Gradient CKV and the CMC condition in LRS spacetimes

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Some background . . . 1/4

- Role of symmetries in general relativity dates back to the 1920's when Brinkmann examined the usefulness of such transformations to obtain new exact solutions to the field equations (**Brinkmann, Math. Ann.**, 1924, 1925).

- Killing and conformal Killing symmetries, the action for which the metric is left either unchanged, or scaled, are usually assumed.

- If the vector field generating the symmetry is timelike, one has a choice of observers.

- Then, the spacetime admitting the symmetry is conformal to a stationary one, i.e. it scales a stationary spacetime by some factor. These spacetimes are termed *conformally stationary* (CS) spacetimes (**Alías** *et al.*, 1997, **Caballero** *et al.*, 2011).

Some background . . . 2/4

- If the timelike conformal Killing vector field is gradient, these spacetimes are called gradient CS (GCS) spacetimes. A sufficient condition to ensure causal stability is that the gradient condition be global.
- In the timelike region, the orthogonal distribution of the GCKV induces a foliation of the spacetime into spacelike hypersurfaces.
- Stability and uniqueness of these hypersurfaces have been examined in for example (**Caballero** *et al.*, **Nonlinear Anal.: Theory Methods Appl.**, 2011, **Romero** *et al.*, **CQG**, 2013, **de la Fuente** *et al.*, **GRG**, 2017).

Some background . . . 3/4

- A particular subclass of GCS spacetimes is the set of generalized Robertson-Walker (GRW) spacetimes, a warped product spacetime that has been extensively studied for various reasons. See the references (**Romero** *et al.*, **CQG**, 2013, **Romero** *et al.*, **IJGMMP**, 2013, **Romero** *et al.*, **J. Math. Anal. App.**, 2014, **Mantica & Molinari**, **JMP**, 2019). For an indepth review in the case of GRW spacetimes, see (**Mantica** *et al.*, **IJGMMP**, 2017.)

- Given a symmetry of a spacetime, there are implications for the presence and location of trapped and marginally outer trapped surfaces (to be defined later), and hence the character of horizons that they foliate.

Some background . . . 4/4

- The relationship between symmetries of spacetimes and trapped and marginally trapped surfaces have previously been investigated, for example, in (**Mars & Senovilla**, **CQG**, 2003, **Ashtekar & Galloway**, **Adv. Theor. Math. Phys.**, 2005.).
- These studies have primarily focused on the Killing case, where the existence of trapped surfaces, MOTS and certain horizon types were investigated under the Killing symmetries assumptions, considering different causal characters of the Killing vector.
- The stability of MOTS contained in an initial data, which admits a Killing symmetry, have been investigated by (**Booth** *et al.*, **arXiv:2311.02063v1**, 2023, **Carrasco & Mars**, **CQG**, 2009). In particular, in the former, several results demonstrating the instability of exotic MOTS were obtained.

What is done . . . 1/3

- Broadly put, we are interested in the existence of GCKV and what implications these necessary and/or sufficient conditions have on the presence and character of MOTS in the spacetime. Our interest is in the class of *locally rotationally symmetric* (LRS) spacetimes (which we define shortly).

- We employ the so-called 1+1+2 formalism which extends the 1+3 covariant approach. The approach is partularly well suited for the class of spacetimes we are considering.

- Conformal symmetries in LRS spacetimes have been studied variously under several assumptions and adaptation to the 1+1+2 formulation can be found in these recent works (**Bergh**, **PRD**, 2017, **Singh** *et al.*, **JMP**, 2019, **Cheverra** *et al.*, **GRG**, 2020, **Cheverra** *et al.*, **IJGMMP**, 2023).

What is done . . . 2/3

- Existence of gradient CKV in vacuum and perfect fluid spacetimes was first examined in (**Daftardar & Dadhich**, **GRG**, 1994).

- In a recent work by (**Koh** *et al.*, **arXiv:2305.05148v2**, 2023), this was generalized to all of LRS spacetimes, with the existence in the LRS II case characterized by a wave-like PDE for associated the potential function. The timelike condition for the gradient CKV was also discussed.

- We revisit the existence of GCKV for LRS spacetimes. We obtain an identity relating the curvature to the divergence of the CKV (henceforth, *conformal divergence*), which provides an alternative set of equations in the potential function, from which we extract information about the existence of GCKV.

What is done . . . 3/3

- We make several observations about the set of equations. We further obtain a uniqueness result when restricting to perfect fluid spacetimes.
- The spacelike hypersurfaces resulting from the splitting induced by the GCKV have CMC. We obtain the CMC condition(s) . These are formulated here as algebraic relations on the matter and Weyl curvature variables.
- The CMC condition also allows us to draw some conclusions about the existence about GCKV. From this we strengthen the previous uniqueness result.
- CMC condition is used to study existence of trapped surfaces and MOTS, the evolution of the surface in the Case of MOTS, and how a black hole is observed in the conformal frame.

