Mode Amplitudes: Linear and nonlinear dynamics in ringdowns

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Flow of the talk

1. Introduction

2. Start of the linear regime and non-linearity

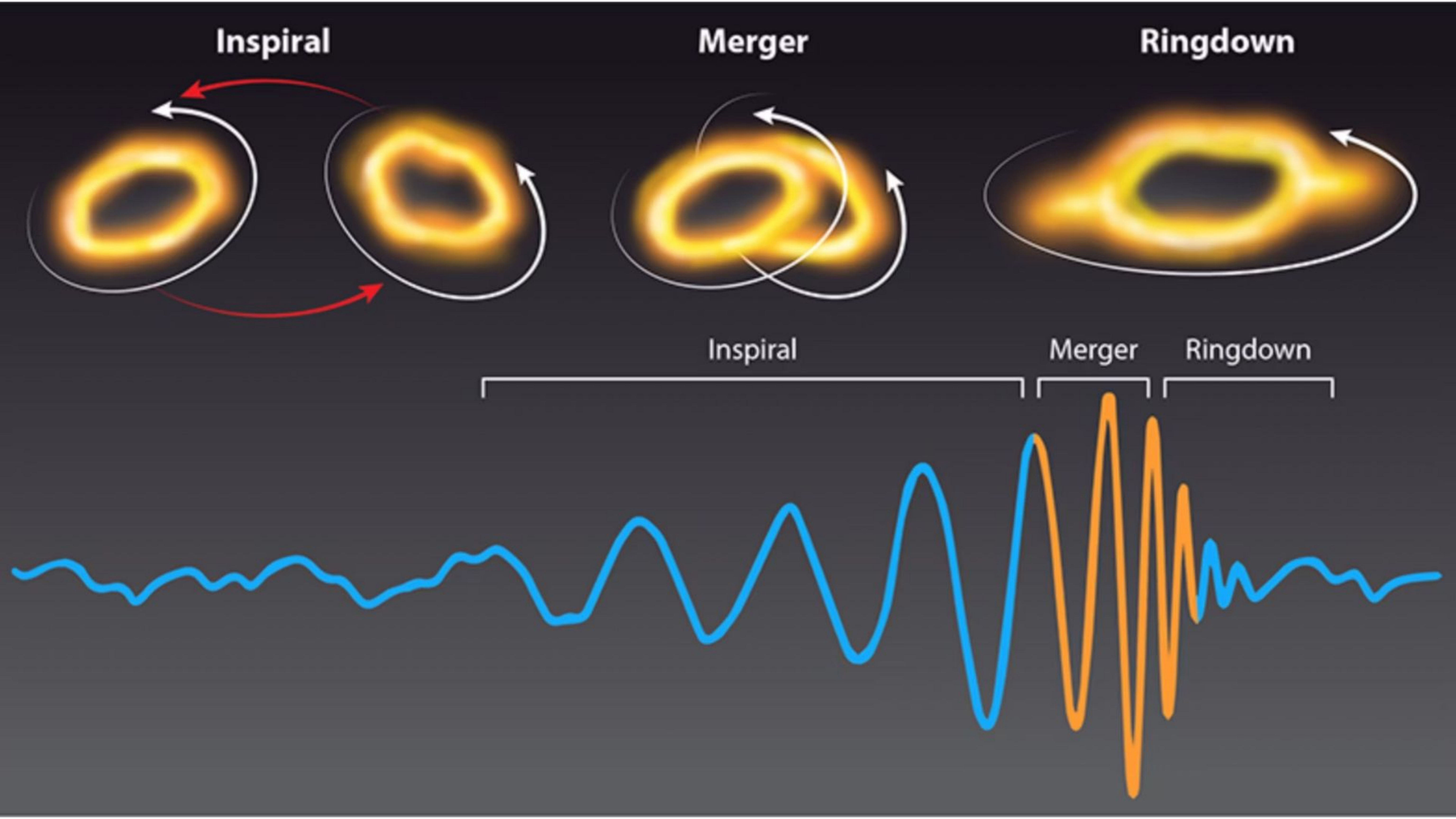
3. Black hole perturbation theory

   1. 1st order
   2. 2nd order

4. Amplitudes and amplitude fits

   1. aligned spin systems
   2. **precessing spin systems (NEW!!!)**

Introduction


Inspiral
Merger
Ringdown
Inspiral
Merger
Ringdown

Ringdown

In perturbation theory, ringdown can modelled as superposition of characteristic modes of a black hole

$$h_{lm}(t) = A_{lm}\, e^{-(t-t_0)/\tau_{lm}}\, e^{-i\omega_{lm}(t-t_0)+\phi_{lm}}$$

Black Hole Spectroscopy & Quasi-Normal Modes (QNMs)

- **Frequencies** and **damping times** of QNMs are uniquely determined by the black hole's **mass** and **spin** → Basis of **black hole spectroscopy**
- **Amplitudes** and **phases** depend on the **initial perturbation** → Complementary to intrinsic mode properties

Can be used to **test General Relativity** → *E.g.*, **Amplitude-Phase Consistency Test** [Forteza, Bhagwat *et al.*, *Phys. Rev. Lett.* 130.021001 (2022)]

A brief review of BH perturbation theory

Black hole perturbation theory and 1st order QNMs

Teukolsky Master Equation (non-vacuum, spin-s):

$$\mathcal{T}\psi_s = 4\pi\Sigma T_s$$

In first-order perturbation theory (vacuum):

$$\mathcal{T}\psi_s = 0 \quad \text{(no source term, linearized theory)}$$

We focus on ψ_4, the outgoing radiative Weyl scalar ($s = -2$)

Ansatz: Decompose $\psi_4(t, r, \theta, \phi)$

$$e^{-i\omega t}\ S_{-2}(\theta; a\omega)\ R_{-2}(r)\ e^{im\phi}$$

The master equation separates into:

Time dependence: $e^{-i\omega t}$ Angular equation for $S_{-2}(\theta)$ Radial equation for $R_{-2}(r)$

Boundary Conditions for QNMs:

$$R_{-2}(r) \sim \Delta^{-2} e^{-ikr_*} \quad \text{(purely ingoing at the horizon)}$$

$$R_{-2}(r) \sim r^3 e^{+i\omega r_*} \quad \text{(purely outgoing at infinity)}$$

These conditions yield a discrete set of complex frequencies

$$\omega_n = \omega_{R,n} + i\omega_{I,n} \quad \text{(damped oscillations — Quasinormal Modes)}$$

Black hole perturbation theory and 1st order QNMs

$$h(t,\theta,\phi) = \sum_{\ell,m,n} A_{\ell mn}\, e^{-i\tilde{\omega}^{\mathrm{QNM}}_{\ell mn}(t-t_0)}\ {}_{-2}Y_{\ell m}(\theta,\phi)$$

Theory:

- Perturbation theory computes QNMs.
- QNMs depend only on mass & spin (no-hair).
- Spectrum fully set by mass & spin.

Observations:

- Black hole spectroscopy tests QNMs.
- Measure 3+ QNMs:
  1. Use 2 to find mass & spin,
  2. Use others to test no-hair.

However, the amplitude for these mode excitations **cannot be determined from perturbation theory — full numerical relativity** is required to capture the nonlinear dynamics.

- QNM amplitudes fitted to numerical relativity simulations
- Surrogates are built to predict amplitudes from binary parameters
- Amplitudes can be used as GR consistency tests

Schematics for 2nd order perturbation

Second-Order Black Hole Perturbation: Explicit Structure and Interpretation

Metric Expansion: $g_{ab} = g_{ab}^{(0)} + \epsilon h_{ab}^{(1)} + \epsilon^2 h_{ab}^{(2)} + \cdots$

Radiative Weyl Scalar (Spin $s = -2$): $\psi_4 = \psi_4^{(1)} + \psi_4^{(2)} + \cdots$

Second-Order Teukolsky Equation: $\mathcal{T}\psi_4^{(2)} = \mathcal{S}[\psi_4^{(1)}, h^{(1)}]$

$\mathcal{T}$ is the same operator from the first-order theory (Teukolsky Operator)

Structure of the Source Term:

$$\mathcal{S}[\psi_4^{(1)}] \sim \psi_4^{(1)} \cdot \psi_4^{(1)}, \quad h^{(1)} \cdot \partial\psi_4^{(1)}, \quad \partial\psi_4^{(1)} \cdot \partial\psi_4^{(1)}, \text{ etc.}$$

Quadratic in first-order quantities ⇒ drives nonlinear radiation

Ansatz for $\psi_4^{(2)}$: (separated form)

$$e^{-i\omega^{(2)}t} \; {}_{-2}S_{\ell m}^{(2)}(\theta) \; R_{\ell m}^{(2)}(r) \; e^{im\phi}$$

Structure same as first-order, but now driven by source term

(1) Solve First-Order Teukolsky Equation $\mathcal{T}\psi_4^{(1)} = 0$

$\psi_4^{(1)} = e^{-i\omega^{(1)}t} \cdot {}_{-2}S_{\ell m}(\theta) \cdot R_{\ell m}(r) \cdot e^{im\phi}$

(2) Angular Structure in Source $\mathcal{S}[\psi_4^{(1)}]$ contains products of angular harmonics

Decompose these into spin-weighted spheroidal harmonic basis using Clebsch–Gordan–like expansions

(3) Solve Second-Order Equation $\mathcal{T}\psi_4^{(2)} = \mathcal{S}$

Use method of Green's functions or mode-sum decomposition (frequency domain)

(4) Decompose Into Second-Order Modes

Perform angular projection: $\langle {}_{-2}S_{\ell m}^{*} | \psi_4^{(2)} \rangle$ to isolate each angular mode

Schematics for 2nd order perturbation: Frequency

Origin of $\omega^{(2)} \approx 2\omega^{(1)}$:

$$\psi_4^{(1)}(t,r,\theta,\phi) \sim e^{-i\omega^{(1)}t} \cdot {}_{-2}S_{\ell m}^{(1)}(\theta) \cdot R_{\ell m}^{(1)}(r) \cdot e^{im\phi}$$

