Flux compactifications in String Theory and Connections to Modularity

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IBS-CTPU Welcome Workshop

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Number Theory and Physics?

Physics: Geometric realizations are usually defined on spaces over $\mathbb{R}$ or $\mathbb{C}$

Number Theory: Framework of finite fields $\mathbb{F}_p$ (for p prime)

Local-to-Global principle:
Geometries defined over finite fields contain information of those defined over $\mathbb{C}$

Example: *Modularity of Calabi-Yau manifolds*

- relates certain Calabi-Yau geometries (in a unique way) to modular forms
- based on tools from arithmetic geometry
- has strong implications on physics (e.g.: flux compactifications)

[Weil, Deligne, Dwork, Serre, Wiles, Taylor, Moore, Bönisch, Candelas, de la Ossa, Elmi, Fischbach, Hulek, Kachru, Klemm, Kuusela, McGovern, Nally, Rodrigues-Villegas, van Straten, Verrill, Yang,...]

String Theory

String Theory...

- ... is a UV complete quantum theory of gravity
- ... yields a large landscape of low energy effective field theories
- ... provides a framework to study consistent theories of quantum gravity

String Theory in a Nutshell:

- Fundamental degrees of freedom: one-dimensional extended objects (strings) evolving in spacetime
- Quantum fields: Oscillation modes of the string
- Fermionic states are included via supersymmetric extension $\Rightarrow$ superstrings
- Superstring theory is anomaly-free only in $D = 10$ spacetime dimensions

String Compactifications

Superstring theory is consistent only in $D = 10$ spacetime dimensions

Compactification: Decompose spacetime

$$M^{10} = \mathbb{R}^{1,3} \times X_6$$

with X_6 a compact “internal“ space

Spectrum of EFT on $\mathbb{R}^{1,3}$ is determined by the geometry of X_6

e.g.: Massless states $\longleftrightarrow$ harmonic modes of X_6

Calabi-Yau condition:

- Generically, supersymmetry in 10d is broken by compactification to 4d EFT
- For special choices of X_6, supersymmetry is restored after compactification

$\Rightarrow$ Calabi-Yau Manifolds

Moduli Stabilization and Fluxes

Spectrum of 4d EFT contains massless scalar fields

- Geometric interpretation: Moduli of the internal space X_6
- phenomenological: Massless scalar fields have not been observed!
- Add a potential W to “stabilize“ these fields ($\Rightarrow$ Adding mass terms)

UV complete theory $\Rightarrow$ Need “background fluxes“

- Background flux: Non-trivial (topological) field strength of a higher form gauge field on the internal space X_6

Flux superpotential for type IIB string compactifications: [Gukov,Vafa,Witten, 2000]

$$W = \int_{X_6} \Omega\left(z^i\right) \wedge (F - \tau H) \qquad \Omega \in H^{3,0}(X_6, \mathbb{C})$$

- scalar fields z^i and τ (the complex structure moduli and the axio-dilaton)
- Internal three-form fluxes $F, H \in H^3(X_6, \mathbb{Z})$ (Field strengths of two-form fields)

Flux Compactification of Type IIB String Theory

Flux superpotential:

$$W = \int_{X_6} \Omega\left(z^i\right) \wedge (F - \tau H) \qquad \Omega(z^i) \in H^{3,0}(X_6, \mathbb{C})$$

for background fluxes $F, H \in H^3(X_6, \mathbb{Z})$

Supersymmetric vacuum constraints:

$$\partial_{z^i} W = 0\ ,\ \partial_\tau W = 0\ ,\ W = 0$$

Flux Compactification of Type IIB String Theory

Flux superpotential:

$$W = \int_{X_6} \Omega\left(z^i\right) \wedge (F - \tau H) \qquad \Omega(z^i) \in H^{3,0}(X_6, \mathbb{C})$$

for background fluxes $F, H \in H^3(X_6, \mathbb{Z})$

Supersymmetric vacuum constraints:

$$\int_{X_6} \Omega(z^i) \wedge F = 0\ ,\ \int_{X_6} \Omega(z^i) \wedge H = 0\ ,\ \int_{X_6} \partial_{z^i} \Omega\left(z^i\right) \wedge (F - \tau H) = 0$$