LRS spacetimes . . . 1/2

- Formally, *LRS* spacetimes are those admitting a multiply transitive isometry group, with a continuous isotropy group at each point of the spacetime. In other words, admit *local* axis of symmetry (**Stewart & Ellis, JMP**, 1968).

- In local coordinates LRS metric is (**Stewart & Ellis, JMP**, 1968)

$$
\begin{aligned}
ds^2 = &-b_1^2dt^2 + b_2^2dr^2 + b_3^2dy^2 + \left((Db_3)^2 + \left(b_2\bar{h}\right)^2 - (b_1\bar{g})^2\right)dz^2 \\
&+ 2\left(b_1^2\bar{g}dt - A_2^2hdr\right)dz,
\end{aligned} \tag{1}
$$

where $b_j = b_j(t,r)$, $\bar{g} = \bar{g}(y), \bar{h} = \bar{h}(y)$, and

$$
\begin{cases}
D &= \sinh y \quad \text{for hyperbolic } M_2, \\
D &= y \quad \text{for planar } M_2, \\
D &= \sin y \quad \text{for spherical } M_2.
\end{cases}
$$

LRS spacetimes . . . 2/2

- In addition to the 4-velocity $u^a = -b_1^{-1}\partial_t^a$, the metric (1) admits a preferred local axis of symmetry, the unit field $n^a = b_2^{-1}\partial_r^a$.

- A particular class is LRS II with $g = 0 = h$, which are irrotational (the temporal and spatial congruences have vanishing vorticities) with the Weyl contribution coming only from the electric part.

- The LRS II class generalizes spherically symmetric (SS) solutions. Also contains exact black holes and cosmological solutions: **Schwarzschild, SS LTB, FRW**, and background of many **stellar** models.

- Few cases with at least one of the vorticities of u^a and n^a nonvanishing, have been considered. A well known example is the *Gödel* rotating solution.

A decomposition of LRS spacetimes: 1+1+2 . . . 1/4

We follow these references (**Clarkson, CQG**, 2003; **PRD**, 2007).

- The presence of the unit spatial vector $n^a \perp u^a$ introduces the projector tensor

$$N_{ab} = g_{ab} + u_a u_b - n_a n_b.$$

- N_{ab} is the induced metric on surfaces of constant t and r, to which u^a and n^a are orthogonal.

- Accordingly, the energy momentum tensor assumes the form

$$T_{ab} = \rho u_a u_b + 2Q u_{(a} n_{b)} + p h_{ab} + \Pi \left(n_a n_b - \frac{1}{2} N_{ab} \right) :$$

$\rho = T_{ab} u^a u^b$ is energy density; $3p = T_{ab} h^{ab}$ is pressure; $Q = T_{ab} n^a u^b$ is heat flux; and $\Pi = T_{ab} n^a n^b - p$ encodes deviation from isotropy.

A decomposition of LRS spacetimes: 1+1+2 . . . 2/4

- The Ricci tensor, from the field equations, take the form

$$R_{ab} = g_1 u_a u_b + g_2 h_{ab} + 2Qu_{(a}n_{b)} + \Pi\left(n_a n_b - \frac{1}{2}N_{ab}\right),$$

where we have set

$$g_1 = \frac{1}{2}(\rho + 3p - 2\Lambda); \quad g_2 = \frac{1}{2}(\rho - p + 2\Lambda).$$

- The decomposition introduces two convective derivatine:
  - Dot derivative: $\dot{\psi}_{a\dots b}{}^{c\dots d} = u^f \nabla_f \psi_{a\dots b}{}^{c\dots d}$ (evolution);
  - Hat derivative: $\hat{\psi}_{a\dots b}{}^{c\dots d} = n^f \nabla_f \psi_{a\dots b}{}^{c\dots d}$ (propagation).

