The spectral reconstruction problem for thermal dilepton and photon production rates from lattice QCD

Anthony Francis

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陽明交大物理研究所
NYCU Institute of Physics

partly based on: [2403.11647]

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The inverse problem challenge - a mismatch in information

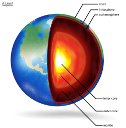
At this conference many talks addressing different physics, but facing a similar core issue: An inverse problem.

In fact, inverse problems are common across science, in general:

- A mismatch between available and desired information
- Need of a robust map / transformation from one into the other
- Inverse problem: This transformation is (numerically) ill-posed or ill-conditioned

Examples outside and inside of (thermal) particle physics:

available information	desired information
seismic waves	geological tomogram
electromagnetic waves	medical images
Thermal meson correlation functions	dilepton/photon production rate
	electrical conductivity

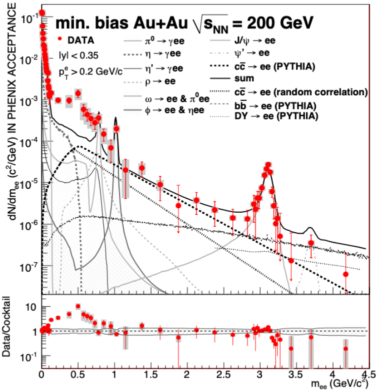



Electromagnetic source imaging: Backus–Gilbert resolution spread function-constrained and functional MRI-guided spatial filtering
Xiaohong Wan,[1] Atsushi Sekiguchi,[2,3] Satoru Yokoyama,[3,4] Jorge Riera,[3,5] and Ryuta Kawashima[3,5]

Dilepton and photon production rates - identifying the problem

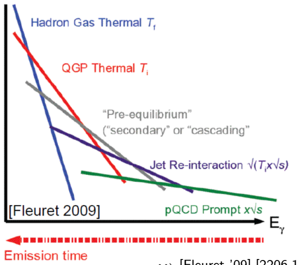
Probes of the quark gluon plasma

- Dileptons and photons are produced at all stages of Heavy-Ion Collisions.
- Weak coupling to QGP constituents
  $\rightarrow$ they decouple after production
  $\rightarrow$ probes of the full thermal medium evolution



[PHENIX '10]

Sketch of different photon sources



$\rightsquigarrow$ [Fleuret '09],[2206.1467]

Thermal QCD goals

- thermal dileptons: understand contribution for $m_{ee} \sim 0.5\,\mathrm{GeV}$
- thermal photons: understand dominant contribution for $p_T \in [1:2]\,\mathrm{GeV}$

Dilepton and photon production rates - identifying the problem

The photon emissivity is related to thermal vector-vector current spectral functions

$$\rho^{\mu\nu}(\mathcal{K}) = \int d^4x e^{i\mathcal{K}\cdot x} \frac{1}{Z} \sum_n e^{-E_n/T} \langle n | [j^\mu(x), j^\nu(0)] | n \rangle$$

- Rate of dilepton production per unit volume plasma: ⇝[McLerran, Toimela '85]

$$d\Gamma_{\ell^+\ell^-}(\mathcal{K}) = \alpha^2 \frac{d^4\mathcal{K}}{6\pi^3\mathcal{K}^2} \frac{-\rho^\mu{}_\mu(\mathcal{K})}{e^{\beta\mathcal{K}^0} - 1} \qquad (\mathcal{K}^2 \equiv \omega^2 - k^2)$$

- Rate of photon production per unit volume plasma:

$$d\Gamma_\gamma(\boldsymbol{k}) = \alpha \frac{d^3k}{4\pi^2 k} \frac{-\rho^\mu{}_\mu(k, \boldsymbol{k})}{e^{\beta k} - 1}.$$