When constructing the source term $\mathcal{S}[\psi_4^{(1)}]$, you get products like:

$$\mathcal{S} \sim \left(e^{-i\omega^{(1)}t}\right)^2 = e^{-2i\omega^{(1)}t}$$

$$\Rightarrow \psi_4^{(2)} \sim e^{-i\omega^{(2)}t} \quad \text{with} \quad \omega^{(2)} \approx 2\omega^{(1)}$$

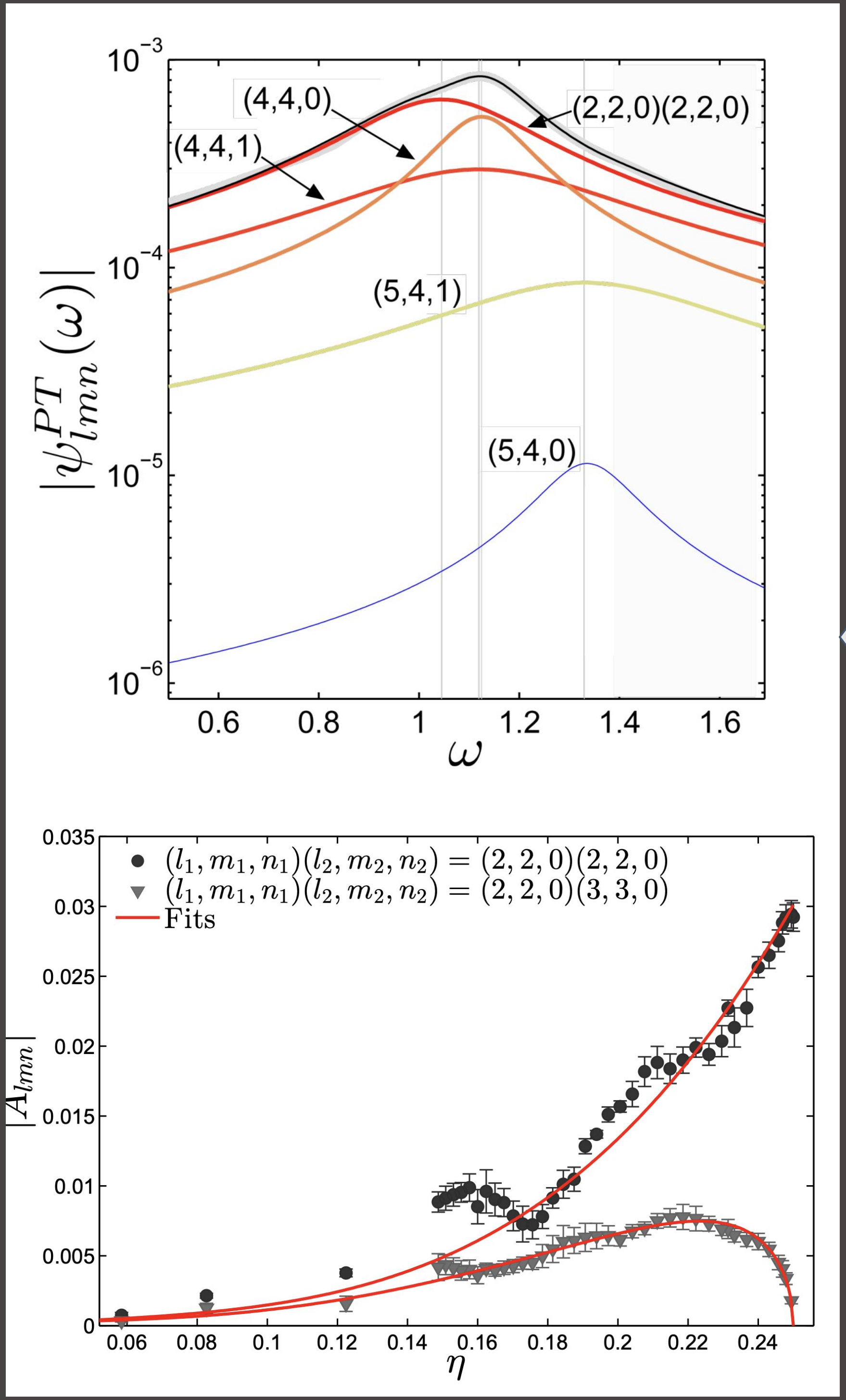
$\Rightarrow$ A direct consequence of the quadratic source built from linear fields

From the quadratic sourcing by the linear (2,2,0) mode, we expect $A_{(4,4)}^{(2,2,0)\times(2,2,0)} \propto \left(A_{(2,2,0)}\right)^2$.

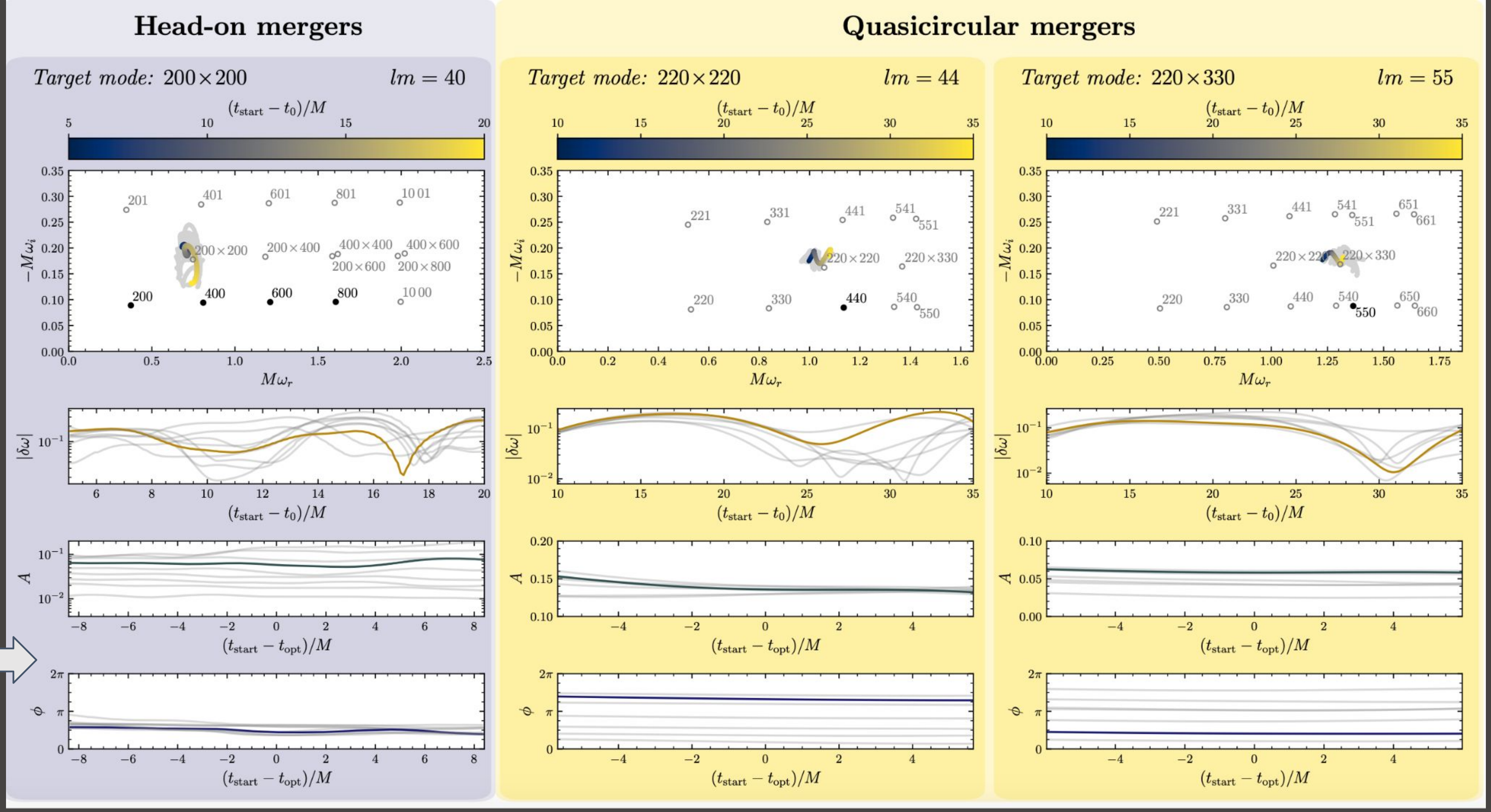
CAN WE SEE THIS IN DATA?

- Loud, clean data might enable direct testing of 2nd-order QNM frequencies and amplitudes.
- We might be able to verify 2nd-order perturbation effects observationally.