- Any 4-vector ψ^a can then alway be written in the decomposed form

$$\psi^a = \psi_1 u^a + \psi_2 n^a. \qquad (\psi_1 = -u_a\psi^a,\ \psi_2 = n_a\psi^a,\ N^a{}_b\psi^b = 0)$$

A decomposition of LRS spacetimes: 1+1+2 . . . 3/4

- Similarly, the gradient of a scalar ψ takes the decomposed form

$$\nabla_a\psi = -\dot{\psi}u_a + \hat{\psi}n_a.$$

- The covariant derivatives of the unit vector fields can also be written covariantly as

$$\nabla_a u_b = -\mathcal{A}u_a n_b + \left(\frac{1}{3}\theta + \sigma\right) n_a n_b + \frac{1}{2}\left(\frac{2}{3}\theta - \sigma\right) N_{ab} + \Omega\varepsilon_{ab},$$

$$\nabla_a n_b = -\mathcal{A}u_a u_b + \left(\frac{1}{3}\theta + \sigma\right) n_a u_b + \frac{1}{2}\phi N_{ab} + \xi\varepsilon_{ab} :$$

$\mathcal{A} = n_a\dot{u}^a$ is acceleration; $\theta = \nabla_a u^a$ is expansion of u^a, $\sigma = \sigma_{ab}n^a n^b$ is shear scalar; $\phi = N^{ab}\nabla_a n_b$ is expansion of n^a (surface expansion henceforth); $\Omega = (1/2)\varepsilon^{ab}\nabla_a u_b$; is vorticity of u^a, $\xi = (1/2)\varepsilon^{ab}\nabla_a n_b$ is vorticity of n^a; ε_{ab} is 2-dimensional alternating tensor.

A decomposition of LRS spacetimes: 1+1+2 . . . 4/4

- Generally, 'dot' and 'hat' derivatives do not commute. Rather, they obey the following commutation relation when acting on a scalar ψ:

$$\dot{\hat{\psi}} - \hat{\dot{\psi}} = -\mathcal{A}\dot{\psi} + \left(\frac{1}{3}\theta + \sigma\right)\hat{\psi}.$$

- Furthermore, the electric and magnetic parts of the Weyl tensor are

$$E_{ab} = \mathcal{E}\left(n_a n_b - \frac{1}{2}N_{ab}\right), \qquad H_{ab} = \mathcal{H}\left(n_a n_b - \frac{1}{2}N_{ab}\right).$$

- LRS spacetimes specified by the set

$$Z \equiv: \{\mathcal{A}, \theta, \sigma, \phi, \Omega, \xi, \rho, p, \mathcal{E}, \mathcal{H}, Q, \Pi\} \quad (\text{Eg. } Z_{\text{Sch}} \equiv: \{\mathcal{A}, \phi, \mathcal{E}\}).$$

- Finally, one writes the field equations as a set of evolution and propagation, as well as constraint equations for Z.

Conformal symmetry

- A spacetime $(\mathcal{M}, g_{ab})$ admits conformal symmetry if $\exists\ \eta^a$ s.t.

$$\mathcal{L}_\eta g_{ab} = \nabla_a \eta_b + \nabla_b \eta_a = 2\varphi g_{ab}. \tag{2}$$

- η^a is Killing vector (KV), homothetic Killing vector (HKV), or (proper) conformal Killing vector (CKV) if $\varphi = 0$, (nonzero) constant, or nonconstant, respectively.

- η^a is gradient if $\exists$ scalar Ψ s.t. $\eta^a = \nabla^a \Psi$.

- η^a is CKV $\iff \nabla_b \eta_a = \varphi g_{ab} + \mathcal{F}_{ab}$, where $\mathcal{F}_{ab} = \mathcal{F}_{[ab]}$.

- Thus, η^a is gradient $\iff \mathcal{F}_{ab}$ vanishes.

- If η^a timelike, $\mathcal{M}$ is **gradient conformally stationary** (GCS).

Conformal symmetry in LRS spacetimes

- We consider vector field of the form

$$x^a = \alpha_1 u^a + \alpha_2 n^a .$$

- x^a gradient $\iff$ $\alpha_1 = -\dot{\Psi}, \alpha_2 = \hat{\Psi}$.

$$\mathcal{F}_{ab} = -2\left(\dot{\alpha}_2 + \mathcal{A}\alpha_1\right) u_{[a} n_{b]} + \left(\alpha_1 \Omega + \alpha_2 \xi\right) \varepsilon_{ab} = 0$$

- In (**Koh** *et al.*, **arXiv:2305.05148v2**, 2023), it was established that only subclasses of LRS spacetimes admitting a GCKV are
  - $\Omega = 0$ and $\xi \neq 0$;
  - LRS II class: $\Omega = 0 = \xi$.

- For the latter, the necessary and sufficient condition was given by the following wave-like PDE:

$$\Box \Psi = 4\varphi .$$

Alternative characterization for gradient CKV . . . 1/7

Let us immediately state the following Lemmas.

Lemma

If a spacetime $\mathcal{M}$ admits a CKV x^a with conformal divergence φ, then, the following hold:

$$\Box x_a = -R_{ab}x^b - 2\varphi_a, \tag{3a}$$

$$\Box \varphi = -\frac{1}{6}x^a\nabla_a R - \frac{1}{3}R\varphi, \tag{3b}$$

where R denotes the scalar curvature.