- Electrical conductivity of the quark gluon plasma: ⇝see e.g. [1104.3708]

$$\sigma_{el} = e^2 \sum_{f=1}^{N_f} Q_f^2 \lim_{k\to 0^+} \frac{\rho^i{}_i(k, 0)}{k}$$

⇝from Meyer, Lattice@CERN '24

Dilepton and photon production rates - identifying the problem

We are interested in three properties of the vector spectral function:

- conductivity: $\sigma_{el} \sim \rho^{i}{}_{i}(0,0)$ — note: $\rho^{0}{}_{0}(0,0) = const. =: \chi_q$
- dilepton rate: $d\Gamma_{\ell^+\ell^-}(\mathcal{K}) \sim -\rho^{\mu}{}_{\mu}(\mathcal{K}) \rightsquigarrow \rho^{\mu}{}_{\mu}(\omega, \boldsymbol{k}=0)$
- photon rate: $d\Gamma_{\gamma}(\boldsymbol{k}) \sim -\rho^{\mu}{}_{\mu}(k, \vec{k}) \rightsquigarrow \rho^{\mu}{}_{\mu}(\omega = k, \boldsymbol{k} \neq 0)$

The most interesting are in the low-energy, non-perturbative regime of QCD.

$\rightsquigarrow$ Where can we trust perturbative calculations?

$\rightsquigarrow$ Access through lattice QCD?

On the lattice and at $T > 0$:

- Based on imaginary-time path-integral rep. of QFT (Matsubara formalism).
- Only **imaginary-time** vector correlators are accessible ($j^\mu = \sum_f Q_f \bar{\psi}_f \gamma^\mu \psi_f$):

$$G^{\mu\nu}(x_0, \boldsymbol{k}) = \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle j^\mu(x) j^\nu(0)\rangle_T = \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{Tr}\left\{ \frac{e^{-\beta H}}{Z(\beta)} j^\mu(x) j^\nu(0) \right\},$$

- Their spectral representation is: $\rightsquigarrow$ inverse problem(!)

$$G^{\mu\nu}(x_0, \boldsymbol{k}) = \int_0^\infty \frac{d\omega}{2\pi} \underbrace{\rho^{\mu\nu}(\omega, \boldsymbol{k})} \frac{\cosh\left[\omega\left(\beta/2 - x_0\right)\right]}{\sinh(\beta\omega/2)} \quad , \ \beta = 1/T.$$

Nature of the inverse problem - setting expectations

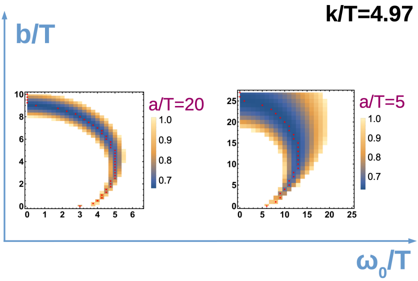
Jacques Hadamard established three conditions for a well-posed problem

1. Existence
2. Uniqueness
3. Stability (solution changes continuously with the initial conditions)

The problems we consider fail in the sense of 3. This is due to **discrete sampling plus finite precision** and a method-independent statement.

An illustrative example: Extract $\rho(\omega)$ by motivated photon production rate fit-Ansatz with 3 parameters a, b, ω_0, and minimize:

$$\chi_{fit} = \left[G(\tau, p) - \int d\omega \; \rho_{fit}\,(\omega, a, b, \omega_0)\, K(\omega, \tau, \beta) \right]$$



- no clear global minimum visible
- each point represents one "acceptable" spf solution describing the Euclidean data.
- brute force accuracy increase to arrive at a more constrained result realistic?

Figure from [1710.07050]

Anthony Francis, afrancis@nycu.edu.tw

Approaching the inverse problem - three basic strategies

Strategy I: Accept the premise and focus on information in the data

Challenges

- Optimal bases for sparse modeling / correlation structures for Gaussian processes?
- Constraints (positivity, sum rules)?