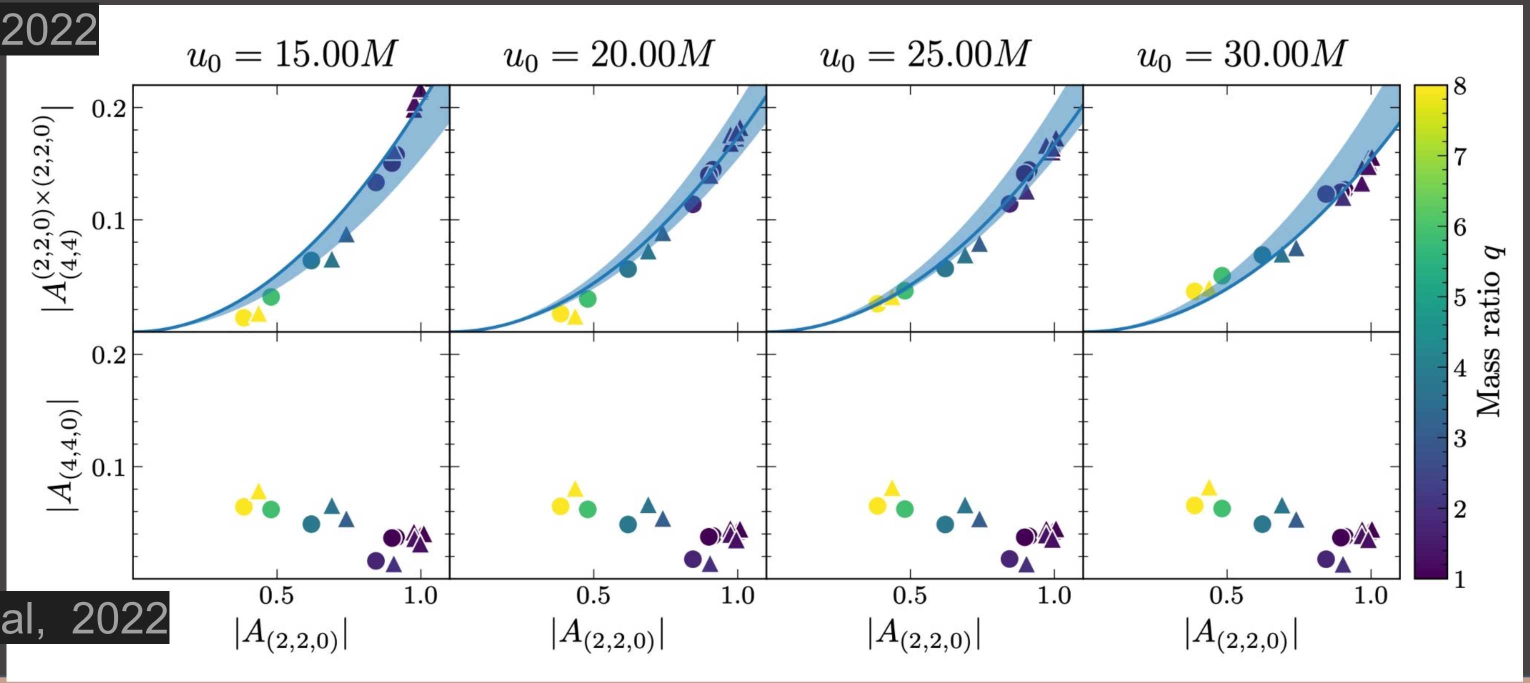
2nd order modes



London et al, 2014



Cheung et al, 2022

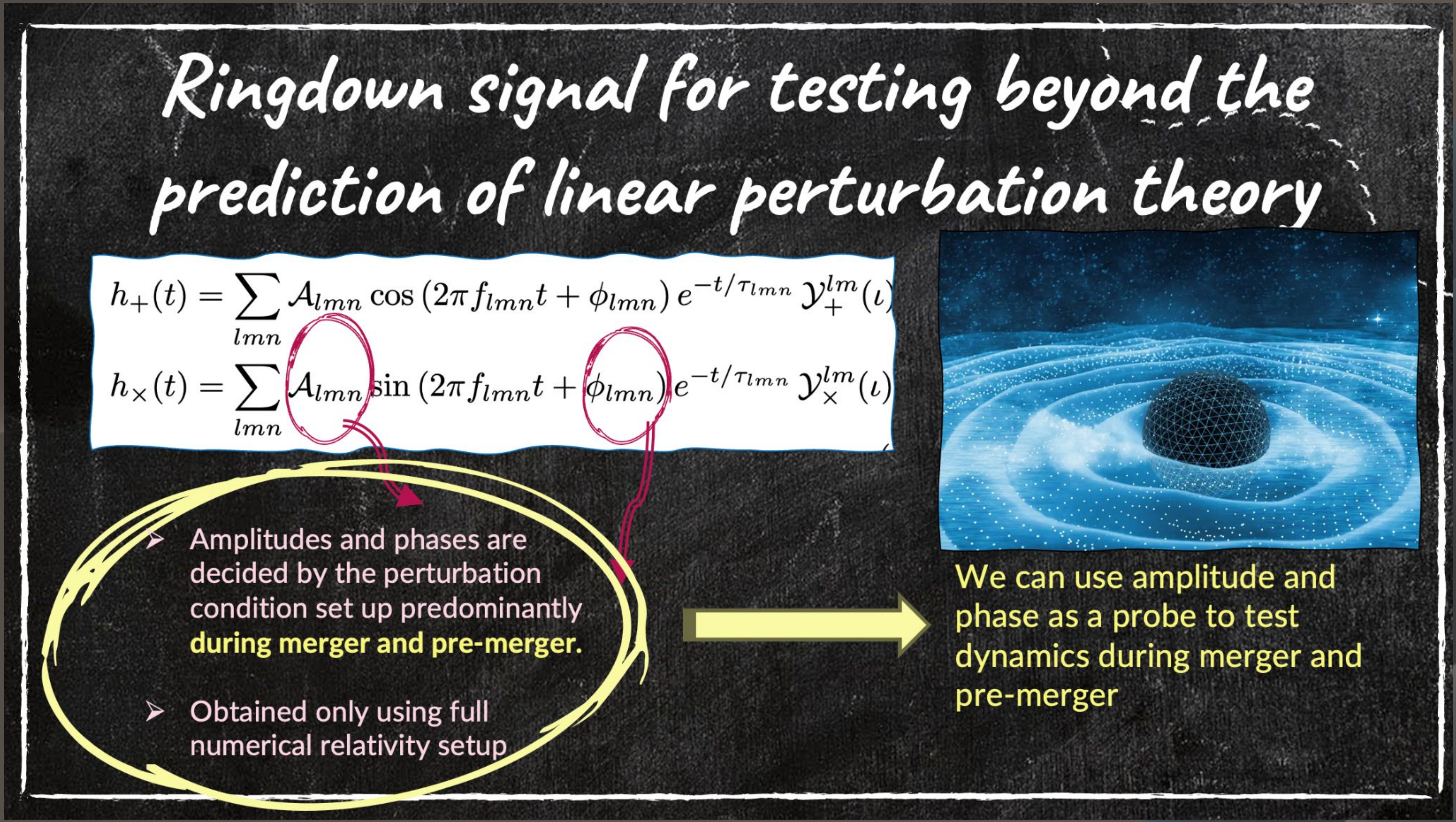


Mitman et al, 2022

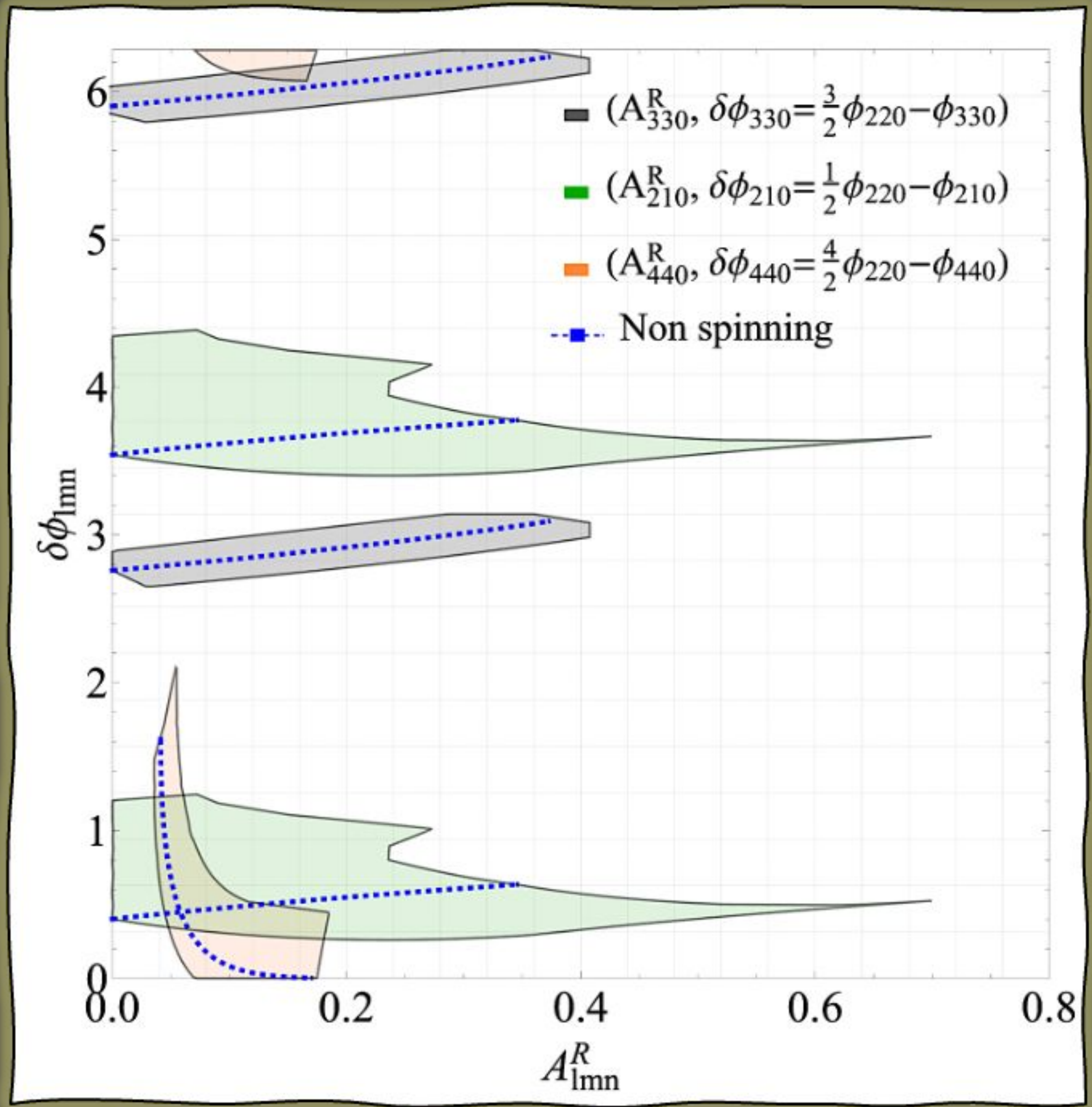
Ringdown mode amplitudes

Ringdown mode amplitudes

- Ringdown amplitudes are important in signal modelling and parameter estimation
- You can also use them to test GR!