Statement of proof: Use Ricci identities for x^a and the CKE to establish

$$\nabla_c\nabla_a x_b = R_{bacd}x^d + g_{ab}\varphi_c + g_{cb}\varphi_a - g_{ca}\varphi_b, \tag{4}$$

Then, (3a) follows from (4), and (3b) follows from the definition of φ.

Alternative characterization for gradient CKV . . . 2/7

Next,

Lemma

If a spacetime $\mathcal{M}$ admits a gradient CKV x^a, then, it holds that

$$R_{ab}x^b = -3\varphi_a. \tag{5}$$

Proof.

If $\mathcal{M}$ admits a gradient CKV x^a, $\nabla_a x_b = \varphi g_{ab}$. Thus,

$$\Box x_a = \varphi_a, \tag{6}$$

which upon inserting into (3a) gives (5). □

Alternative characterization for gradient CKV . . . 3/7

- We expand out (5):

$$g_1\dot{\Psi} + Q\hat{\Psi} = 3\dot{\varphi}, \tag{7}$$

$$-\left[Q\dot{\Psi} + (g_2 + \Pi)\,\hat{\Psi}\right] = 3\hat{\varphi}. \tag{8}$$

- We make a few observations from the above:
  - If $g_1 = \rho + 3p - 2\Lambda > 0 \neq 0$, x^a must be proper for otherwise, x^a is trivial.
  - Any gradient KV in a spacetime strictly obeying the weak and strong energy conditions (WEC and SEC), respectively $g_1 + g_2 = \rho + p > 0$ and $g_1 > 0$, must lie along the timelike congruence.

- Suppose x^a is a Ricci principal direction, and denote by η the eigenvalue associated to x^a. We analyze this as the two distinguished cases $Q = 0$ and $Q \neq 0$.

Alternative characterization for gradient CKV . . . 4/7

- Begin with $Q = 0$. We reduce the above system to

$$(g_1 + \eta)\,\dot{\Psi} = 0, \tag{9}$$

$$(g_2 + \Pi - \eta)\,\hat{\Psi} = 0. \tag{10}$$

- Timelike case: $\dot{\Psi} \neq 0$. Thus, $\eta = -g_1$, and the strict SEC ensures $\eta < 0$.

Proposition

Any timelike gradient CKV of a perfect fluid LRS spacetime with nonvanishing pressure and a non-negative cosmological constant, and obeying the SEC ($g_1 \geq 0$), must lie along u^a.

Proof.

Assume $\hat{\Psi} \neq 0$. It follows that $g_1 + g_2 = 0 \implies g_1 < 0$, thereby failing the SEC. □

Alternative characterization for gradient CKV . . . 5/7

- Of course, as $\hat{\Psi} = 0$, $\mathcal{A}$ must vanish by the vanishing of $\mathcal{F}_{ab}$. Furthermore, it is easily checked that the spacetime is shear-free. Then, from the field equations this imposes that the spacetime is necessarily conformally flat, i.e. $\mathcal{E} = 0$. It therefore follows that

Theorem

The only GCS perfect fluid LRS II spacetime with a nonvanishing pressure and non-negative cosmological constant is the Robertson-Walker type solution.

- In the $Q \neq 0$ case, one can also comment on the bound on the eigenvalue η:

$$Q^2 = (g_1 + \eta)(g_2 + \Pi - \eta) \implies \eta < \frac{g_1 (g_2 + \Pi)}{g_1 - g_2 - \Pi}.$$

Alternative characterization for gradient CKV . . . 6/7

- Returning to the original system, we can write down the integrability condition:

$$2Q\ddot{\Psi} + (\rho + p + \Pi)\,\hat{\dot{\Psi}} + F_1\dot{\Psi} + F_2\hat{\Psi} = 0, \tag{11}$$

where we have defined

$$F_1 = \hat{g}_1 + \mathcal{A}\,(\rho + p + \Pi) + \left[\dot{Q} + \left(\frac{2}{3}\theta - \sigma\right)Q\right],$$
$$F_2 = (g_2 + \Pi)^{\cdot} + \left(\hat{Q} + \phi Q\right).$$

- If we restrict to the perfect fluid case, assuming the SEC, it is easily verified that (11) is satisfied.

