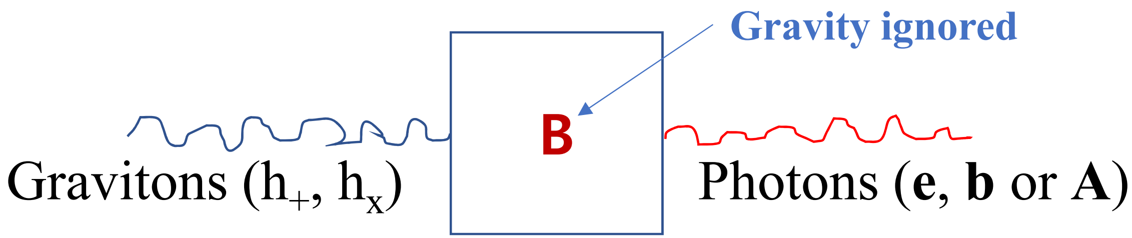
On graviton-photon conversions in magnetic environments

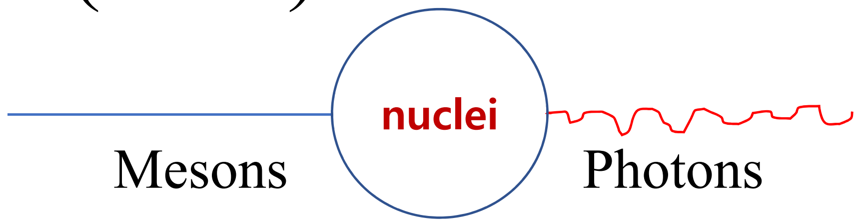
J. Hwang (IBS) & H. Noh (KASI)
CTPU workshop on
Particle Physics and Cosmology
20-25 October 2023

JH, Noh, On graviton-photon conversions in magnetic environments, 2310.04150

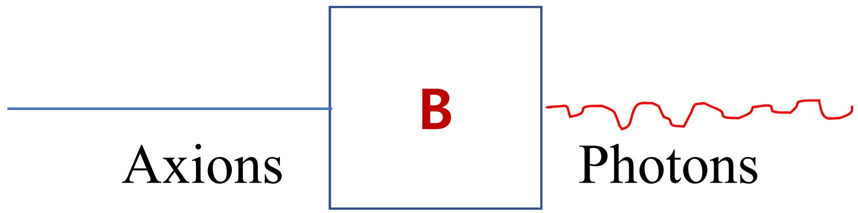
Gertsenshtein effect (1961)



Primakoff effect (1951)



Axion-photon mixing (Sikivie 1983)



Gertsenshtein, Wave resonance of light and gravitational waves (1962);
Primakoff, Photo-production of neutral mesons in nuclear electric fields and the mean life of the neutral meson (1951);
Sikivie-Experimental tests of the invisible axion (1983).

To linear orders in gravitons and photons

Graviton conversion (Einstein equation):

$$(\partial_0^2 - \Delta)h_{ij} = \frac{16\pi G}{c^4}\delta T_{ij}^{\mathrm{TT}}. \qquad T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^{cd}F_{cd}.$$

Metric (gravitons) in T_{ab} is **ignored** in the literature

with unknown reason

Photon conversion (Maxwell equations):

$$(\sqrt{-g}F^{ab})_{,b} = \eta^{ac}\eta^{bd}F_{cd,b} + \left(\frac{1}{2}h_c^c F^{ab} - F^{ac}h_c^b + F^{bc}h_c^a\right)_{,b} = \frac{1}{c}\sqrt{-g}J^a.$$

$$\eta^{abcd}F_{bc,d} = 0.$$

Metric (gravitons) in F_{ab} (in terms of EM fields) is **ignored** in the literature

by mistake

Covariant decomposition

(Møller 1952, Lichnerowicz 1967, Ellis 1973)

❖ The electric and magnetic (EM) fields and charge and current densities are **defined** using a time-like frame four-vector u_a

$F_{i0} \neq E_i$

$$F_{ab} = u_a E_b - u_b E_a - \eta_{abcd} u^c B^d, \quad E_a \equiv F_{ab} u^b, \quad B_a \equiv F^*_{ab} u^b.$$