Methods - (non)linear

- Sparse modeling
- Neural Networks
- Gaussian processes

Strategy II: Accept the premise and try to supply as much extra information as possible

Challenges

- Effectively encode more specific information in priors/ Ansätze?
- Control over prior bias and systematics?

Methods - (non)linear

- χ^2-fits
- Maximum entropy methods
- Stochastic inference / optimization

Strategy III: Reject the premise and focus on smeared spf's or Euclidean quantities

Challenges

- Identify the physics observables that ...
  - can benefit from smeared spf input.
  - simplify / avoid inverse problems altogether.

Methods - linear

- Backus-Gilbert / Hansen-Lupo-Tantalo methods
- Gaussian processes

Each method has its own problem-dependent pros and cons. Unlikely there is a single best solution.

Results

Roadmap:

1. **dilepton rates / electrical conductivity:** *brief overview of community status*
2. **photon rates:** *new research shown [2403.11647]*

Our lattice setup (HotQCD ensembles)

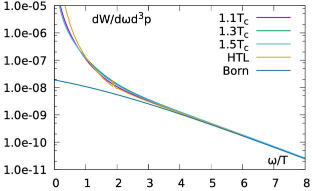
- $n_f = 0$ quenched QCD, $a^{-1} = 9.4, 11.3, 14.1\,\text{GeV}$, Wilson-Clover
- $n_f = 2 + 1$ full QCD, $a^{-1} = 7.04\,\text{GeV}$, Wilson-Clover on HISQ sea.
- $m_\pi = 320\,\text{MeV}$, $m_s = 5 \cdot m_\ell$

Dilepton rate and electrical conductivity

- Dilepton rates available in quenched and full QCD from multiple groups (last update 2019).
- Methods used:
  - χ^2-fits with additional constraints
  - MEM
  - BG method with Tikhonov regulator
- Fits: Transport (+ BW) + Asymptotic
  $\Rightarrow$ Data well described by Ansatz
  $\Rightarrow$ Difficult to distinguish Transport and BW
- MEM prior: Positivity + default model

Consistent picture emerging.

Dilepton rate in quenched QCD



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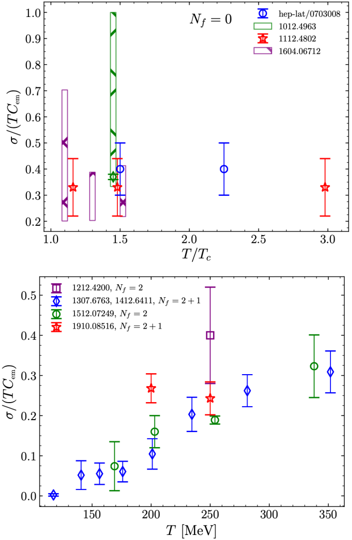
Electrical conductivity

[2008.12326]



Photon production rate from the lattice - a better estimator

⇝ see [Cè et al '20], [Ali, AF et al '24], [Meyer, Lattice@CERN '24]

Back to:

$$\rho^{\mu\nu}(\mathcal{K}) = \int d^4x e^{i\mathcal{K}\cdot x} \frac{1}{Z} \sum_n e^{-E_n/T} \langle n | [j^\mu(x), j^\nu(0)] | n \rangle$$

at $T > 0$, there are two independent components (longitudinal and transverse):

$$\rho_L(\omega, k) \equiv \left(\hat{k}^i \hat{k}^j \rho^{ij} - \rho^{00} \right), \quad \text{and} \quad \rho_T(\omega, k) \equiv \frac{1}{2} \left(\delta^{ij} - \hat{k}^i \hat{k}^j \right) \rho^{ij} \ ,$$

where $\left(k \equiv |\boldsymbol{k}|, \quad \hat{k}^i = k^i / k \right)$

- Due to current conservation: $\omega^2 \rho^{00}(\omega, k) = k^i k^j \rho^{ij}(\omega, k)$
- ρ_L vanishes at light like kinematics, $\mathcal{K}^2 = 0$