E.g.,: Amplitude phase consistency test



Aligned spin ringdown : 1st order mode amplitude

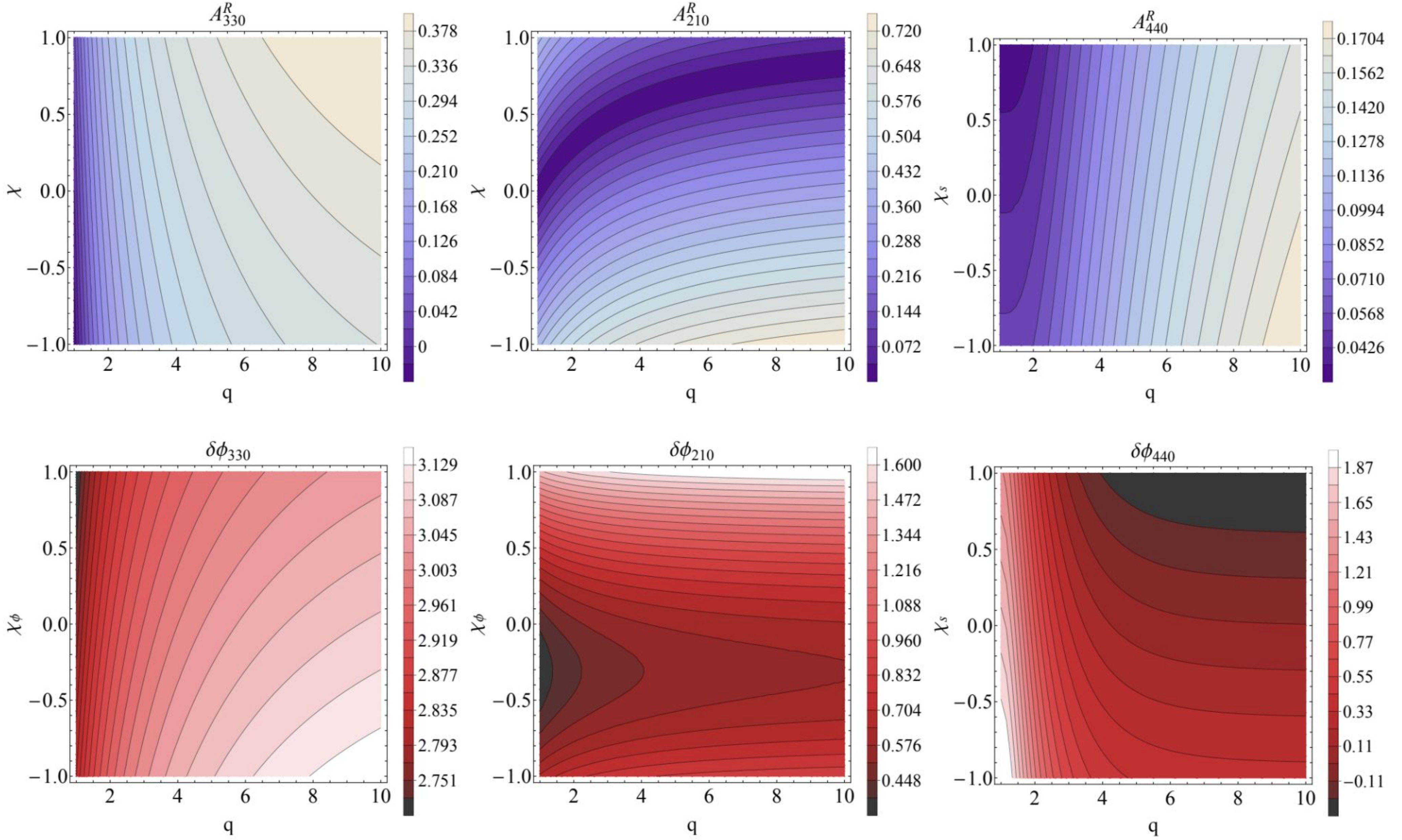
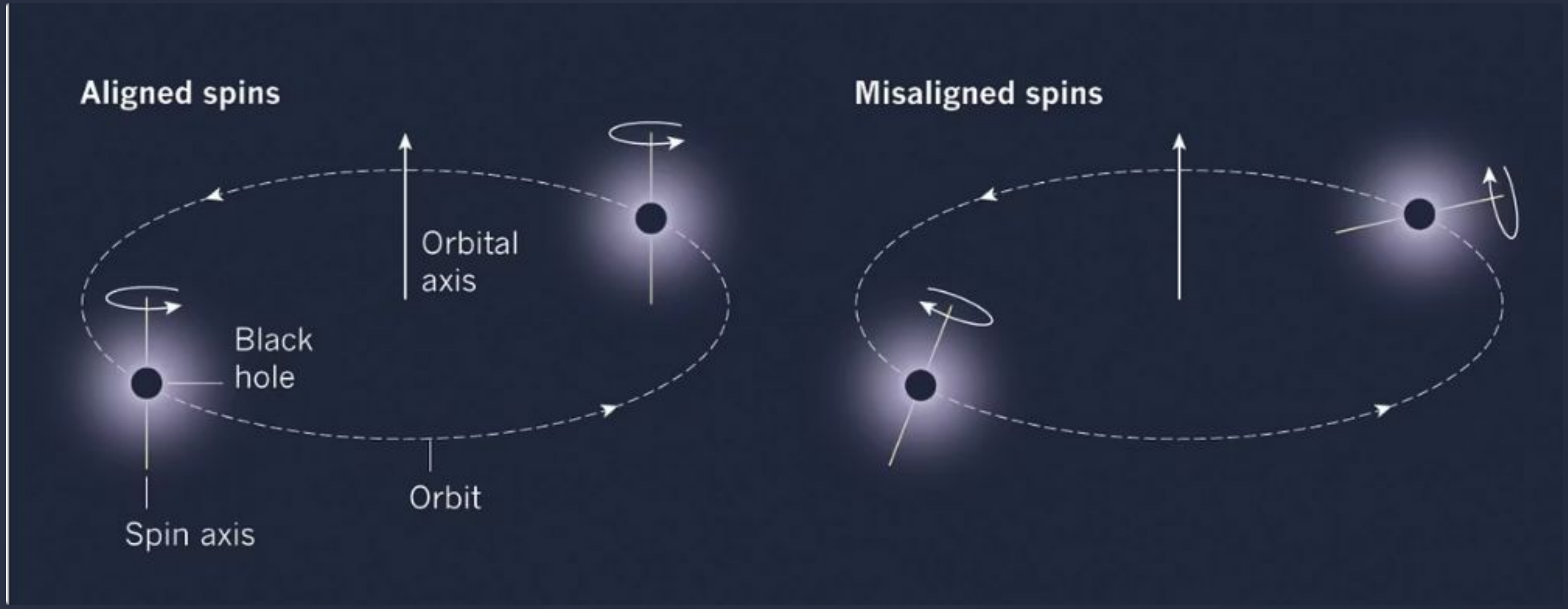


Figure from Forteza, Bhagwat et al, 2023

Aligned Spin Amplitude Fits

- London et al, 2022, 2018
- Chung et al, 2023
- Forteza, Bhagwat et al, 2023
- Pacillio, Bhagwat et al, 2024
- Zertuche et al,2024

Amplitude for precessing binary black hole ringdowns (NEW work!!!)



Mode Amplitude Fits for Black Hole Ringdown

Precessing Spin Amplitude Fits

- Zhu eta al, 2024; Some study on Precessing amplitude
- For Precessing spin, we present the **first-ever fit** and some detailed study on phenomenology.
- See Nobili, Bhagwat et al, 2025 arxiv:2504.17021

Step 0: Rotate all the waveform to the frame of remnant

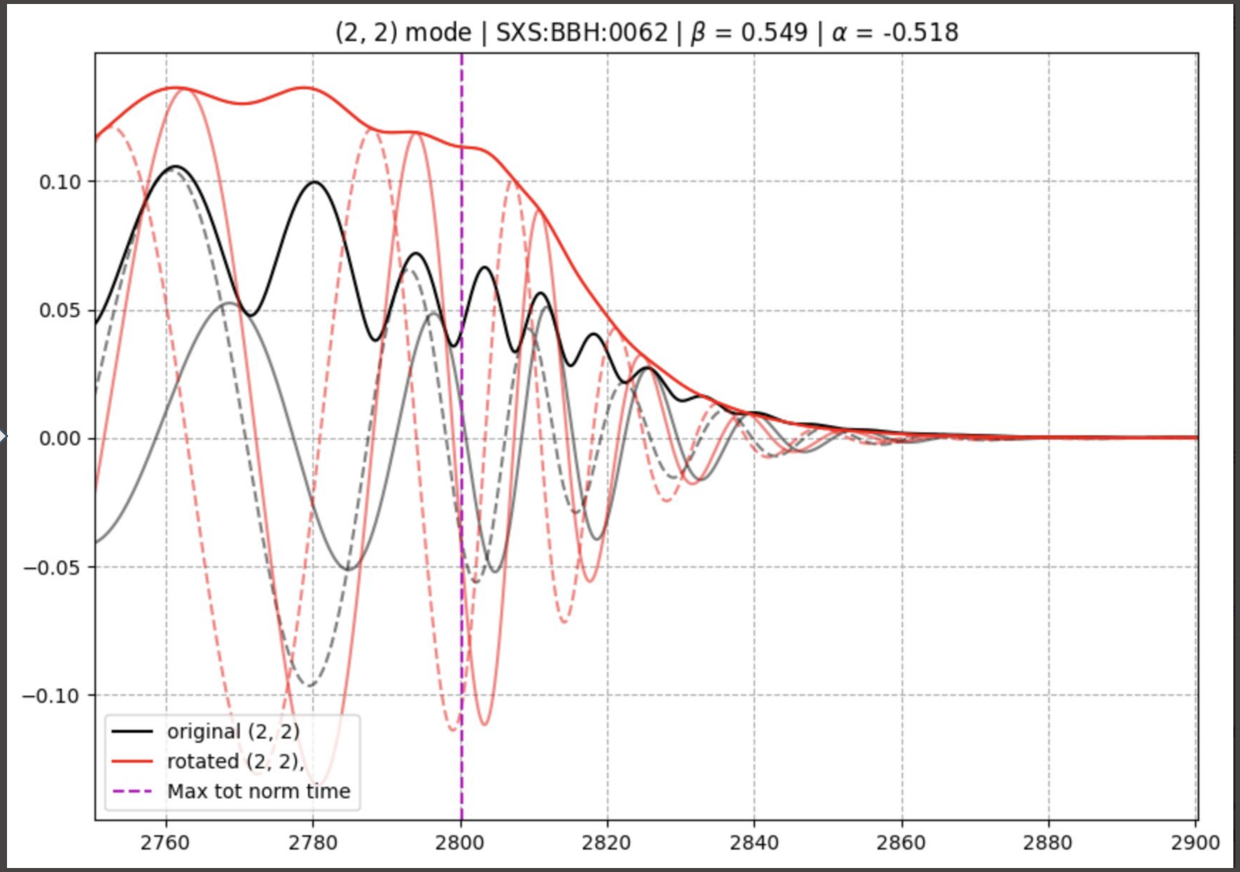
Numerical relativity waveform are provided in coordinate frame where the z axis is aligned with the orbital angular momentum at the start of the simulation