In curved spacetime

$$J^a \equiv \varrho c u^a + j^a, \quad j_a u^a \equiv 0, \quad \varrho \equiv -\frac{1}{c} J^a u_a, \quad j^a \equiv h^a_b J^b.$$

Four current; Charge density; Current density; Projection tensor $h_{ab} \equiv g_{ab} + u_a u_b$

❖ Fluid quantities are similarly **defined** using the four-vector

$T_{00} \neq \mu$

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab},$$

No meaning in curved spacetime

$$\mu = T_{ab} u^a u^b, \quad p = \frac{1}{3} T_{ab} h^{ab}, \quad q_a = -T_{cd} u^c h^d_a, \quad \pi_{ab} = T_{cd} h^c_a h^d_b - p h_{ab}.$$

Energy density; Pressure; Flux vector; Anisotropic stress tensor

➢ Comoving four-vector (Lagrangian observer): $u_a =$ fluid four-vector.

➢ Normal four-vector (Eulerian observer): n_a with $n_i \equiv 0$.

➢ Coordinate four-vector: $\bar{n}_a$ with $\bar{n}^i \equiv 0$.

JH, Noh, Definition of electric and magnetic fields in curved spacetime, 2303.07562

Graviton conversion

GWs in TT gauge

To linear order, indices can be raised using η^{ab}.

Perturbed metric: $g_{ab} = \eta_{ab} + h_{ab}.$

$$h^c_{a,bc} + h^c_{b,ac} - h^c_{c,ab} - h_{ab}{}^{,c}{}_{c} - \eta_{ab}(h^{cd}{}_{,cd} - h^{c,d}_{c}{}_{d}) = \frac{16\pi G}{c^4}\delta T_{ab}.$$

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^{cd}F_{cd}.$$

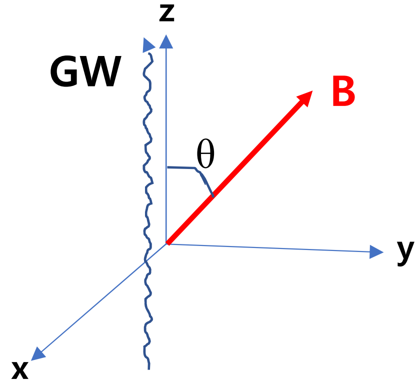
Metric is everywhere, but **ignored** in the literature

Transverse-tracefree (TT) gauge conditions: $h^j_{i,j} \equiv 0 \equiv h^i_i, \quad h_{00} \equiv 0 \equiv h_{0i}.$

$$(\partial_0^2 - \Delta)h_{ij} = \frac{16\pi G}{c^4}\delta T^{\mathrm{TT}}_{ij}.$$

TT projection operator

$$T^{\mathrm{TT}}_{ij} \equiv P_{ik}T^{k\ell}P_{\ell j} - \frac{1}{2}P_{ij}P_{k\ell}T^{k\ell}, \quad P_{ij} \equiv \delta_{ij} - \hat{n}_i\hat{n}_j, \quad \hat{n}^i\hat{n}_i \equiv 1.$$



$B_x = 0,\ B_y = B\sin\theta,\ B_z = B\cos\theta.$

Assume a plane GW propagating in z-direction: $h_{ij} = h_{ij}(x^0 - z)$

$$h_{11} = -h_{22} \equiv h_+, \quad h_{12} = h_{21} \equiv h_\times.$$

In terms of EM fields

Metric involved in F_{ab}:

Metric (gravitons)

$$F_{0i} = -E_i, \quad F_{ij} = \eta_{ijk}(B^k - h^{k\ell}B_\ell).$$

Photons

Energy-momentum tensor: $B_i \to B_i + b_i,\ E_i \to e_i$

$$\delta T_{ij} = -B_i b_j - B_j b_i + \delta_{ij} B^k b_k + \frac{1}{2}(h_{ij}B^2 - \delta_{ij}h^{k\ell}B_k B_\ell).$$