→ We can rewrite the photon rate when introducing the estimator:

$$\rho(\omega, k, \lambda) = 2\rho_T + \lambda \rho_L \quad \overset{\lambda=1}{=} -\rho^\mu{}_\mu$$

→ For any value of λ we have

$$\boxed{d\Gamma_\gamma(\boldsymbol{k}) = \alpha \frac{d^3k}{4\pi^2 k} \frac{\rho(k, k, \lambda)}{e^{\beta k} - 1}}$$

→ Choose λ such that the reconstruction becomes particularly easy.

Lattice photon production rate - estimator and perturbative result

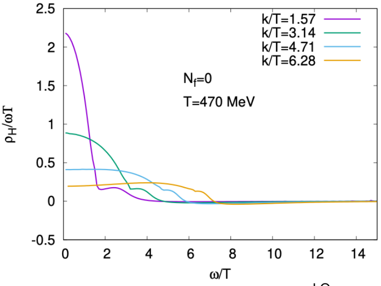
⇝ [Ali, AF et al '24]

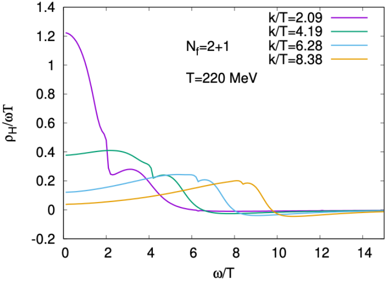
We choose $\lambda = 2$ such that the spf we want to reconstruct becomes:

$$\rho_H(\omega, \vec{k}) = 2\left\{\rho_T(\omega, \vec{k}) - \rho_L(\omega, \vec{k})\right\}$$

- → At $T = 0$ this estimator is $\rho_H(\omega, \vec{k}) = 0$. (due to restoration of Lorentz symmetry)
- → Purely thermal effects contribute.
- → Aysmptotically $\rho_H(\omega, \vec{k}) \sim 1/\omega^4$ for $\omega \gg k$ (πT) .
- → Sum rule $\int_0^\infty \mathrm{d}\omega\omega\rho_H(\omega) = 0$ is a possible extra constraint.

Can be worked out in perturbation theory at NLO + LPM^{LO}. ⇝ [1910.09567]



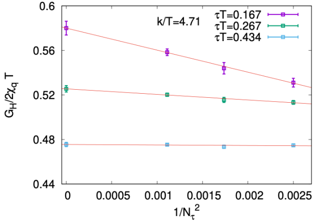


LPM ^{LO} means that near the light cone LPM resummation is performed at leading order.

Euclidean correlators - NLO+LPM vs. lattice data

Quenched results

- Continuum limit with $N_\tau = 20, 24, 30$
- non-pert. tuned WCF



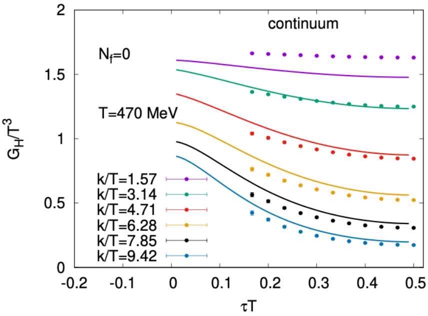
- General behavior reproduced, apart from lowest k/T

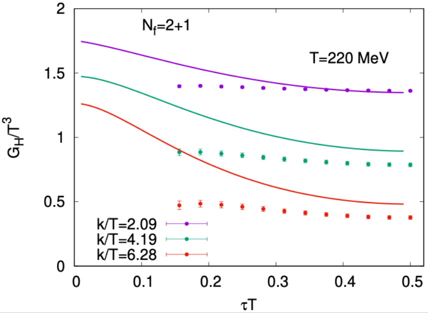
Full QCD results

→ Single lattice spacing

→ Differences more visible

→ Improvement after continuum limit?