Alternative characterization for gradient CKV . . . 7/7

- In fact, in general, one may use the vanishing of $\mathcal{F}_{ab}$ and the CKE to bring (11) into the form

$$2Q\ddot{\Psi} + F_1\dot{\Psi} + \bar{F}_2\hat{\Psi} = 0, \tag{12}$$

with

$$\bar{F}_2 = F_2 + (\rho + p + \Pi)\left(\frac{1}{3}\theta + \sigma\right).$$

- For example, if one considers a u^a-directed x^a, i.e. $\hat{\Psi} = 0$ for this radiating spacetime, one has

$$2Q\ddot{\Psi} + F_1\dot{\Psi} = 0. \tag{13}$$

The above admits a solution if F_1 and Q vanish simultaneously, imposing

$$\hat{g}_1 + \mathcal{A}(\rho + p + \Pi) = 0. \tag{14}$$

Hypersurface geometry . . . 1/2

- We have a foliation $\mathcal{W}_x$ by spacelike hypersurfaces, to which the CKV x^a is orthogonal.

- The unit normal to the leaves is the normalized timelike gradient CKV $\tilde{x}^a = fx^a$ (with $f = 1/\sqrt{-x_a x^a}$). On such a spacelike hypersurface, denote $\mathcal{T}$, is induced the Riemannian metric and projected covariant derivative

$$z_{ab} = g_{ab} + \tilde{x}_a \tilde{x}_b, \qquad \mathcal{D}_a = z^b{}_a \nabla_b.$$

- It can be checked that both f and φ are constant on $\mathcal{T}$. We then have the second fundamental form on $\mathcal{T}$ as

$$\bar{\chi}_{ab} = \frac{1}{2}\bar{\mathcal{L}}_{\tilde{x}} z_{ab} = f\varphi z_{ab}.$$

Hypersurface geometry . . . 1/2

- $\implies \mathcal{T}$ is totally umbilical. This gives the mean curvature as

$$\bar{\chi} = -\frac{1}{3} z^{ab} \chi_{ab} = -f\varphi, \tag{15}$$

which is constant on $\mathcal{T}$, i.e. $\mathcal{T}$ is a CMC hypersurface.

- Also, $\mathcal{T}$ is maximal ($\bar{\chi} = 0$), i.e. totally geodesic, if and only if x^a is a true KV.

- The constancy of the mean curvature essentially constrains the character of the CKV. We now obtain this CMC criterion.

The CMC condition . . . 1/5

- For an arbitrary scalar ψ in an LRS solution, the $\mathcal{T}$-gradient

$$\begin{aligned}\mathcal{D}_a\psi &= \nabla_a\psi + f^2\left(x^b\nabla_b\psi\right)x_a\\ &= \left[\left(f^2\alpha_1^2-1\right)\dot{\psi}+f^2\alpha_1\alpha_2\hat{\psi}\right]u_a\\ &+\left[\left(f^2\alpha_2^2+1\right)\hat{\psi}+f^2\alpha_1\alpha_2\dot{\psi}\right]n_a.\end{aligned}$$

- Then, we can write down the constancy of ψ as the vanishing condition:

$$\alpha_2\dot{\psi}+\alpha_1\hat{\psi}=0.$$

- That is, exactly one of the following is true on $\mathcal{T}$:

1) $\alpha_2=\hat{\psi}=0$ & $\dot{\psi}\neq 0$; 2) $\hat{\psi}=\dot{\psi}=0$; or 3) $\hat{\psi}\neq 0$ & $\dot{\psi}\neq 0$.

The CMC condition . . . 2/5

We examine the constancy conditions for the mean curvature $\bar{\chi}$. We begin by stating the following

Lemma

Let x^a be a timelike gradient CKV in a LRS spacetime $\mathcal{M}$, and let $\mathcal{T}$ be a hypersurface in $\mathcal{M}$ to which integral curves of x^a are orthogonal. Then, the evolution and propagation equations of the mean curvature $\bar{\chi}$ of $\mathcal{T}$ are

$$\dot{\bar{\chi}} = \frac{\left[2\alpha_1\varphi^2 - \left(\alpha_1^2 - \alpha_2^2\right)\left(\alpha_1 X_1 + \alpha_2 X_2\right)\right]}{2\left(\alpha_1^2 - \alpha_2^2\right)^{3/2}}, \tag{16a}$$

$$\hat{\bar{\chi}} = -\frac{\left[2\alpha_2\varphi^2 + \left(\alpha_1^2 - \alpha_2^2\right)\left(\alpha_1 X_2 + \alpha_2 X_3\right)\right]}{2\left(\alpha_1^2 - \alpha_2^2\right)^{3/2}}, \tag{16b}$$

where we have defined

$$X_1 = -\frac{2}{3}g_1 + \mathcal{E} - \frac{1}{2}\Pi, \quad X_2 = Q, \quad X_3 = 2\xi^2 - \frac{2}{3}\rho - \mathcal{E} - \frac{1}{2}\Pi.$$