The metric in T_{ab} is **ignored** in previous literature

TT projection:

$$\delta T_{ij}^{\mathrm{TT}} = -B_i b_j - B_j b_i + \delta_{ij}(B^k b_k - B^k \hat{n}_k b^\ell \hat{n}_\ell) + 2\hat{n}_{(i}(B_{j)} b^k \hat{n}_k + b_{j)} B^k \hat{n}_k)$$
$$-\hat{n}_i \hat{n}_j (B^k \hat{n}_k b^\ell \hat{n}_\ell + B^k b_k) + \frac{1}{2}B^2 h_{ij}.$$

Graviton conversion:

Tachyonic instability term caused by the metric in T_{ab}

$$(\partial_0^2 - \partial_z^2 - \frac{8\pi G}{c^4}B^2)h_+ = -\frac{16\pi G}{c^4}(B_1 b_1 - B_2 b_2),$$
$$(\partial_0^2 - \partial_z^2 - \frac{8\pi G}{c^4}B^2)h_\times = -\frac{16\pi G}{c^4}(B_1 b_2 + B_2 b_1).$$

Graviton instability?

Exponential instability: $h_{+,\times} \propto \exp\Big(\pm\sqrt{\frac{8\pi G}{c^2}B^2 - c^2k^2}\,t\Big).$

Instability criterion: $L>\lambda_B$

$$\lambda_B \equiv \frac{2\pi}{k_B} = \sqrt{\frac{\pi}{2}}\frac{c^2}{\sqrt{G}B} = 1.4\frac{1\text{gauss}}{B}\text{Mpc} \sim 4\times 10^{11}\frac{10^{13}\text{gauss}}{B}\text{cm}.$$

$>> R_{\text{Neutron Star}} \sim 10^6$ cm

∴ Not likely to be realized in Nature.

Gravitational strength of background B:

$$\left(\frac{L}{\lambda_B}\right)^2 = \frac{2}{\pi}\frac{L^2GB^2}{c^4} \sim \frac{GM}{Lc^2}\frac{B^2}{\varrho c^2} \sim 5.2\times 10^{-9}\left(\frac{L}{10^7\text{cm}}\right)^2\left(\frac{B}{10^{13}\text{gauss}}\right)^2$$

← Astrophysics

$$\sim 5.2\times 10^{-35}\left(\frac{L}{10^2\text{cm}}\right)^2\left(\frac{B}{10^5\text{gauss}}\right)^2.$$

← Laboratory

❖ $L>\lambda_B$ **violates** the hidden assumption of ignoring the gravity of background **B**.

❖ The instability term cannot dominate in our analysis and is small.

In terms of the potential

No metric involved in F_{ab}:

Photons

$$\delta F_{0i} = A_{i,0} - A_{0,i}, \quad \delta F_{ij} = A_{j,i} - A_{i,j}.$$

Energy-momentum tensor:

$$\delta T_{ij} = 2B_{(i}\eta_{j)k\ell}A^{k,\ell} - \delta_{ij}B^m\eta_{mk\ell}A^{k,\ell} + \frac{1}{2}(h_{ij}B^2 + \delta_{ij}h^{k\ell}B_kB_\ell) - 2h^k_{(i}B_{j)}B_k.$$

Graviton conversion:

Instability term, or effective mass term

$$\left[\partial_0^2 - \partial_z^2 - \frac{8\pi G}{c^4}(B_3^2 - \cancel{B_1^2} - B_2^2)\right] h_+ = \frac{16\pi G}{c^4}[\cancel{B_1}(A_{2,3} - A_{3,2}) - B_2(A_{3,1} - A_{1,3})],$$

$$\left[\partial_0^2 - \partial_z^2 - \frac{8\pi G}{c^4}(B_3^2 - \cancel{B_1^2} - B_2^2)\right] h_\times = \frac{16\pi G}{c^4}[\cancel{B_1}(A_{3,1} - A_{1,3}) + B_2(A_{2,3} - A_{3,2})].$$

- The instability terms **differ** from the ones using EM fields

Relations between the potential and EM fields involve the metric:

$$E_i = \partial_i A_0 - \partial_0 A_i, \quad B_i = (\delta_{ij} + h_{ij})\eta^{jk\ell}\partial_k A_\ell.$$