BUT, QNMs frequencies and damping times are computed in a coordinate frame where the z-axis is aligned with the spin direction of BH

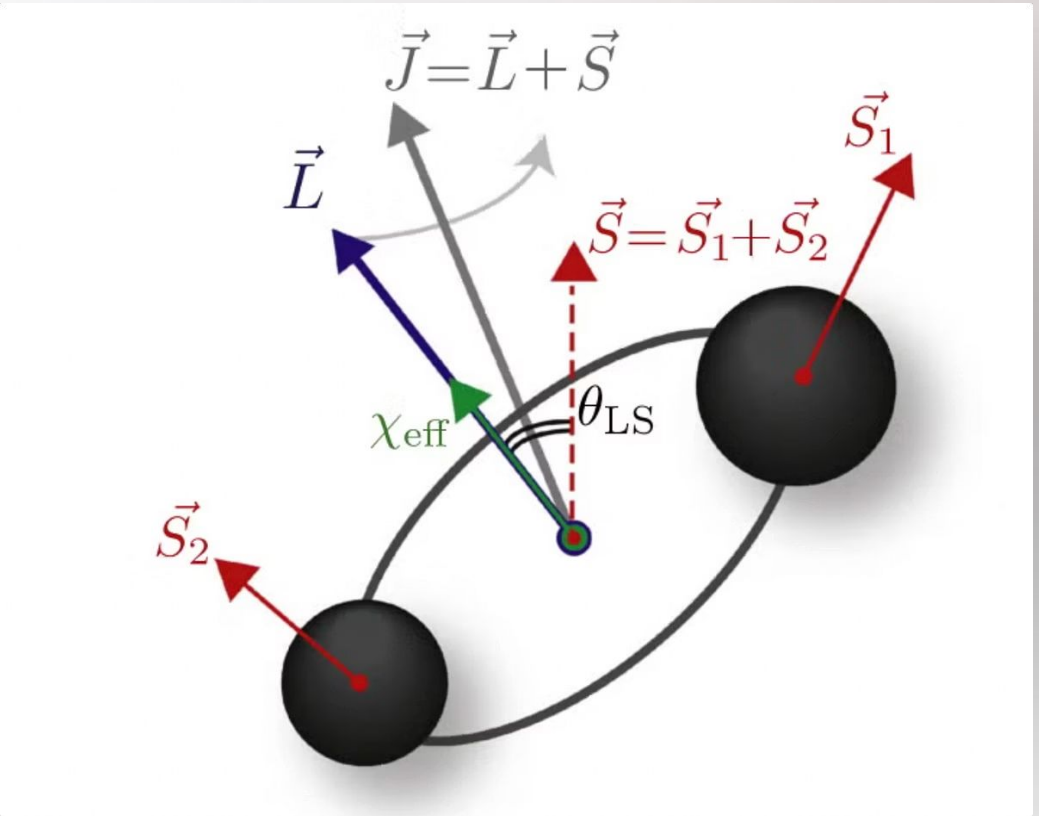
$$h'_{\ell m}(t) = \sum_{m'} D^{(\ell)}_{mm'}(\alpha, \beta, \gamma) h_{\ell m'}(t)$$

Where:

- (α, β, γ) are Euler angles (z-y-z convention) corresponding to the rotation from original to the remnant-aligned frame.
- $D^{(\ell)}_{mm'}$ is the Wigner D-matrix of degree ℓ.



All descriptions henceforth is in this ringdown frame



Challenges with Precessing Ringdown

1. Aligning the waveform - reference time for ringdown
2. Parametrization and degrees of freedom
3. Analytical anzarts for the fits ix intractable
4. Multiple modes (e.g., (2,1), (3,3), (4,4)) can be equally strong—hierarchies shift

1. Alignment: reference time for identifying ringdown

- To fit for amplitudes of the modes -- align the waveforms at a reference point and decide when ringdown starts.

- For aligned spin systems:
  - peak of 22 strain is a typical choice

- For precessing system:
  - you don't know which mode will be loudest
  - you have an amplitude modulation
  - the +m and -m modes are differently excited

We choose the point at which the energy in the ringdown maximises as the point to align the waveform: EMOP time

1. Mode-by-Mode Parallel Energy

For each (ℓ, m) and start time t_i,

$$E_{\|,\ell m}(t_i) = \frac{1}{8\pi} \frac{\left|\int_{t_i}^{\infty} \dot{h}_{\mathrm{NR},\ell m}\, \dot{h}^{*}_{\mathrm{QNM},\ell m}\, dt\right|^2}{\int_{t_i}^{\infty} |\dot{h}_{\mathrm{QNM},\ell m}|^2\, dt}.$$

2. Total ℓ=2 Energy

Combine all m for the dominant quadrupole:

$$E_{\|,\,\ell=2}(t_i) = \sqrt{\sum_{m=-2}^{2} E_{\|,2m}(t_i)}.$$

3. EMOP Time & Energy

Pick t_{EMOP} that maximizes $E_{\|,\ell=2}(t_i)$:

$$t_{\mathrm{EMOP}} = \arg\max_{t_i} E_{\|,2}(t_i), \quad E_{\mathrm{EMOP}} = E_{\|,2}(t_{\mathrm{EMOP}})$$

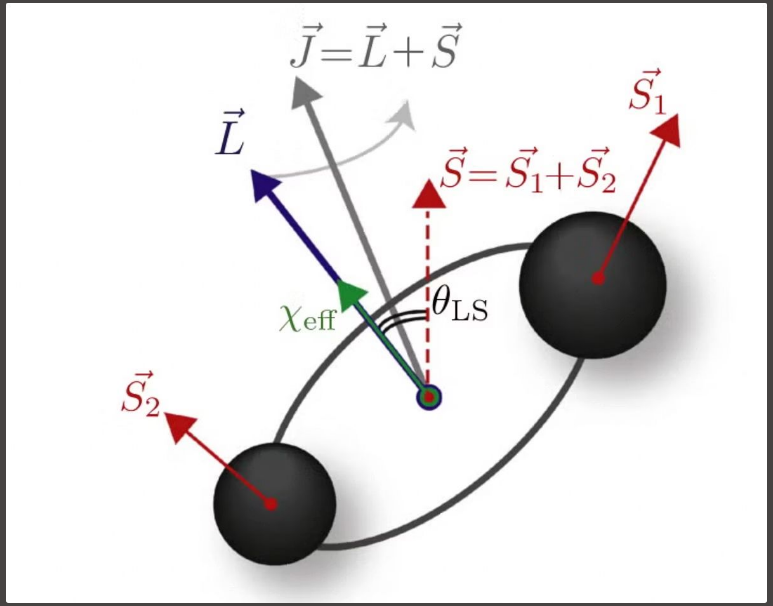
2. Parametrization and degrees of freedom

After lots of experimentations, inspiration from precessing Phenom IMR models and parameterization used in Zhu, et al 2024

1

6-dimensional (6D) Physically Motivated Space

- Asymmetric mass ratio δ = (q-1)/(q+1)
- Symmetric and antisymmetric aligned spins (χs, χa)
- Angle between orbital angular momentum (at ISCO) and remnant spin (θf)
- Angle between remnant spin and recoil direction (ϕk)
- Magnitude of recoil velocity (vk)