The CMC condition . . . 3/5

- $\alpha_2 = \hat{\chi} = 0; \dot{\chi} \neq 0$: In this case, we know that $\hat{\chi} = -X_2/2 = 0$, and since the spacetime is necessarily shear-free. The CMC condition reduces to the pair

$$X_1 \neq \frac{2}{9}\theta^2 \iff \frac{2}{3}\dot{\theta} \neq \mathcal{A}\phi., \qquad X_2 = 0.$$

In the irrotational and twisting LRS case, this is the only applicable CMC condition. (Note: $\mathcal{A}$ is necessarily zero $\implies$ θ is nonconstant.)

- $\hat{\chi} = \dot{\chi} = 0$: In this case, it can be shown that the CMC condition reduces to the pair of constraints

$$X_3 + X_1 = -(\rho + p + \Pi) = 0, \qquad X_2 = 0.$$

The CMC condition . . . 4/5

- $\dot{\hat{\chi}} \neq 0; \dot{\bar{\chi}} \neq 0$: For this case, is can be shown that the CMC condition reduces to the pair of constraints

$$X_3 - X_1 = -\frac{1}{3}R - 2\mathcal{E} = 0, \qquad X_2 = 0.$$

We will say a bit more here as we have the following:

$$\frac{\dot{\bar{\chi}}}{\dot{\hat{\chi}}} = -\frac{\alpha_1}{\alpha_2} \implies \frac{\dot{\bar{\chi}}^2}{\dot{\hat{\chi}}^2} > 1$$

$$\varphi^2\left(\alpha_1^2 - \alpha_2^2\right)\left[\varphi^2 - \left(\alpha_1^2 - \alpha_2^2\right)X_1\right] > 0 \iff \varphi^2 - \left(\alpha_1^2 - \alpha_2^2\right)X_1 > 0.$$

Indeed, the condition $X_1 < 0$ is sufficient for the above inequality to hold. In fact, this condition is quite reasonable on most physical grounds, and a strictly positive scalar curvature R: If the electric Weyl scalar $\mathcal{E} < 0$, and one then insists that the pressure is non-negative, $p \geq 0$, it is easily checked that this will ensure that the SEC holds, a physically reasonable condition. It then follows that $X_1 < 0$.

The CMC condition . . . 5/5

- Set $Q = 0$ and assume SEC. Further, suppose $\alpha_i \neq 0, \dot{\chi} \neq 0$, and $\hat{\dot{\chi}} \neq 0$. Then, one has the following ratio

$$\hat{\chi}/\dot{\chi} = -g_1/(g_2 + \Pi). \quad (17)$$

  Consider a perfect fluid. If $p = 0$, $g_1 = g_2$. Thus, $\alpha_1 = \alpha_2$, i.e. x^a is null. It follows **an inhomogeneous pressureless LRS perfect fluid cannot be GCS**. An immediate consequence of the above proposition is that *Lemaitre-Tolman-Bondi solution is not GCS*.

- Stay with the perfect fluid case, but $p \neq 0$. Applying the timelike criterion to (17), if the weak energy condition is imposed, necessarily $p < 0$: $g_1^2 - g_2^2 = 2p\,(\rho + p) < 0$. This is, **an inhomogeneous LRS perfect fluid which obeys the weak energy condition is not GCS**.
- $\implies$ *The only GCS LRS perfect fluid obeying standard energy conditions is the Robertson-Walker type solution.*

MOTS geometry and dynamics . . . 1/5

- Consider a 2-surface $\mathcal{S}$ embedded in a 4-dimensional spacetime $(\mathcal{M}, g_{ab})$, and we assume that the surface $\mathcal{S}$ has a well defined notion of an 'inside' and an 'outside'.

- If we notate the pullback (induced) metric to $\mathcal{S}$ by F_{ab}, the compatible covariant derivative, $\tilde{\mathcal{D}}_a$, for F_{ab}, is just the projection of the full covariant derivative ∇_a onto $\mathcal{S}$: $\tilde{\mathcal{D}}_a = F_a{}^b \nabla_b$.

- The normal space of $\mathcal{S}$ is spanned by a pair of null normal vector fields, which we denote by k^a and l^a, and for our purpose we normalize here to $k_a l^a = -1$. These vector fields are respectively the unit tangents to null rays leaving and entering $\mathcal{S}$. The surface metric $\mathcal{F}_{ab}$ can then be decomposed as

$$\mathcal{F}_{ab} = g_{ab} + k_{(a} l_{b)}. \tag{18}$$

MOTS geometry and dynamics . . . 2/5

- The particular frame choice that defines k^a and l^a depends on how the spacetime is sliced. This problem is related to the (non)-uniqueness of dynamical horizons which was investigated in detail by (**Ashtekar & Galloway**, **Adv. Theor. Math. Phys.**, 2005).

