\gamma_5 N \qquad \text{[peskin]}$$



$$\Gamma(\pi^0 \to \gamma\gamma) = g_A^2 \left(\frac{m_\pi}{4\pi}\right)^3 \left(\frac{\alpha}{f_\pi}\right)^2 \qquad m_N\text{-indep.}$$

$$(*) \quad P_\mu \langle \bar{q}\gamma^\mu\gamma_5 q \rangle = -2m \langle \bar{q}\gamma_5 q \rangle_{div}$$

$$+2M \langle \bar{q}\gamma_5 q \rangle_{div}^{M\to\infty}$$

$$\partial_\mu [\bar{q}\gamma^\mu\gamma_5 q]_{Reg} = 2im\,\bar{q}\gamma_5 q + \frac{e^2}{16\pi^2}\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}.$$

$U(1)_A$ is anomalous

$U(1)_A$ transformation $q_{L,R} \to e^{\pm i\alpha_A} q_{L,R}$

$$q \to e^{i\gamma_5 \alpha_A} q.$$

is a chiral symmetry which is anomalous.

not a symmetry at quantum level.

Axial vector current is not conserved.

$$J_5^\mu = \bar{q}\gamma^\mu\gamma_5 q \qquad \text{No GB}$$

$$\partial_\mu J_5^\mu = 2 i m_q \bar{q}\gamma_5 q + \frac{g_3^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu}$$

$$\tilde{G}^{a\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} G^a_{\alpha\beta}$$

$\neq 0$ even in the chiral limit

$$m_q = 0.$$

Note) $G\tilde{G}$ is a total derivative

$$G^a_{\mu\nu}\tilde{G}^{a\mu\nu} = \partial_\mu K^\mu,$$

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma} A^a_\nu \left(G^a_{\rho\sigma} - \frac{g_3}{3} f^{abc} A^b_\rho A^c_\sigma \right). \quad A^a_\nu \Big|_\infty = 0$$

$$\delta S = \alpha_A \int d^4x\, \partial_\mu J_5^\mu \propto \int d^4x\, \partial_\mu K^\mu = \int d\Sigma_\mu K^\mu \overset{?}{=} 0$$

Vaccum structure of QCD

- Nontrivial configuration at the boundary ∞

$$G^a_{\mu\nu}\big|_\infty = 0 \quad \text{but} \quad A^a_\mu\big|_\infty \neq 0.$$

Gauge transformation U

$$A_\mu = A^a_\mu T_a \to U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1}$$

$$G_{\mu\nu} = G^a_{\mu\nu} T_a \to U G_{\mu\nu} U^{-1}$$

pure gauge configuration

$$A_\mu = \frac{i}{g} U \partial_\mu U^{-1} \quad \text{non-physical?}$$

ex) $SU(2)$ $\quad T_a = \frac{1}{2}\sigma_a$

Consider $A_\mu = A^a_\mu \frac{\sigma_a}{2} = \frac{i}{g} U_1 \partial_\mu U_1^{-1}$

with $U_1 = \dfrac{x_4 + i\vec{\sigma}\cdot\vec{x}}{\sqrt{r^2}}$, $\quad x_4 = ict$, $\quad r^2 = \vec{x}^2 + x_4^2$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

$$K_\mu = \varepsilon_{\mu\nu\alpha\beta} \operatorname{Tr}\left[A_\nu \partial_\alpha A_\beta - \frac{2}{3} i g A_\nu A_\alpha A_\beta\right]$$

$$\Rightarrow \int d^4x\, G^a_{\mu\nu} \tilde{G}^{a\mu\nu} = \int d^4x \frac{1}{2} \operatorname{Tr}\left[G_{\mu\nu} \tilde{G}^{\mu\nu}\right]$$

$$= \int d^4x\, \partial_\mu K^\mu = \int d\sigma_\mu K^\mu = \underbrace{\int dS^3}_{\substack{\parallel \\ 2\pi^2 r^3}} \hat{x}_\mu \underbrace{K^\mu}_{\substack{\parallel \\ \frac{16}{g^2}\frac{x_\mu}{r^4}}} = \frac{32\pi^2}{g^2}$$

$\therefore$ For $A_\mu^{(1)} = \frac{i}{g} U_1 \partial_\mu U_1^{-1}$

"winding #"

$$\frac{g^2}{32\pi^2}\int d^4x\, G^a_{\mu\nu} G^{a\mu\nu} = \underbrace{1}_{} \qquad S^3 \longrightarrow \underset{\substack{\wr \\ SU(2)}}{S^3}$$

- n-vacuum $|n\rangle$ with winding # n.
  - $A_\mu^{(n)} = \frac{i}{g} U_n \partial_\mu U_n^{-1}$