2

7-dimensional (7D) Cartesian Spin Space

Includes δ and the two spin vectors

2. An Aside: Why pick at ISCO? Is this a good choice?

6-diensional (6D) Physically Motivated Space

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- Angle between orbital angular momentum (at ISCO) and remnant spin (θf)
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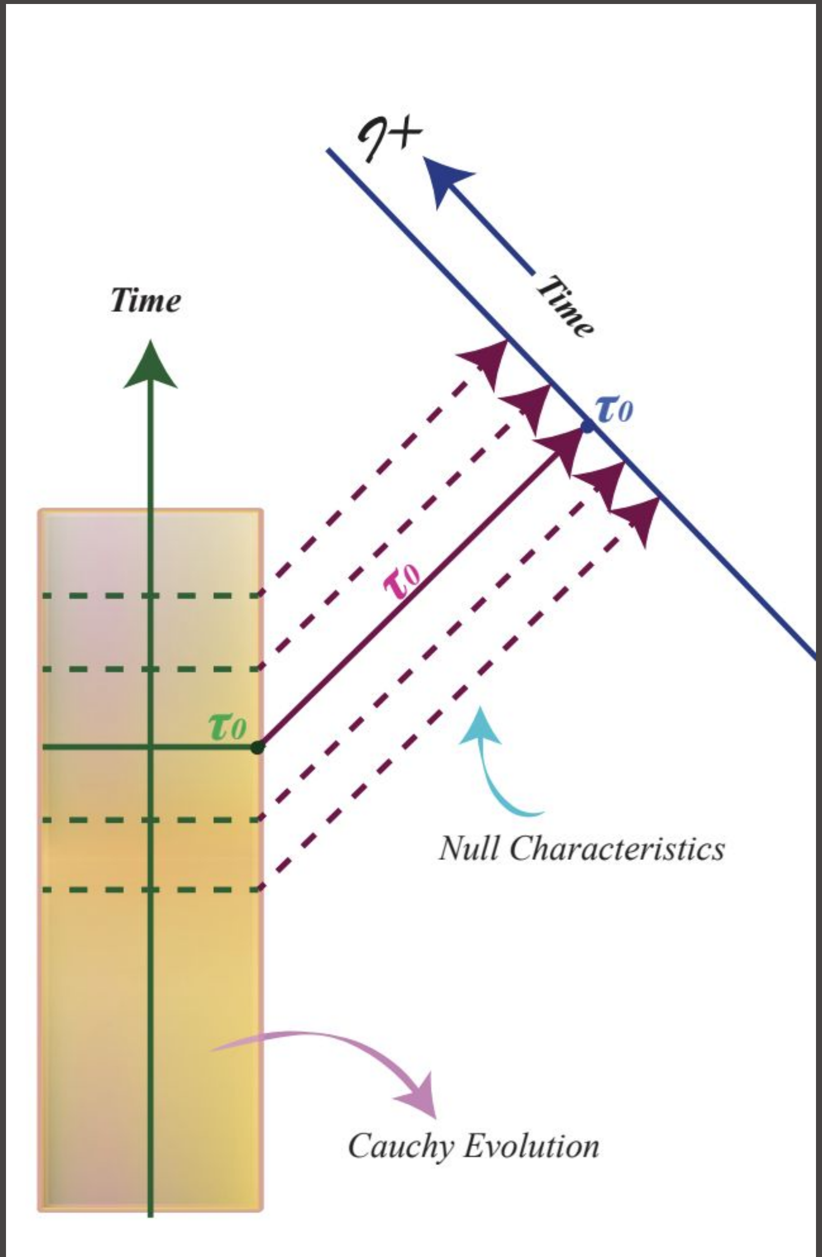
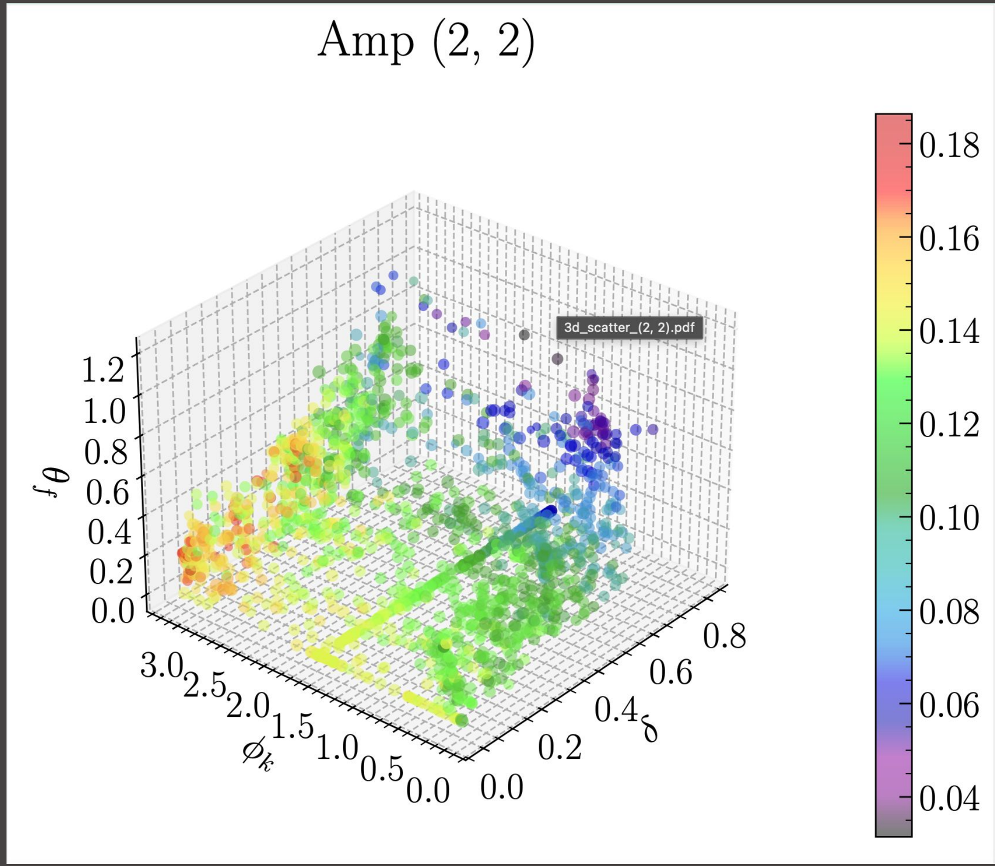
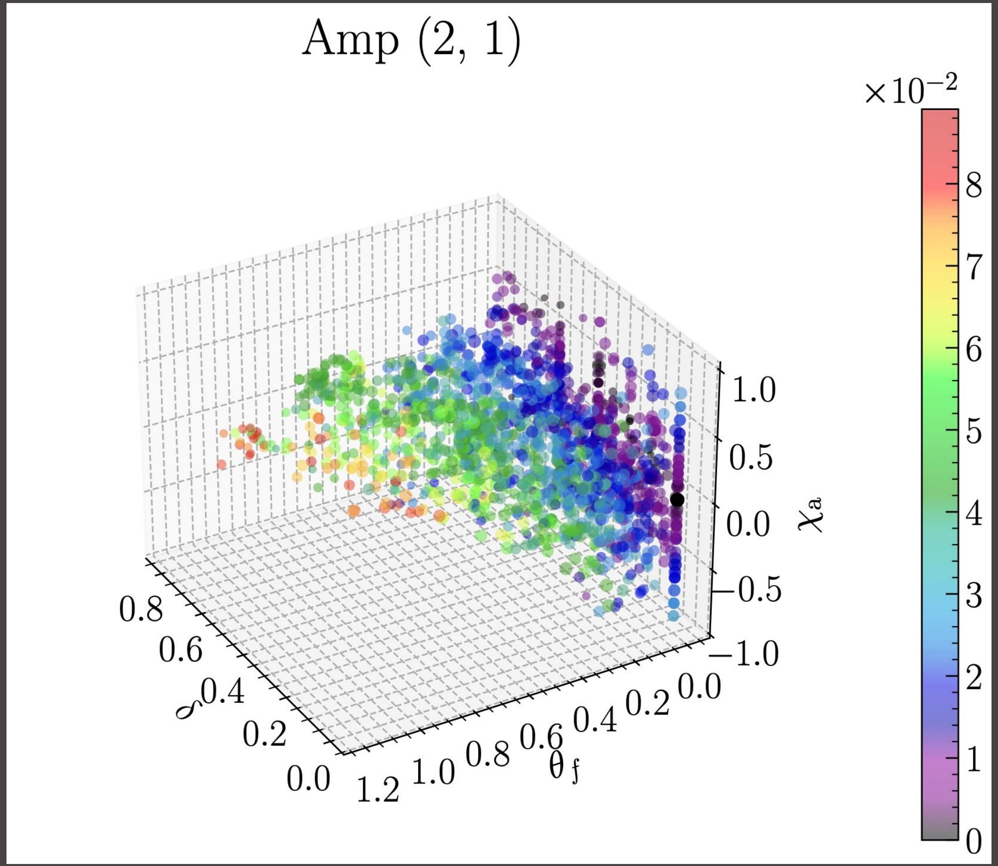
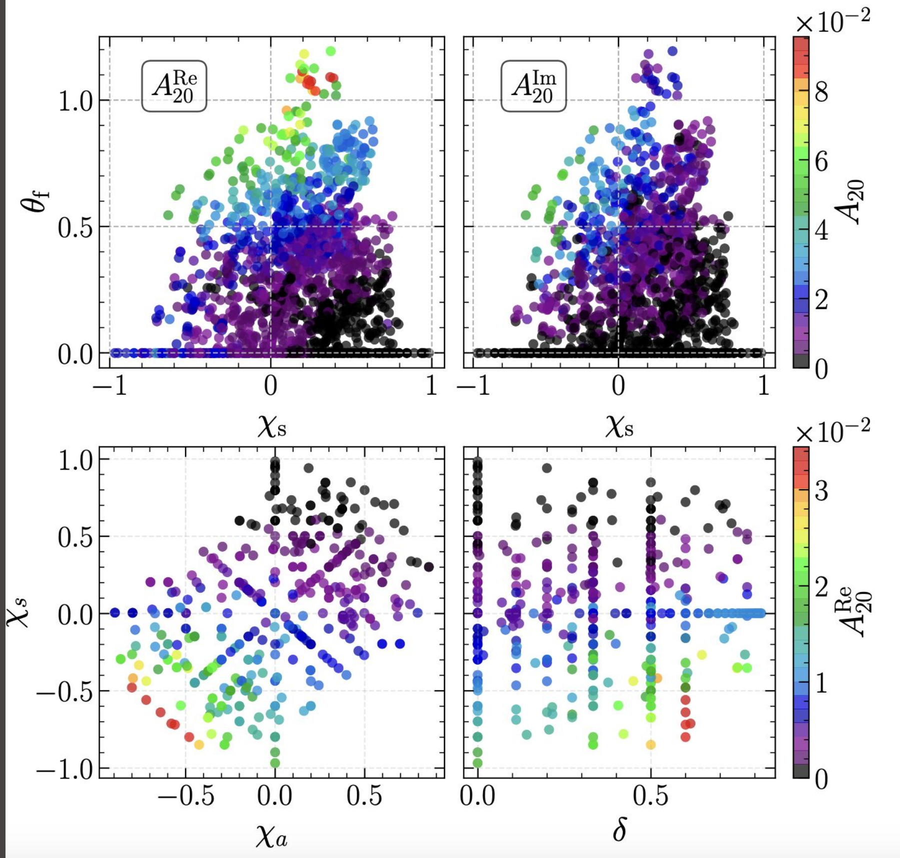
ALL these quantities evolve with time; need to pick a point to define them