X_6 supports a non-trivial flux configuration with a supersymmetric vacuum only if

$$\langle F, H\rangle_{\mathbb{Z}} \subset \left[H^{2,1}(X_6, \mathbb{C}) \oplus H^{1,2}(X_6, \mathbb{C})\right] \cap H^3(X_6, \mathbb{Z})$$

defines a two-dimensional sublattice

Flux Compactification of M-Theory

Setup: 11d M-theory (or 12d F-theory) compactified on a Calabi-Yau fourfold X_8

Flux superpotential:

$$W = \int_{X_8} \Omega(z^i) \wedge G \qquad \Omega(z^i) \in H^4(X_8, \mathbb{C})$$

- z^i complex structure moduli
- $G \in H^4(X_8, \mathbb{Z})$ internal topological four-form flux

Supersymmetric vacuum constraints imply

X_8 supports a non-trivial flux configuration with a supersymmetric vacuum only if

$$\langle G \rangle_{\mathbb{Z}} \subset \left[H^{4,0}(X_8, \mathbb{C}) \oplus H^{2,2}(X_8, \mathbb{C}) \oplus H^{0,4}(X_8, \mathbb{C}) \right] \cap H^4(X_8, \mathbb{Z})$$

$\Rightarrow$ one-dimensional sublattice

Hodge Substructures

We have seen:

Supersymmetric Flux Vacua of string or M-theory compactifications require a non-trivial sublattice $\Lambda \subset H^n(X, \mathbb{Z})$ for $n = 3, 4$ such that

$$\Lambda \otimes \mathbb{C} = \bigoplus_{p+q=n} \Lambda^{p,q}$$

with

$$\Lambda^{q,p} = \overline{\Lambda^{p,q}} \quad , \quad \Lambda^{p,q} \subset H^{p,q}(X, \mathbb{C})$$

Such sublattices are called **Hodge substructures**

Question: Given a Calabi-Yau n-fold X, does there exists a Hodge substructure?

$\Rightarrow$ **Modularity**

Arithmetic Geometry

- Assume that
$$X = \{f_i(x_k) = 0\} \quad f_i(x) \in \mathbb{Z}[x_1, \ldots, x_m]$$
is some (affine or projective) complex variety
- Treat X to be defined over the finite field $\mathbb{F}_{p^r}$ with p^r elements (p prime, $r \in \mathbb{N}$)
$$X/\mathbb{F}_{p^r} := \{\bar{f}_i(x) = 0\} \subset (\mathbb{F}_{p^r})^m$$
- (Finite) Number of points
$$N_{p^r}(X) := |X/\mathbb{F}_{p^r}|$$
collected in the generating **local zeta function**
$$\zeta_p(X, T) = \exp\left(\sum_{r=1}^{\infty} N_{p^r}(X) \frac{T^r}{r}\right)$$
- **"Local-to-global principle":**
$\zeta_p(X, T)$ contain information about the Hodge structure of $H^k(X, \mathbb{Z})$

Weil conjectures: Constrain $\zeta_p(X,T)$ strongly [Weil, 1949]

Rationality:

$$\zeta_p(X,T) = \frac{R_1(X,T)\cdots R_{2n-1}(X,T)}{R_0(X,T)\cdots R_{2n}(X,T)} \quad , \; n = \dim_{\mathbb{C}}(X)$$

$R_k(X,T)$ are polynomials of degree $b^k = \dim(H^k(X,\mathbb{Q}))$

In particular: $R_k(X,T) = \det(\mathbb{1} - T\mathrm{Fr}_p^{-1})$ for linear maps

$$\mathrm{Fr}_p : H^k(X,\mathbb{Q}_p) \to H^k(X,\mathbb{Q}_p)$$

$H^k(X,\mathbb{Q}_p)$: p-adic cohomology groups

Important fact:
If $H^k(X,\mathbb{Z})$ has a Hodge substructure, Fr_p becomes block-diagonal
$\Rightarrow R_k(X,T)$ factorizes for (almost) all primes p!