(§3.2.1) - $U_n = U(1)^n$

- $A_\mu^{(n)} \to A_\mu^{(n+1)} = U_1 A_\mu^{(n)} U_1^{-1} + \frac{i}{g} U_1 \partial_\mu U_1^{-1}$

- $|n\rangle$ is not gauge-invariant

$$U_1 |n\rangle = |n+1\rangle$$

- Gauge invariant "θ-vacuum":

$$|\theta\rangle \equiv \sum_n e^{-in\theta} |n\rangle$$

$$U_m |\theta\rangle = \sum_n e^{-in\theta} |n+m\rangle = e^{im\theta} |\theta\rangle$$

- Effective action for θ-vacuum

$$ {}_{+\infty}\langle \theta | \theta \rangle_{-\infty} = \sum_{m,n} e^{i(m-n)\theta} \, {}_{+\infty}\langle m | n \rangle_{-\infty} $$

$$ = \sum_{\nu} e^{i\nu\theta} \sum_{n} {}_{\infty}\langle n+\nu | n \rangle_{-\infty} $$

$$ = \sum_{\nu} \int [dA]_{\nu} \, e^{i\int d^4x \left[\mathcal{L}^{0}_{QCD}(x) + \mathcal{L}_{\theta}(x) \right]} $$

where $\mathcal{L}_{\theta}(x) = \theta \dfrac{g_3^2}{32\pi^2} G^{a}_{\mu\nu} \tilde{G}^{a\mu\nu}$

gauge invariant

θ = arbitrary real number

Strong CP problem

- $\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + i \bar{q}_L \not{D} q_L + i \bar{q}_R \not{D} q_R$

$$- \bar{q}_L M q_R - \bar{q}_R M^\dagger q_L + \theta \frac{g_3^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}$$

$M = N_f \times N_f$ complex matrix diagonalizable to

$M \longrightarrow \mathrm{Diag}\left[m_q e^{i\varphi_q} \right]$

The phase φ_q can be rotated away

by $\quad q_{L,R} \rightarrow e^{\pm i \varphi_q / 2} q_{L,R}$

This chiral rotation is anomalous

and changes $\theta \longrightarrow \theta + \sum_q \varphi_q$

Thus, we have

$$\boxed{\begin{aligned} \mathcal{L}_{QCD} = {} & -\frac{1}{4} G^2 + i \bar{q}_L \not{D} q_L + i \bar{q}_R \not{D} q_R \\ & - m_q \left(\bar{q}_L q_R + \bar{q}_R q_L \right) + \bar{\theta} \frac{g_3^2}{32\pi^2} G \tilde{G} \end{aligned}}$$

where $\boxed{\bar{\theta} = \theta + \mathrm{Arg\,Det}(M_q)}$

- $GG \sim \vec{E}^2 - \vec{B}^2$ CP-even

  $G\tilde{G} \sim \vec{E}\cdot\vec{B}$ CP-odd.

	$\vec{E}$	$\vec{B}$	$\vec{\mu}$	$\vec{d}$
C	$-$	$-$	$-$	$-$
P	$-$	$+$	$+$	$-$
T	$+$	$-$	$-$	$+$

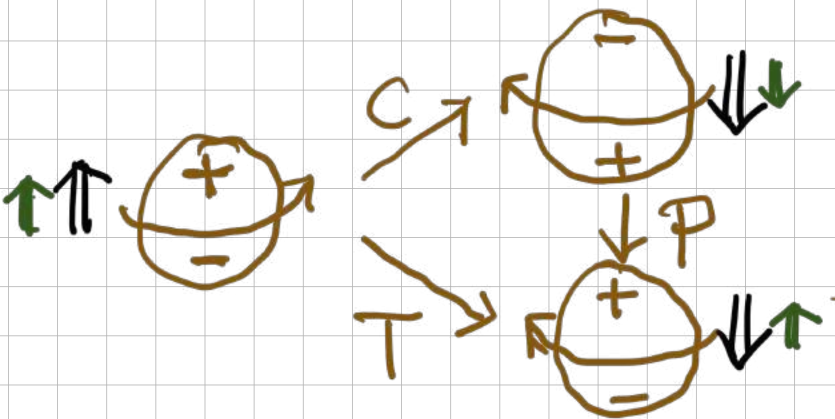
Maxwell eqs.