Smoothness in the parameter space







- Most mode have a dominant dependence on 2 or 3 parameters;

3. Analytical anzarts for fit is intractable: need for more flexible and data-driven method

Analytic Ansatz for Ringdown Mode Amplitudes with Aligned Spins

(e.g., Forteza, Bhagwat et al. 2023; London et al. 2018, 2022)

$$A^R_{330} = 0.572\,\sqrt{1-4\eta} - 0.144\,(1-4\eta) + 0.035\,\chi\,,$$
$$A^R_{210} = \left|\,0.328\,\sqrt{1-4\eta} + 0.115\,(1-4\eta) - 0.414\,\chi\,\right|\,,$$
$$A^R_{440} = 0.251\,\left(1 + 59.773\,\eta^3 - 16.307\,\eta^2 - 3\eta\right) - 0.011\,\chi_s\,.$$

RELATIVELY SIMPLE ANSATZ WORKS!

- BUT we have NO tractable analytical ansatz for precessing amplitude

- **We use GPR as a way to do regression**

DISCLAIMERS!

When we fit it at 20M after t_EMOP, then --

Simplifying Assumptions

🟦 **Spherical–spheroidal mixing is neglected**

Due to its minimal impact on low to moderate spin black hole remnants.

🟨 **Only fundamental modes ($n = 0$) are considered**

Fits assume all overtones ($n > 0$) have decayed by the start of the fit. The first overtone decay time (τ_1) is roughly ¼ of the fundamental mode decay time (τ_0), i.e., $\tau_1 \approx \frac{1}{4} \tau_0$.

🟥 **Nonlinear (second-order) modes are excluded**

Their contribution is negligible, particularly for (2, m) and (3, ±3) modes.

🟪 **Retrograde modes are ignored**

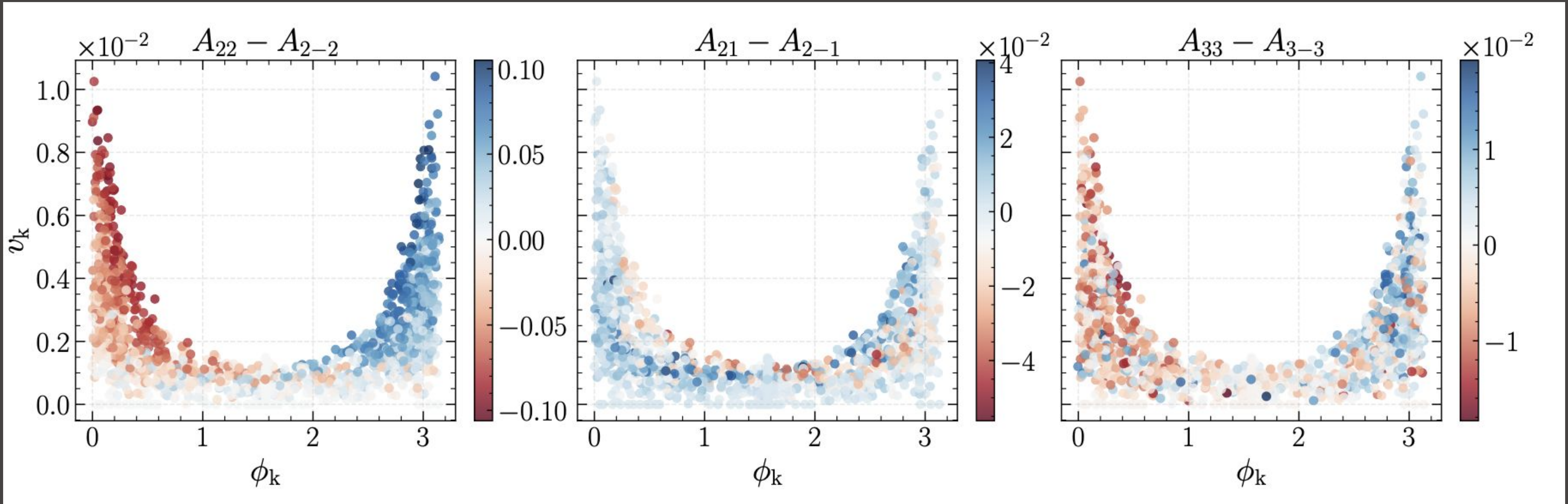
They are weakly excited unless there is angular momentum reversal.

Precessing amplitude trends are fuzzy but to the first order

Mode	δ	θ_f	χ_s	χ_a	ϕ_k	v_k
(2,2)	Negative, strong	Negative, strong	-	-	Positive, strong	$\phi_k \sim \pi$: Positive, strong $\phi_k \sim 0$: Negative, strong
(2,-2)	Negative, strong	Negative, strong	-	-	Negative, strong	$\phi_k \sim \pi$: Negative, strong $\phi_k \sim 0$: Positive, strong
(2,1)	Negative, weak	Positive, strong	Negative, strong	Negative, strong	-	$\phi_k \sim \frac{\pi}{2}$: Positive
(2,-1)	Negative, weak	Positive, strong	Negative, strong	Negative, strong	-	$\phi_k \sim \frac{\pi}{2}$: Positive
(3,3)	Positive, strong	for large δ: Negative, strong	for large δ: Positive	for large δ: Positive	for large δ: Positive	-
(3,-3)	Positive, strong	for large δ: Negative, strong	for large δ: Positive	for large δ: Positive	for large δ: Negative	-
Re(2,0)	for $\chi_s \to -0.8$: Positive, weak	Positive, strong	Negative	Negative, weak	-	-
Im(2,0)	for $\chi_s \to -0.8$: Positive, weak	Positive, strong	Negative, strong	Negative, strong	-	-

Aside:

Breaking of equatorial symmetry depends dominantly on recoil



Training the GPR Models

Kernel: Standard RBF + white noise

Data Selection

- ~1866 numerical relativity (NR) simulations from SXS catalog
- Exclude high-eccentricity and "spin-flip" systems
- Exclude system with abnormal numerical noise

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Preparing the data

Rotate the SXS waveform to **ringdown frame**, aligned with final BH spin

- Waveforms are truncated at $t=100,M$ after merger to avoid numerical noise

- Use fixed QNM frequencies from perturbation theory; Least square fit only amplitude and phase

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Some specifics:

- Each mode is trained separately.
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Training data

A labeled dataset consists of input–output pairs $(\mathbf{x}_i, A_{\ell m}^{(i)})$, where each input $\mathbf{x}_i \in \mathcal{X}$ represents binary parameters, and $A_{\ell m}^{(i)} \in \mathcal{Y}$ is the corresponding QNM mode amplitude.

$$\{(\mathbf{x}_i, A_{\ell m}^{(i)})\}_{i=1}^{N}, \quad \mathbf{x}_i \in \mathcal{X}, \quad A_{\ell m}^{(i)} \in \mathcal{Y}$$

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Testing: In a leave one out framework