Metric

- In terms of EM fields is physical

Photon conversion

Maxwell's equations

Maxwell's equations with linear metric perturbations:

Inhomogeneous eq:

$$(\sqrt{-g}F^{ab})_{,b} = \eta^{ac}\eta^{bd}F_{cd,b} + \left(\frac{1}{2}h^c_c F^{ab} - F^{ac}h^b_c + F^{bc}h^a_c\right)_{,b} = \frac{1}{c}\sqrt{-g}J^a.$$

Ignore the current

Homogeneous eq:

$$\eta^{abcd}F_{bc,d} = 0.$$

In terms of EM fields, F_{ab} involves the metric!
Often missed in the literature (Berlin et al. 2022, Domcke et al. 2022, Palessandro, Rothman 2023)
Thus, indices of F_{ab} cannot be raised using η^{ab}.

- In terms of the potential, $F_{ab} \equiv \partial_a A_b - \partial_b A_a$, F_{ab} does not involve the metric, and the homogeneous equation is identically valid.
- In the literature, this fact is often **confused** with F_{ab} free from the metric, even using EM fields. [See below Eq. (13) in Berlin et al. (2022), and see below Eq. (S13) in Domcke et al. (2022).]
- The relation between the potential and EM fields involves the metric.

JH, Noh, Maxwell equations in curved spacetime, 2307.14555

In terms of EM fields

Maxwell's equations:

$$E^i{}_{,i} = (h^{ij}E_j)_{,i},$$
$$E^i{}_{,0} - \eta^{ijk}\nabla_j B_k = (h^{ij}E_j)_{,0},$$
$$B^i{}_{,i} = (h^{ij}B_j)_{,i},$$
$$B^i{}_{,0} + \eta^{ijk}\nabla_j E_k = (h^{ij}B_j)_{,0}.$$

Background B + photons: $B_i \to B_i + b_i,\ E_i \to e_i$

Uniform and constant, assumed

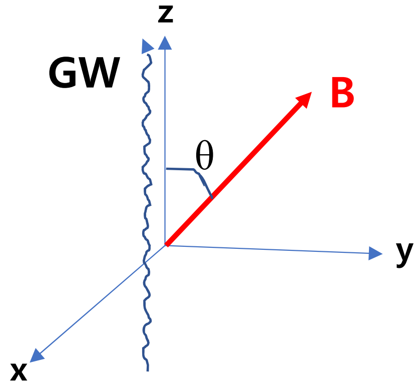
Photons

$$(\partial_0^2 - \Delta)b_i = (h_i^j B_j)_{,00} - \nabla_i(h^{jk}B_{j,k}),$$
$$(\partial_0^2 - \Delta)e_i = \eta_{ijk}\nabla^j(h_\ell^k B^\ell)_{,0}.$$

For a uniform and constant B:

$$(\partial_0^2 - \Delta)b_i = h^j_{i,00}B_j,$$
$$(\partial_0^2 - \Delta)e_i = \eta_{ijk}\nabla^j(h^k_{\ell,0})B_\ell.$$

Align:



$B_x = 0,\ B_y = B\sin\theta,\ B_z = B\cos\theta.$

$$(\partial_0^2 - \Delta)b_1 = h_{+,00}B_1 + h_{\times,00}B_2,$$
$$(\partial_0^2 - \Delta)b_2 = h_{\times,00}B_1 - h_{+,00}B_2,$$
$$(\partial_0^2 - \Delta)b_3 = 0,$$
$$(\partial_0^2 - \Delta)e_1 = -h_{\times,z0}B_1 + h_{+,z0}B_2,$$
$$(\partial_0^2 - \Delta)e_2 = h_{+,z0}B_1 + h_{\times,z0}B_2,$$
$$(\partial_0^2 - \Delta)e_3 = 0.$$

- Ignoring background **E** already implies a uniform and constant **B**.

Wrong EM fields

The EM fields are physical (measurable) quantities.
No observer can measure these as the EM fields.
Thus, these are **wrong** definitions!