Spectral reconstruction - a 3-pronged approach

Strategy

We employ all three basic strategies:

- χ^2-fits (strategy II)
- Backus-Gilbert method (strategy III)
- Gaussian processes (strategy I)

This makes visible and enables the study of:

⇒ Different systematics in all approaches.

⇒ Maximal view of possible outcomes.

⇒ Aim for robust, conservative final result.

Gaussian process regression

- Gaussian kernel (related to NN)
- Simultaneous reconstruction in $(\omega,\ k)$
- Continuity only constraint

→ Use mildest possible constraints

χ^2-fits

Two sets of Ansätze

(a) Polynomial fit to reproduce IR and UV results (OPE)

(b) Padé fit with sum rule incorporated (OPE and AdS/CFT)

→ Include as much extra info as possible

BGM

We stabilize/improve the reconstruction by rescaling the spf by asymptotic behavior

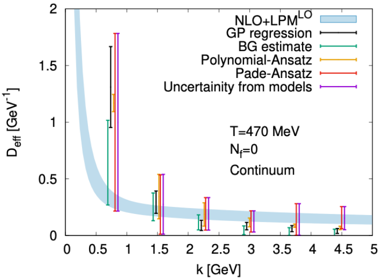
$$\frac{\rho_H^{\mathrm{BG}}(\omega,\vec{k})}{f(\omega,\vec{k})} = \sum_i q_i(\omega,\vec{k}) G_H\left(\tau_i,\vec{k}\right)$$

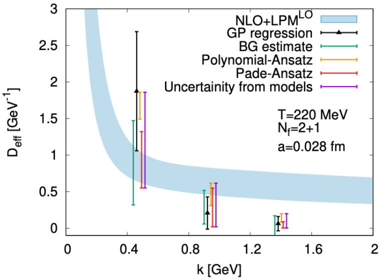
Rescaling function:

$$f(\omega,\vec{k}) = \left(\frac{\omega_0}{\omega}\right)^4 \tanh\left(\frac{\omega}{\omega_0}\right)^5$$

→ Work with a smeared spf

Photon rate from the lattice - final results





Cè et al [2205.02821]

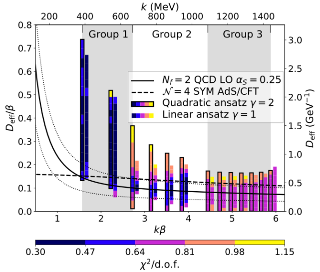
Plotted:

$$D_{\text{eff}}(k) \equiv \frac{\rho_H(\omega = k, k)}{2\chi_q k}$$

Connection to full photon rate:

$$\frac{d\Gamma_\gamma(\boldsymbol{k})}{d^3k} = \frac{\alpha}{4\pi^2 k}\frac{\rho(k, k, \lambda = 2)}{e^{\beta k} - 1}$$

$$= \frac{\alpha n_b(k)\chi_q}{\pi^2}\left(\sum_{i=1}^{N_f} Q_i^2\right) D_{\text{eff}}(k)$$



compatible results, see further material

Summary - inverse problem for thermal spfs from the lattice

I. Inverse problem for thermal spectral functions

- Presented in general terms the inverse problem to obtain (thermal) spectral functions from lattice correlators
- Broadly highlighted applications and connection to dilepton / photon rates
- Gave general approach strategies and their challenges

II. Dilepton rates and electrical conductivity

- Collected a brief overview (last update 2019). Consistent picture emerging.