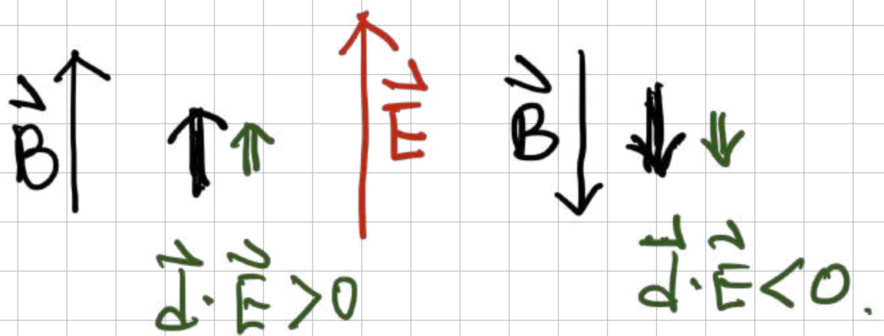
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{B} = \vec{j} + \frac{\partial\vec{E}}{\partial t}$$

$\vec{\mu} \propto \vec{r}\times\vec{j}$

$\vec{S} \propto \vec{r}\times\vec{v}$

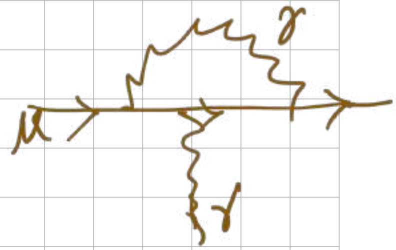




Magnetic Dipole Moment : $\vec{\mu} = g \dfrac{e}{2m} \vec{S}$

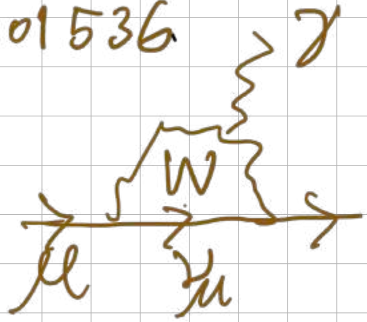
$$g = 2\Big(1 + \underbrace{\frac{\alpha}{2\pi}}_{\parallel\,\mu} + \cdots\Big)$$

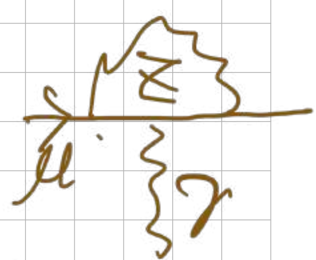
"Schwinger"



0.001 165 920 62(41)

0.000 000 001 536





Electric Dipole Moment : $\vec{d} \neq 0$ ← signal of CPV.

$$h\nu_+ = 2(\mu B + dE)$$

$$h\nu_- = 2(\mu B - dE)$$

$$\Rightarrow d = \frac{h\Delta\nu}{4E}$$

Current limit on neutron EDM

$d_n \lesssim 1.8\times10^{-26}\,e\,cm$ (PSI), $\quad d_n^{SM} \approx 10^{-32}\,e\,cm$.

- $\bar{\theta} \neq 0 \rightarrow d_n \approx \bar{\theta}\times10^{-16}\,e\,cm$

$\therefore \ \underbrace{\bar{\theta} \lesssim 10^{-10}}$ "Why so small?"

(*) To see how EDM arises from $\bar{\theta}$, consider the chiral rotation removing $\mathcal{L}_{anomaly} = \bar{\theta} G\tilde{G}$ term

$$f_{L,R} \longrightarrow e^{\mp i \frac{c_f}{2}\bar{\theta}} f_{L,R} \quad \text{where } f = u, d.$$

under which

$$\mathcal{L}_{anomaly} = \underbrace{\left(\bar{\theta} - (c_u + c_d)\bar{\theta}\right)}_{=0} G\tilde{G}$$

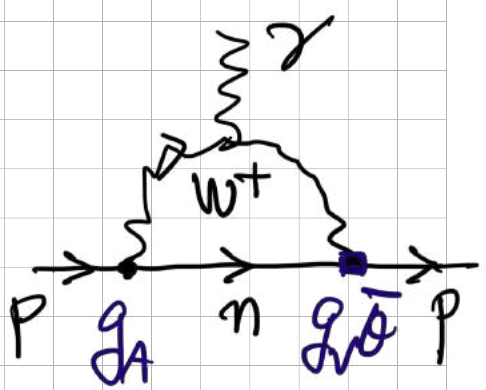
$\leftarrow c_u + c_d = 1$

$\bar{\theta}$ appears in the mass term

$$\mathcal{L}_{mass} = m_u e^{i c_u \bar{\theta}}\, \bar{u}_L u_R + h.c$$
$$+ m_d e^{i c_d \bar{\theta}}\, \bar{d}_L d_R + h.c.$$
$$\simeq m_u \bar{u}u + m_d \underbrace{\bar{d}d}_{\text{CP-even}}$$
$$+ m_u c_u \bar{\theta}\, \bar{u} i\gamma_5 u + m_d c_d \bar{\theta}\, \underbrace{\bar{d} i\gamma_5 d}_{\text{CP-odd}}$$

CPV

$$\Rightarrow \mathcal{L}_{\pi NN} \simeq \frac{g_A}{f_\pi} \partial_\mu \pi^a \bar{N}\gamma^\mu \gamma_5 \sigma^a N$$
$$+ \frac{g_V \bar{\theta}}{f_\pi} \partial_\mu \pi^a \bar{N}\gamma^\mu \sigma^a N$$



PQ mechanism

- Solution to the strong CP problem.