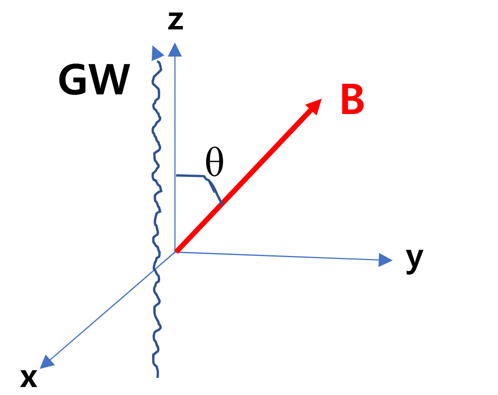
Wrong definitions: $F_{0i} \equiv -\hat{E}_i, \quad F_{ij} \equiv \eta_{ijk}\hat{B}^k. \longrightarrow \hat{E}_i = E_i, \; \hat{B}_i = B_i - h_i^j B_j.$

Maxwell's equations:

$$\begin{aligned}
&\hat{E}^i{}_{,i} = (h^{ij}E_j)_{,i},\\
&\hat{E}^i{}_{,0} - \eta^{ijk}\nabla_j\hat{B}_k = (h^{ij}E_j)_{,0} + \eta^{ijk}(B_\ell h_k^\ell)_{,j},\\
&\hat{B}^i{}_{,i} = 0,\\
&\hat{B}^i{}_{,0} + \eta^{ijk}\nabla_j\hat{E}_k = 0.
\end{aligned}$$

Uniform and constant, assumed

Align:



$B_x = 0, \; B_y = B\sin\theta, \; B_z = B\cos\theta.$

Background B + photons: $\hat{B}_i \rightarrow B_i + \hat{b}_i, \; E_i \rightarrow e_i$

$$(\partial_0^2 - \Delta)\hat{b}_i = \Delta(h_i^j B_j) - \nabla_i(h^{jk}B_{j,k}).$$

For a uniform and constant B:

$$(\partial_0^2 - \Delta)\hat{b}_i = \Delta h_i^j B_j.$$

$$\begin{aligned}
(\partial_0^2 - \Delta)\hat{b}_1 &= h_{+,zz}B_1 + h_{\times,zz}B_2,\\
(\partial_0^2 - \Delta)\hat{b}_2 &= h_{\times,zz}B_1 - h_{+,zz}B_2,\\
(\partial_0^2 - \Delta)\hat{b}_3 &= 0.
\end{aligned}$$

- ❖ The wrong definitions give correct equations only for a plane GW with $h_{+,\times}(x^0 - z)$, **in TT gauge** in a uniform and constant **B**.
- ❖ Case is **unclear** in the FNC (**Fermi Normal Coordinate**). (Berlin et al. 2022, Domcke et al. 2022)

In terms of the potential

Maxwell's equations with Coulomb gauge $\nabla \cdot \mathbf{A} \equiv 0$:

$$\Delta A_0 = h^{ij} E_{i,j}.$$

For uniform background or vanishing background **E**, we may set $A_0 = 0$.

Photons

$$(\partial_0^2 - \Delta) A_i = \eta^{jk\ell} h_{ij,k} B_\ell.$$

For a uniform and constant B:

$$(\partial_0^2 - \Delta) A_1 = h_{\times,z} B_1 - h_{+,z} B_2,$$
$$(\partial_0^2 - \Delta) A_2 = -h_{+,z} B_1 - h_{\times,z} B_2,$$
$$(\partial_0^2 - \Delta) A_3 = 0.$$

Relations to EM fields:

$$b^i = \eta^{ijk} \partial_j A_k + h^i_j B^j, \quad e_i = -A_{i,0}.$$

Metric

Combined equations

Using EM fields:

$$\left(\partial_0^2 - \partial_z^2 - \frac{8\pi G}{c^4} B^2\right) h_+ = \frac{16\pi G}{c^4} B_2 b_2,$$

$$\left(\partial_0^2 - \partial_z^2 - \frac{8\pi G}{c^4} B^2\right) h_\times = -\frac{16\pi G}{c^4} B_2 b_1,$$