- For our purpose, we consider a constant t and r surface $\mathcal{S}$, so that F^{ab} is just $N^{ab} = g^{ab} + u^a u^b - n^a n^b$. Thus, we choose the gauge

$$k^a = u^a + n^a, \quad l^a = \frac{1}{2}\left(u^a - n^a\right).$$

- The expansion of k^a is

$$\chi = \frac{2}{3}\theta - \sigma + \phi.$$

MOTS geometry and dynamics . . . 3/5

- A *marginally outer trapped surface* (MOTS) is that on which, at all points, $\chi = 0$. A 3-dimensional hypersurfaces foliated by MOTS is called a *marginally outer trapped tube* (MOTT).

- The geometry and dynamics of MOTS in LRS spacetimes have been considered in several recent works (**Sherif** *et al.*, **CQG**, 2019, **Sherif**, **EPJ C**, 2021, **Sherif & Dunsby**, **CQG**, 2023).

- On a MOTT $\bar{H}$, one can always find a (constant) function c such that the vector field (**Hayward**, **PRD**, 1994, **Booth & Fairhurst**, **PRL**, 2004, **Booth**, **Can. J. Phys.**, 2005)

$$y^a = k^a - cl^a, \tag{19}$$

is tangent to $\bar{H}$. (Variation of $\mathcal{S}$ along y^a induces the foliation.)

MOTS geometry and dynamics . . . 4/5

- Since $y_a y^a = 2c$, the causal character of $\bar{H}$ at a point is determined by the sign of c: spacelike for $c > 0$, timelike for $c < 0$, and null for $c = 0$. Note that the expansion χ is Lie dragged along y^a, to find

$$c = (\mathcal{L}_k \chi) / (\mathcal{L}_l \chi) \tag{20}$$

- c is constant over $\bar{H}$, so we call the MOTT a 'horizon'. $\mathcal{S}$ evolves into a spacelike horizon (dynamical horizon (DH)), timelike horizon (timelike membrane (TLM)), or null horizon (isolated horizon (IH)) if c is positive, negative, or zero, respectively.

- NEC $\implies \mathcal{L}_k \chi \leq 0 \implies$ characterization of a MOTS is captured in the sign of $\mathcal{L}_l \chi$: If the NEC strictly holds, the horizon is a DH for $\mathcal{L}_l \chi < 0$ and a TLM for $\mathcal{L}_l \chi > 0$. The horizon is an IH iff $\mathcal{L}_k \chi = 0$.

MOTS geometry and dynamics . . . 5/5

- The above is related to the notion of the *stability* of MOTS. This was introduced by (**Andersson** *et al.*, **PRL**, 2005). Roughly put, a strictly stable MOTS in an initial data set, under the assumption of the DEC, will evolve into a horizon containing trapped surfaces just to the 'inside' (DH or IH).

- Containing trapped surfaces just to the inside of a horizon is therefore akin to the condition that $\mathcal{L}_l \chi < 0$, and hence, assuming NEC, only DH and IH will bound black holes.

- It was observed in (**Sherif & Dunsby**, **CQG**, 2022) that while the positivity of c is necessary for the stability of a MOTS in DH, it is not sufficient. It is further required that the bound is obeyed:

$$0 < c < 1.$$

Results . . . 1/6

- The scalars X_j, whose combination characterize the CMC condition, is related to the evolutions of the null expansion χ along the null congruences.

- Because these evolutions characterize the dynamics of the MOTS, it follows that the CMC condition provides a characterization of the MOTS dynamics as well.

- We demarcate our cases according to the character of Q.

- We start with the $Q = 0$ (we consider $\Pi \neq 0$). It follows that

$$\mathcal{L}_k \chi = X_3 + X_1; \qquad \mathcal{L}_l \chi = X_1 - X_3.$$

Results . . . 2/6

Proposition

Suppose a LRS II spacetime $\mathcal{M}$ admits a timelike gradient (proper) CKV x^a and a leaf $\mathcal{T}$ of $\mathcal{W}_x$ contains a marginally outer trapped 2-sphere $\mathcal{S}$. If the mean curvature of $\mathcal{T}$ is constant along u^a and $X_1 < X_3$ on $\mathcal{S}$, then, the spacetime contains a black hole with a null boundary.