III. Photon rate

- Showed the improved estimator for photon rate
- Applied all basic strategies to inverse problem to arrive at new result

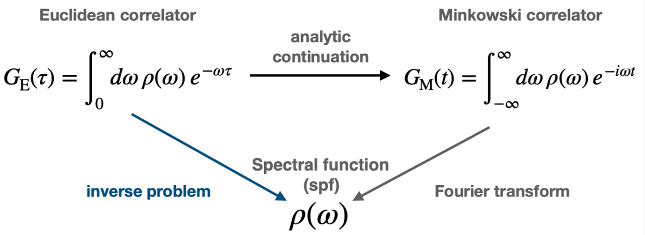
IV. Future questions

- Finding improved estimator has made the study of the photon rate (and heavy quark diffusion) tractable. *Are there more observables like this?*
- In the $T = 0$ community there is a lot of work on using smeared spf's and clarifying the role of the finite volume. *Possibilities also at $T > 0$?*
- Gaussian processes related to NN. Connections between BG-type methods worked out. *New methods that combine the best of all strategies possible?*

Thank you for your attention.

Further material

Inverse problem vs. analytic continuation - a net of connections



From the Euclidean correlator calculated in lattice QCD we want to extract the spf
⇒ Problem can be seen as a **simultaneous Wick rotation and Fourier transform**.

Also, formally:

$$\rho(\omega) = \mathcal{L}^{-1}\{G_E(\tau)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\omega\tau} G_E(\tau) d\tau \quad \text{and} \quad \rho(\omega) = \frac{1}{\pi}\text{Im}\,(G_M(-\omega))$$

,
⇒ Problem is related to having only real data where complex information is required.

The role of the kernel - different physics, different problems

The specific problem sets the kernel entering the inverse problem

$$G_E(\tau) = \int_0^\infty d\omega\, \rho(\omega)\, \kappa(\omega, \tau)$$

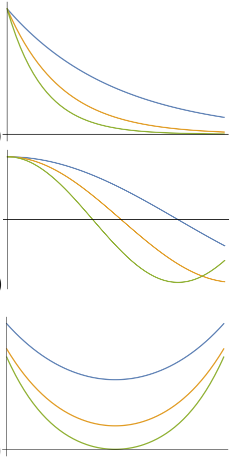
(a) Zero-temperature quantities: $\kappa(\omega, \tau) = e^{-\omega\tau}$
$\rightsquigarrow$ Need to perform the inverse Laplace transform.

(b) qPDFs: $\kappa(\nu, x) = \cos(\nu x)\, \Theta(1 - x)$
$\rightsquigarrow$ Need to perform Fourier transform.

(c) Nonzero-temperature: $\kappa(\omega, \tau) = \frac{\cosh(\omega(\beta/2-\tau))}{\sinh(\omega\beta/2)}$

Focusing on the $T > 0$ case and target spectral information at low energies:

- Laplace transform in limit $\lim_{T\to 0} \kappa(\omega, \tau) = e^{-\omega\tau}$
- Often $T = 1/N_\tau$ short in lattice calculations
  $\rightsquigarrow$ could indicate a benefit of anisotropic calculations
- low-ω contributions suppressed at short τ
- low-ω contributions compete with kernel T-effects at $\tau = N_\tau/2$



(a)
(b)
(c)

Nature of the inverse problem - general approach

All methods to perform a spectral reconstruction can be understood as a master function

$$\mathcal{F}[\mathbf{G}, \mathbf{C}_G] = \left(\boldsymbol{\rho}, \mathbf{C}_\rho \right)$$

where

- $\mathbf{G}$ = discrete samples of $G(\tau)$
- $\mathbf{C}_G$ = covariance of $\mathbf{G}$
- $\boldsymbol{\rho}$ = discrete estimator of $\rho(\omega)$
- $\mathbf{C}_\rho$ = covariance of $\boldsymbol{\rho}$

$\rightarrow$ We want to understand the properties and limitations of the master function $\mathcal{F}$
$\rightarrow$ Crucial to be data focused: The best $\mathcal{F}$ will depend in detail on G, number of slices, properties of C, etc.