$$(\partial_0^2 - \Delta) b_1 = B_2 h_{\times,00},$$

$$(\partial_0^2 - \Delta) b_2 = -B_2 h_{+,00},$$

$$(\partial_0^2 - \Delta) e_1 = B_2 h_{+,z0},$$

$$(\partial_0^2 - \Delta) e_2 = B_2 h_{\times,z0}.$$

Using the potential:

$$\left[\partial_0^2 - \partial_z^2 - \frac{8\pi G}{c^4}(B_3^2 - B_2^2)\right] h_+ = \frac{16\pi G}{c^4} B_2 A_{1,z},$$

$$\left[\partial_0^2 - \partial_z^2 - \frac{8\pi G}{c^4}(B_3^2 - B_2^2)\right] h_\times = \frac{16\pi G}{c^4} B_2 A_{2,z},$$

$$(\partial_0^2 - \Delta) A_1 = -B_2 h_{+,z},$$

$$(\partial_0^2 - \Delta) A_2 = -B_2 h_{\times,z}.$$

Using $A_+ \equiv A_1$, $A_\times \equiv A_2$ with $\lambda = +, \times$:

$$\left[\partial_0^2 - \partial_z^2 - \frac{8\pi G}{c^4}(B_3^2 - B_2^2)\right] h_\lambda = \frac{16\pi G}{c^4} B_2 A_{\lambda,z},$$

$$(\partial_0^2 - \Delta) A_\lambda = -B_2 h_{\lambda,z}.$$

cf., Raffelt, Stodolsky, Mixing of the photon with low-mass particles (1988).

Conversion rates

Graviton conversion rate:

$$P_{\gamma\to g} \equiv \frac{E_{\rm GW}}{E_{\rm EM}} = \frac{c^2}{8\pi G}\frac{|\dot{h}_+|^2+|\dot{h}_\times|^2}{|e_1|^2+|e_2|^2+|b_1|^2+|b_2|^2} \sim \frac{c^2}{16\pi G}\frac{|\dot{h}|^2}{|b|^2} \sim \frac{16\pi G}{c^4}B^2\sin^2\theta L^2.$$

Dimensional estimation

$$T_{00}^{\rm GW} = \frac{c^4}{32\pi G}h^{ij}{}_{,0}h_{ij,0} = \frac{c^2}{16\pi G}(|\dot{h}_+|^2+|\dot{h}_\times|^2). \qquad \mu_{\rm EM} = \frac{1}{2}(E^2+B^2).$$

Photon conversion rate:

Dimensionless gravitational strength of the background **B**

$$P_{\gamma\to g} \sim P_{g\to\gamma} \sim 8\pi^2\sin^2\theta\left(\frac{L}{\lambda_B}\right)^2.$$

❖ Conversion rates are the same and are proportional to the gravitational strength of the background **B**.

Ya.B. Zel'dovich, I.D. Novikov, *The Structure and Evolution of the Universe* (1983) Section 17.9.

Interaction Lagrangian

EM Lagrangian:

$$\mathcal{L}_{\rm EM} = -\frac{1}{4}\sqrt{-g}F^{ab}F_{ab}.$$

To linear order in metric:

$$\mathcal{L}_{\rm EM} = -\frac{1}{4}\eta^{ac}\eta^{bd}F_{ab}F_{cd} + \frac{1}{2}h_{ab}T^{ab}_{\rm EM}.$$

Often called as interaction Lagrangian (Boccaletti et al. 1967)

However, using EM fields, F_{ab} also depends on the metric!

- ❖ F_{ab} also depends on the metric in terms of EM fields.
- ❖ In terms of the potential, F_{ab} is free from the metric, but the relation between the potential and EM fields involves the metric.

Conclusion

- The electric and magnetic fields should be defined by decomposing F_{ab} using the observer's four-vector.
- In a curved spacetime, the relation between EM fields and F_{ab}, with two covariant indices, is inevitably affected by gravity.
- F_{ab} is free from the metric using the potential, but the relation between the potential and EM fields involves the metric.
- Gravity causes modifications in both the homogeneous and inhomogeneous Maxwell's equations.
- Related errors abound in the literature related to detecting gravitational waves using electromagnetic methods, medium interpretation of gravity, and graviton-photon conversions.