Statement of proof:

- $X_1 < X_3$ ensures $\mathcal{L}_l \chi < 0$. (Note that the third criterion is ruled out.)
- $\dot{\chi} = 0$ also rules out the first criterion.
- Hence, the MOTS evolves into a null horizon with trapped surfaces just to the inside.

Results . . . 3/6

Proposition

Suppose a LRS II spacetime $\mathcal{M}$ admits a timelike gradient (proper) CKV x^a, and a leaf $\mathcal{T}$ of $\mathcal{W}_x$ has mean curvature that is non-constant along n^a. Then, a horizon $\bar{H}$ in $\mathcal{M}$ which intersects $\mathcal{T}$ at a marginally trapped 2-sphere $\mathcal{S}$ cannot enclose a black hole.

Statement of proof:

- Imposing χ is non-constant along n^a says only the third criterion is possible.
- And since the hypersurfaces $\mathcal{T}$ and $\bar{H}$ intersect at $\mathcal{S}$, $\mathcal{L}_l\chi$ vanishes along $\bar{H}$.

Corollary

Let a LRS II spacetime $\mathcal{M}$ admits a gradient (proper) CKV x^a. Then, a dynamical horizon in M cannot intersect a leaf of $\mathcal{W}_x$.

Results . . . 4/6

- The above corollary says that an observer with conformal motion does not 'observe' a dynamical black hole changing in area. The corollary is related to Propositions 6.2 and 6.3 of (**Ashtekar & Galloway**, **Adv. Theor. Math. Phys.**, 2005), except that that was the Killing case.

- The behavior of the mean curvature χ of the spacelike hypersurfaces along the spatial vector n^a plays a crucial role in determining the existence of a black hole enclosing horizon.
  - $\hat{\dot{\chi}} = 0$: If a leaf of $\mathcal{W}_x$ contains a MOTS, the MOTS will evolve to a null horizon. Trapped surfaces will be enclosed provided that the scalar and conformal Weyl curvatures satisfy $R < -6\mathcal{E}$.
  - $\hat{\dot{\chi}} \neq 0$: If a horizon intersects a leaf of $\mathcal{W}_x$ it does not enclose trapped surfaces since a leaf of $\mathcal{W}_x$ cannot contain a stable MOTS. Otherwise, if there is no intersection with $\mathcal{W}_x$, independent of the development of the associated black hole, the black hole will be seen as being in equilibrum by conformal observers.

Results . . . 5/6

Let us briefly comment on the case $Q \neq 0$.

- When discussing the previous results of this section, we had simply consider the case with a vanishing heat flux to entertain intersections of a black hole horizon with a leaf of $\mathcal{W}_x$.
- However, Q will generally not be zero. It is then clear that the character of the relationship between X_j and the evolution of the null expansions becomes a bit more intricate. More specifically, there is a modification

$$\mathcal{L}_k\theta_k = X_3 + X_1 + 2Q. \tag{21}$$

Clearly, if $X_3 + X_1 = 0$ (simple & tractable), then, the NEC demands that $Q \leq 0$, with the equality case already covered in the previous subsection.

Results . . 6/6

- Thus, we take $Q < 0$. That is, such a horizon will experience an outward flux. Unless there is flow of some other matter across the horizon from the 'outside', the horizon area must be decreasing and hence a timelike membrane. Of course, depending on the dynamics of the interacting fields of the spacetime to the exterior of the black hole, different horizon characters are possible.

- In any case, horizons considered here will not intersect $\mathcal{W}_x$, and conformal observers will again only see a black hole in equilibrum since there is a non-zero flux. So, for example, for the timelike scenario described, the conformal observers would not observe the collapse of such a black hole.

Summary of results . . . 1/2

- We have studied gradient conformal stationarity of LRS spacetimes and the character of MOTS in these spacetimes.
- We obtained a relationship between the Ricci tensor components along the CKV, and the gradient of its divergence, providing an alternative set of equations, for which the integrability condition is obtain, to analyze the existence of gradient CKV.
- A uniqueness result is obtained in the case of perfect fluids establishing that *The only GCS perfect fluid LRS II spacetime with a nonvanishing pressure and non-negative cosmological constant is the Robertson-Walker type solution.* This result is later strengthened to remove conditions on the pressure.
- Consequences of the alternative characterization were discussed.

Summary of results . . . 2/2

- The constant mean curvature condition(s) is also obtained. This is characterized by three distinct conditions which are specified by a set of three scalars.

- The combinations of the scalars whose vanishing define the constant mean curvature condition, turn out to be related to the evolutions of null expansions of 2-sphere along null congruences. As such, some implications for the presence of MOTS and their evolution are obtained. Crudely put, *dynamical horizons cannot exist in the timelike region of a GCKV.*

THANK YOU!