The Modularity Conjecture

Consider an elliptic curve $\mathcal{E}$:

$$R_1(\mathcal{E}, T) = 1 - a_p T + p T^2 \qquad \text{with } a_p = p + 1 - N_p(\mathcal{E})$$

Modularity: $f(\tau) := \sum_{p \text{ prime}} a_p q^p \ , \ q = e^{2\pi i \tau}$ is a modular form of weight two

For a Calabi-Yau n-fold X: If $H^k(X, \mathbb{Z})$ has a two-dimensional Hodge substructure:

$$R_k(X, T) = R_\Lambda(X, T) \cdot R_\Sigma(X, T) \quad \text{with } R_\Lambda(X, T) = 1 - a_p p^\alpha T + p^\beta T^2$$

for some (fixed) $\alpha, \beta \in \mathbb{N}$

Serre's Modularity Conjecture: [Serre, 1975]

$$f(\tau) := \sum_{p \text{ prime}} a_p q^p \quad , \quad q = e^{2\pi i \tau} \text{ is a modular form}$$

Manifolds of this type are called **modular**

Arithmetic Search for Fluxes

Question: How can we find Calabi-Yau n-folds that admit non-trivial fluxes?
$\Rightarrow$ Use modularity as a necessary condition for Hodge substructures of $H^n(X, \mathbb{Z})$

Setup: X_z a family of Calabi-Yau n-folds, $z \in \mathbb{C}$ modulus

Algorithm: For $p \geq 7$ prime

- The moduli space $z \in \mathbb{C}$ reduces to the finite set $z_p \in \mathbb{F}_p$
- For each $z_p \in \mathbb{F}_p$ compute $R_n(X_{z_p}, T)$
- Count $|\{z_p \in \mathbb{F}_p \mid R_n(X_{z_p}, T) \text{ factorizes quadratically}\}|$

If there is at least one point of factorization per prime p:

- Find $z \in \bar{\mathbb{Q}} \subset \mathbb{C}$ s.t.

  $$z_p \equiv z \mod p$$

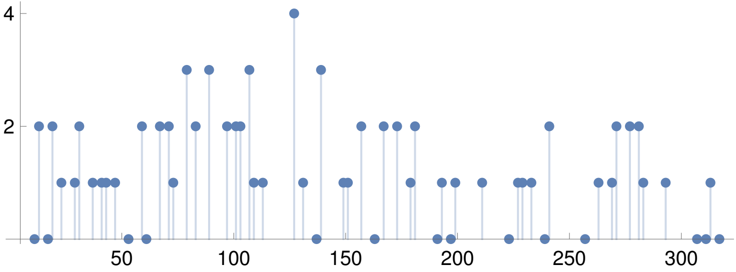
  is a point of quadratic factorization for each prime p
- If such a $z \in \bar{\mathbb{Q}}$ exists, the (complex) variety X_z is a candidate to be modular

[Kachru, Nally, Yang, 2020], [Candelas, de la Ossa, van Straten, 2020],...

Example: Non-Modular Case

The mirror family of the complete intersection $\mathbb{P}^7[2,2,4]$: [Jockers, S.K., Kuusela, '23]

- Family of Calabi-Yau fourfolds X_z dependent on one modulus $z \in \mathbb{C}$
- Number of quadratic factorizations for each prime $7 \leq p \leq 317$:

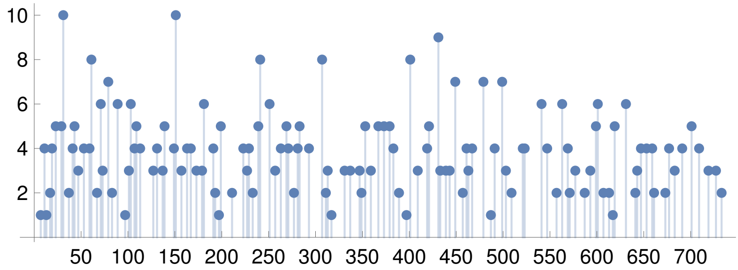


- Many primes p with no point $z_p \in \mathbb{F}_p$ s.t. $R_4(X_{z_p}, T)$ has a quadratic factorization
- The existence of an algebraic modulus $z \in \bar{\mathbb{Q}} \subset \mathbb{C}$ s.t. $H^4(X_z, \mathbb{Z})$ has a two-dimensional sublattice of definite Hodge type is highly unlikely

Example: Modular Case

A one-parameter family of Hulek-Verrill fourfolds HV_z^4: [Jockers, S.K., Kuusela, '23]

- Number of quadratic factorizations for each prime $7 \leq p \leq 733$



- At least one point $z_p \in \mathbb{F}_p$ for each prime s.t. $R_4(\mathrm{HV}_{z_p}^4, T)$ has a quadratic factorization
- There is potentially a modulus $z \in \bar{\mathbb{Q}}$ s.t. HV_z^4 is modular

Example: Modular Case

Reconstruction of possible modular points $z \in \bar{\mathbb{Q}} \subset \mathbb{C}$ from p-adic data:

- Collection of points $z_p \in \mathbb{F}_p$ with quadratic factorization

prime p	$z_p \in \mathbb{F}_p$			
$p = 11$	1	6	8	10
$p = 13$	1			
$p = 17$	1	15		

prime p	$z_p \in \mathbb{F}_p$			
$p = 19$	1	2	7	17
$p = 23$	1	4	5	12
$p = 29$	1	6	11	24

- One (rational) solution $z \in \mathbb{Q}$ s.t. $z_p \equiv z \mod p$ appears for all p:

$$z = 1$$

- $\mathrm{HV}^4_{z=1}$ is a candidate for a modular Calabi-Yau fourfold!

A Modular Calabi-Yau Fourfold

Consistency checks:

- Coefficients a_p of quadratic factor

$$R_\Lambda(\mathrm{HV}^4_1, T) = 1 - a_p p T + p^2 T^2$$

  give the q-expansion of a unique modular form
- Identified generators of the two-dimensional Hodge substructure

$$\Lambda = \left[H^{3,1}(\mathrm{HV}^4_1, \mathbb{C}) \oplus H^{1,3}(\mathrm{HV}^4_1, \mathbb{C})\right] \cap H^4(\mathrm{HV}^4_1, \mathbb{Z})$$

  by suitable covariant derivatives of $\Omega \in H^{4,0}(\mathrm{HV}^4_1, \mathbb{C})$
- Remainder

$$\Sigma = \left[H^{4,0}(\mathrm{HV}^4_1, \mathbb{C}) \oplus H^{2,2}(\mathrm{HV}^4_1, \mathbb{C}) \oplus H^{0,4}(\mathrm{HV}^4_1, \mathbb{C})\right] \cap H^4(\mathrm{HV}^4_1, \mathbb{Z})$$

  defines suitable four-form fluxes
- In particular:

$$G := C \cdot \mathrm{Re}(\Omega(z))\big|_{z=1} \in \Sigma \ , \quad C \in \mathbb{R}$$

Conclusions: Number theory and Physics!

Arithmetic geometry can be used as a tool to investigate varieties which are defined over $\mathbb{C}$

Modularity serves as a necessary condition for (two-dimensional) Hodge substructures, i.e. for

- supersymmetric flux vacua
- rank-two attractor points [Candelas, de la Ossa, Elmi, van Straten, '19]
- topology changing transition loci? [Jockers, S.K., Kuusela, WIP]

The corresponding modular form f_X contains physical information

- For type IIB flux vacua: The axio-dilaton τ
- For rank-two attractor points: The BH entropy S_{BH}