General difficulties

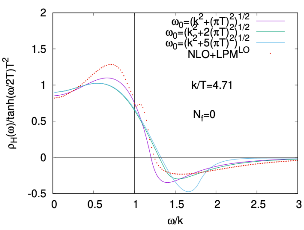
- For $\boldsymbol{\rho}_i = \rho(\omega_i)$

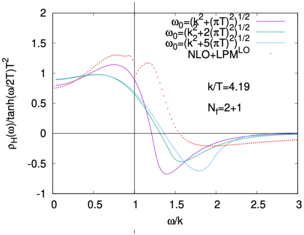
$$\left|\mathcal{F}[\mathbf{G} + \delta\mathbf{G}, \mathbf{C}_G + \delta\mathbf{C}_G] - \mathcal{F}[\mathbf{G}, \mathbf{C}_G]\right| \quad \text{and thus} \quad \left|\mathbf{C}_\rho\right|$$

  explode.
- For cases where $|\mathbf{C}_\rho|$ is under control, relation between $\rho(\omega) \Leftrightarrow \boldsymbol{\rho}$ may be obscured.

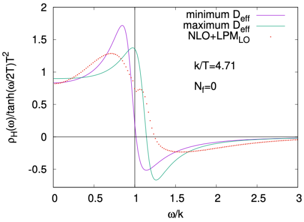
Spectral reconstruction - χ^2-fits

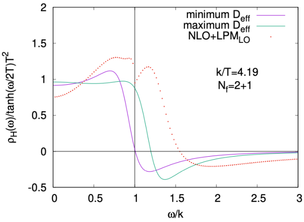
(a) Polynomial fits





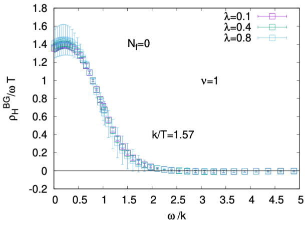
(b) Padé fits

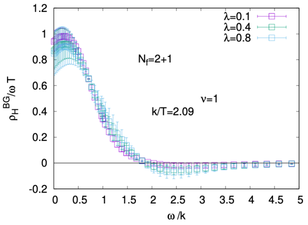




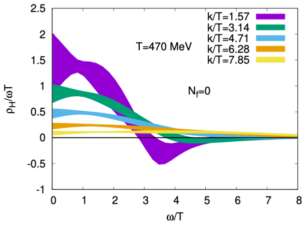
Spectral reconstruction - BGM and GPR

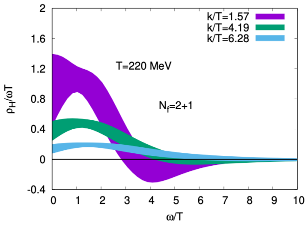
Backus-Gilbert method



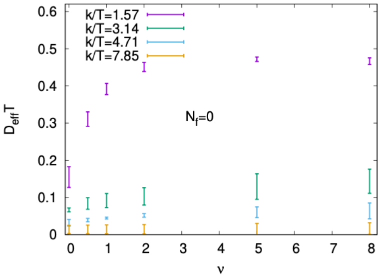


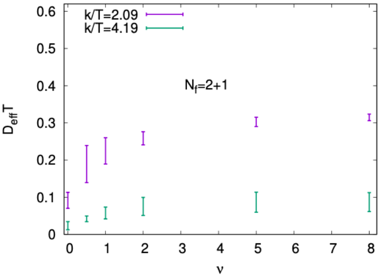
Gaussian processes

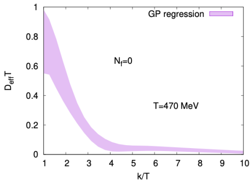


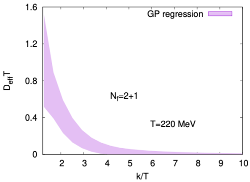


Photon rate from the lattice - $D_{eff}\,T$ comparison









Cè et al [2205.02821]