Best

- Introduce a chiral symmetry which is broken spontaneously.

global U(1) anomalous under QCD $= U(1)_{PQ}$

$$S(x) = \frac{f_a + s(x)}{\sqrt{2}} e^{i\frac{a(x)}{f_a}}$$

PQ-charge 1, Axion = GB of $U(1)_{PQ}$.

$$U(1)_{PQ} : \ a \to a + \alpha f_a$$

$$S \to e^{i\alpha} S.$$

ex) Kim (Heavy quark) axion,

$$\mathcal{L}_{PQ} = y_s \left(S \bar{Q}_L Q_R + S^\dagger \bar{Q}_R Q_L \right) + \lambda_s \left(|S|^2 - \frac{f_a^2}{2} \right)^2 .$$

$$\overset{PQSB}{\Longrightarrow} \ y_s \frac{f_a}{\sqrt{2}} \left(e^{i\frac{a}{f_a}} \bar{Q}_L Q_R + h.c \right)$$

Performing a chiral transformation

$$Q_{L,R} \to e^{\pm i a/2f_a} Q_{L,R}$$

$$\mathcal{L}_{PQ} = M_Q(\bar{Q}_L Q_R + h.c), \quad M_Q = y_5 \frac{f_a}{\sqrt{2}}.$$

$$\mathcal{L}_{anomaly} = \underbrace{\left(\bar{\theta} + \frac{a}{f_a}\right)}_{\text{redefine} \to \frac{a}{f_a}} \frac{g_3^2}{32\pi^2} G\tilde{G}$$

- Axion mass

Rotate away $\frac{a}{f_a}$ from $\mathcal{L}_{anomaly}$ to u, d, s quarks

$$\begin{cases} u_{L,R} \to e^{\mp i \frac{C_u}{2} \frac{a}{f_a}} u_{L,R} \\ d_{L,R} \to e^{\mp i \frac{C_d}{2} \frac{a}{f_a}} d_{L,R}. \end{cases}$$

$$\Rightarrow \mathcal{L}_{anomaly} : \frac{a}{f_a} \to \frac{a}{f_a} - \underbrace{(C_u + C_d)}_{= 1.} \frac{a}{f_a} = 0$$

$$\mathcal{L}^{QCD}_{mass} = m_u e^{i C_u \frac{a}{f_a}} \bar{u}_L u_R + h.c$$
$$+ m_d e^{i C_d \frac{a}{f_a}} \bar{d}_L d_R + h.c$$

$$\Rightarrow \mathcal{L}_{\text{eff.}} = \frac{1}{2} B f_\pi^2 \, \mathrm{Tr}(M\Sigma^\dagger + \Sigma M^\dagger)$$

$$M = \mathrm{Diag.}\left(m_u e^{i c_u \frac{a}{f_a}},\ m_d e^{i c_d \frac{a}{f_a}}\right)$$

$$\Sigma = e^{i\frac{\phi}{f_\pi}}, \quad \phi = \begin{pmatrix} \pi^0 & 0 \\ 0 & -\pi^0 \end{pmatrix}$$

$$M\Sigma^\dagger = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \begin{pmatrix} e^{-i\left(\frac{\pi^0}{f_\pi} - c_u \frac{a}{f_a}\right)} & 0 \\ 0 & e^{+i\left(\frac{\pi^0}{f_\pi} + c_d \frac{a}{f_a}\right)} \end{pmatrix}$$

$$\mathcal{L}_{\text{eff}} = B f_\pi^2 \left[m_u \cos\left(\frac{\pi^0}{f_\pi} - c_u \frac{a}{f_a}\right) + m_d \cos\left(\frac{\pi_0}{f_\pi} + c_d \frac{a}{f_a}\right) \right]$$

$$\approx \frac{1}{2} B f_\pi^2 \left[(m_u + m_d) \frac{(\pi^0)^2}{f_\pi^2} + (m_u c_u^2 + m_d c_d^2) \frac{a^2}{f_a^2} - 2\underbrace{(m_u c_u - m_d c_d)}_{=0} \frac{\pi^0 a}{f_\pi f_a} \right]$$

$$c_u = \frac{m_d}{m_u + m_d}, \quad c_d = \frac{m_u}{m_u + m_d}$$

$$m_{\pi^0}^2 = B(m_u + m_d)$$

$$\boxed{m_a^2 = \frac{m_\pi^2 f_\pi^2}{f_a^2} \frac{m_u m_d}{(m_u + m_d)^2}}$$