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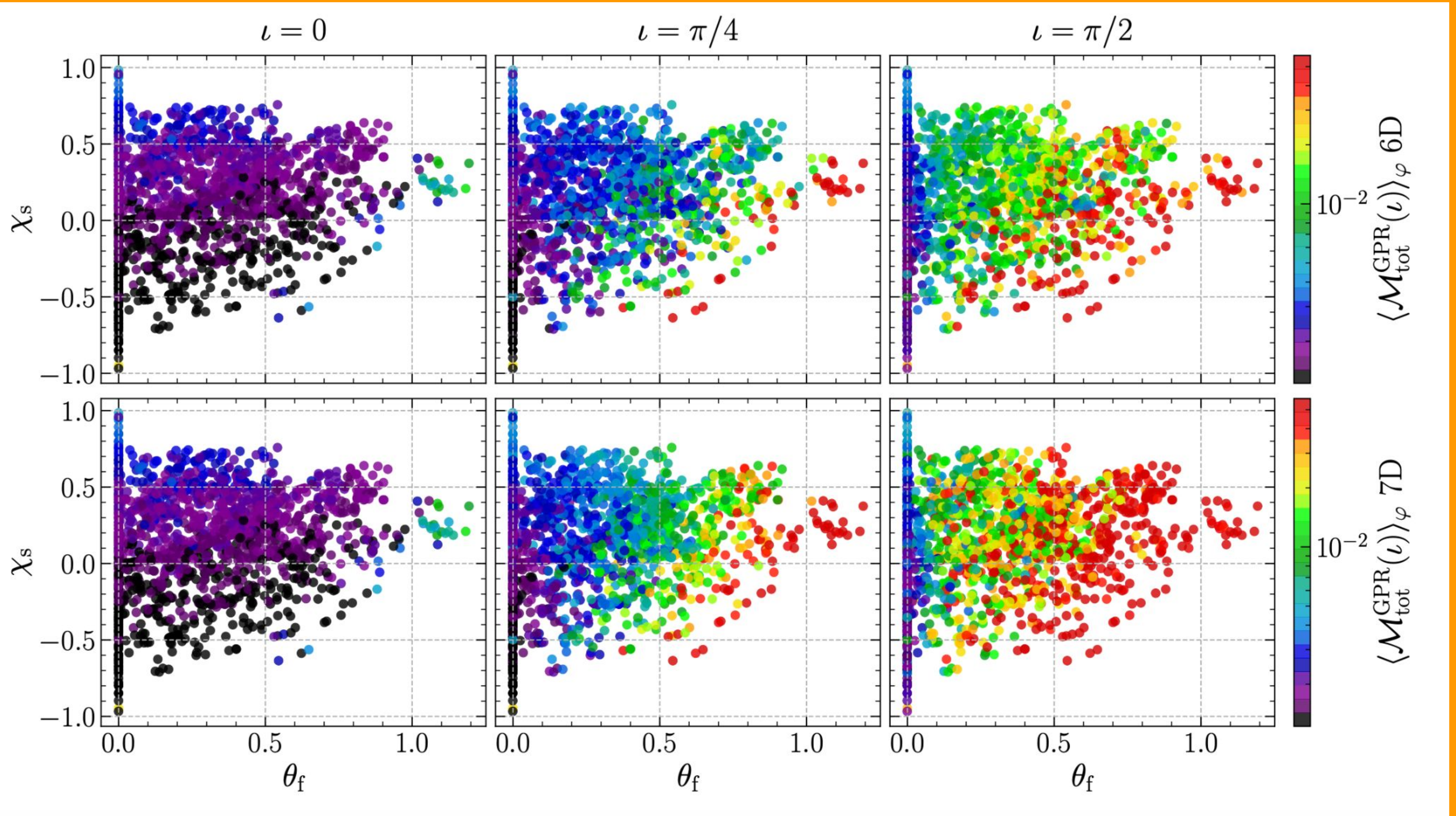
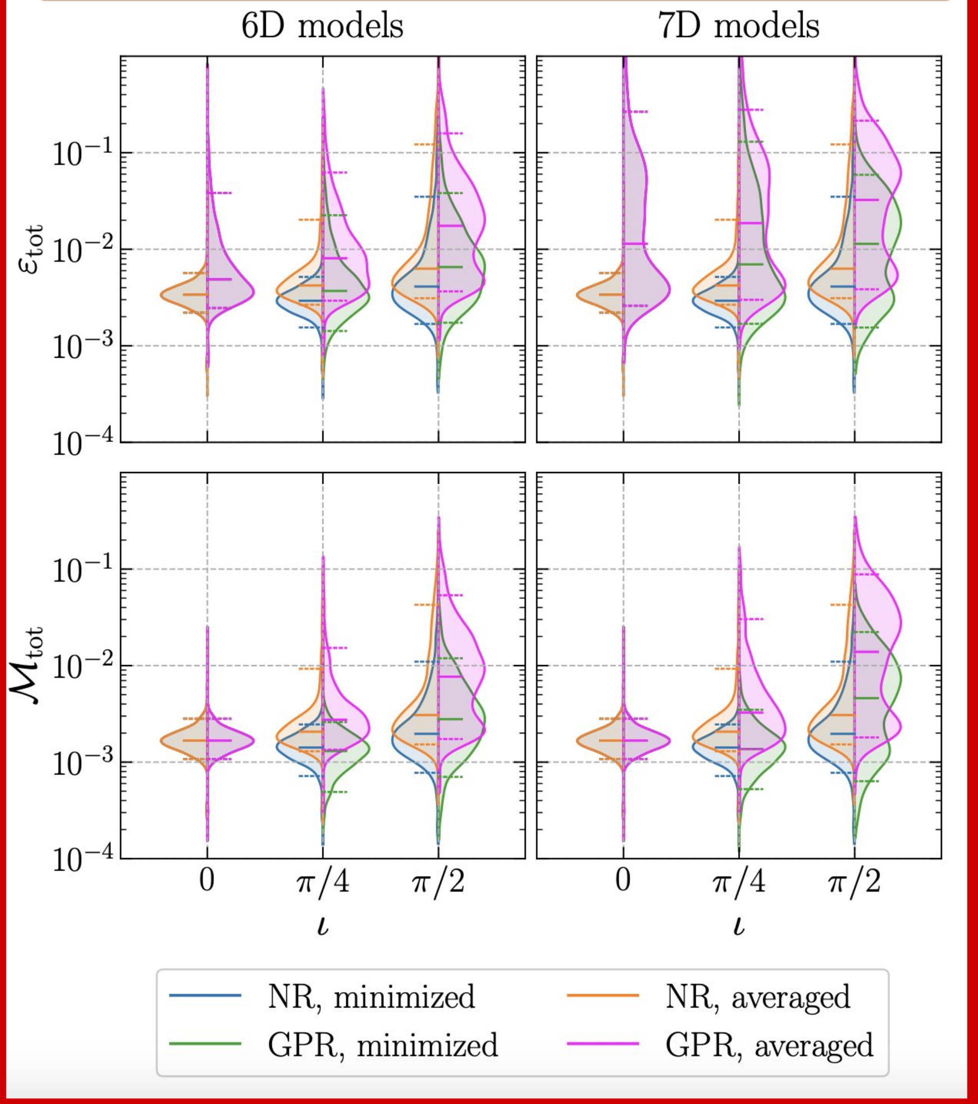
How good are the fits for using in data analysis?

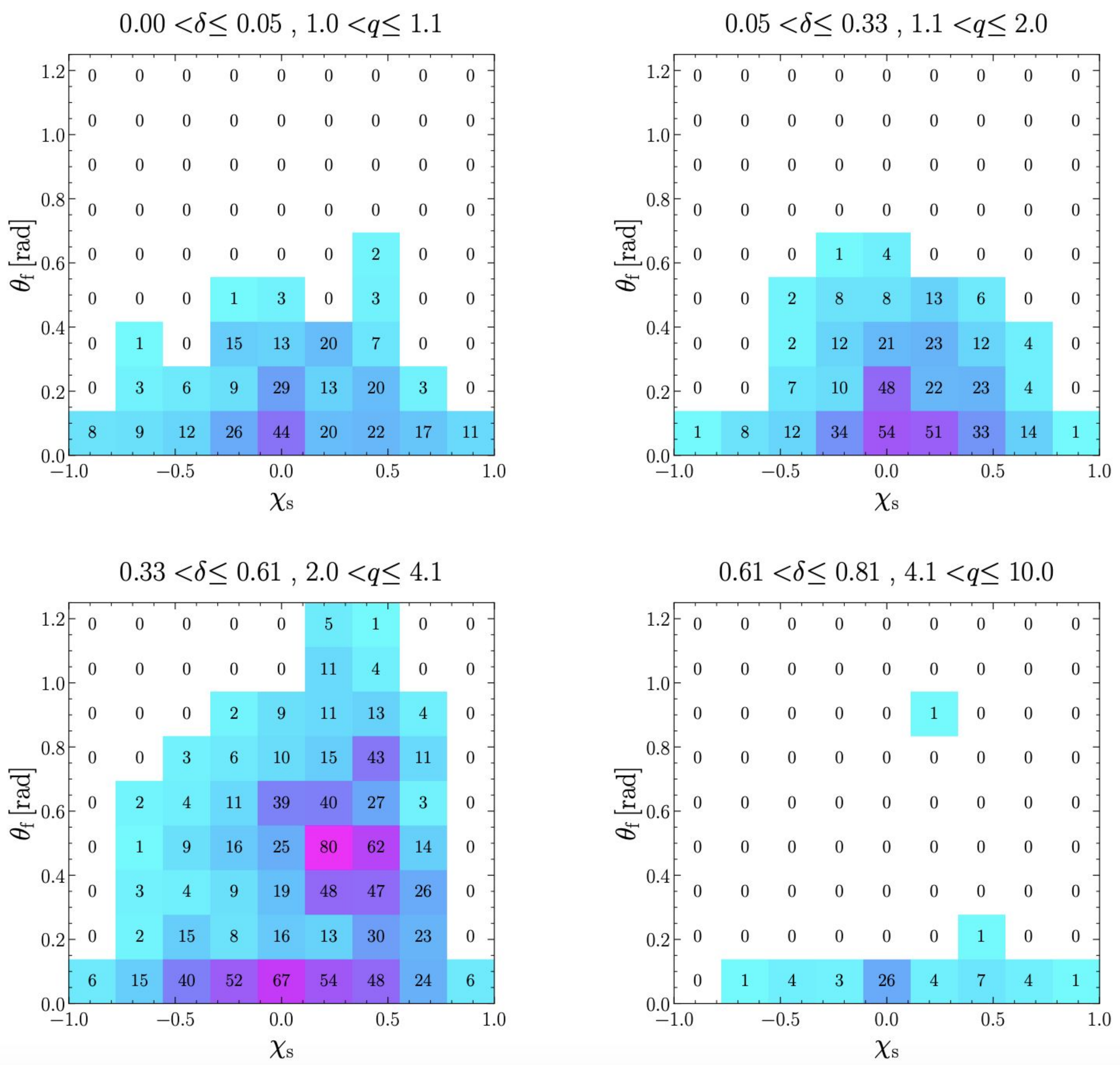
Metric used for judging the domain of validity:

$$\varepsilon_{\rm tot}^{\rm GPR}(\iota,\varphi) = \frac{\int_{t_0}^{100M} \left|h^{\rm NR} - h^{\rm GPR}\right|^2 dt}{\int_{t_0}^{100M} \left|h^{\rm NR}\right|^2 dt},$$

$$\mathcal{M}_{\rm tot}^{\rm GPR}(\iota,\varphi) = 1 - \frac{\langle h^{\rm NR}|h^{\rm GPR}\rangle}{\sqrt{\langle h^{\rm NR}|h^{\rm NR}\rangle \langle h^{\rm GPR}|h^{\rm GPR}\rangle}}$$







Next steps:

1. Model for the phase
2. Model for the non linear effect and include modelling for (4,4) mode.
3. Combine various NR catalogs to and check if the we can populate the parameter space better

Ongoing: Integrate this into PE pipelines and run on interesting events like GW190521!!

A major challenge for accuracy: uneven and sparse coverage of the data

Any advice is welcome!!!

CHECK OUT OUR AMPLITUDE FIT PACKAGE: POSTMERGER

THANK YOU!! And any questions!!

EXTRA SLIDES

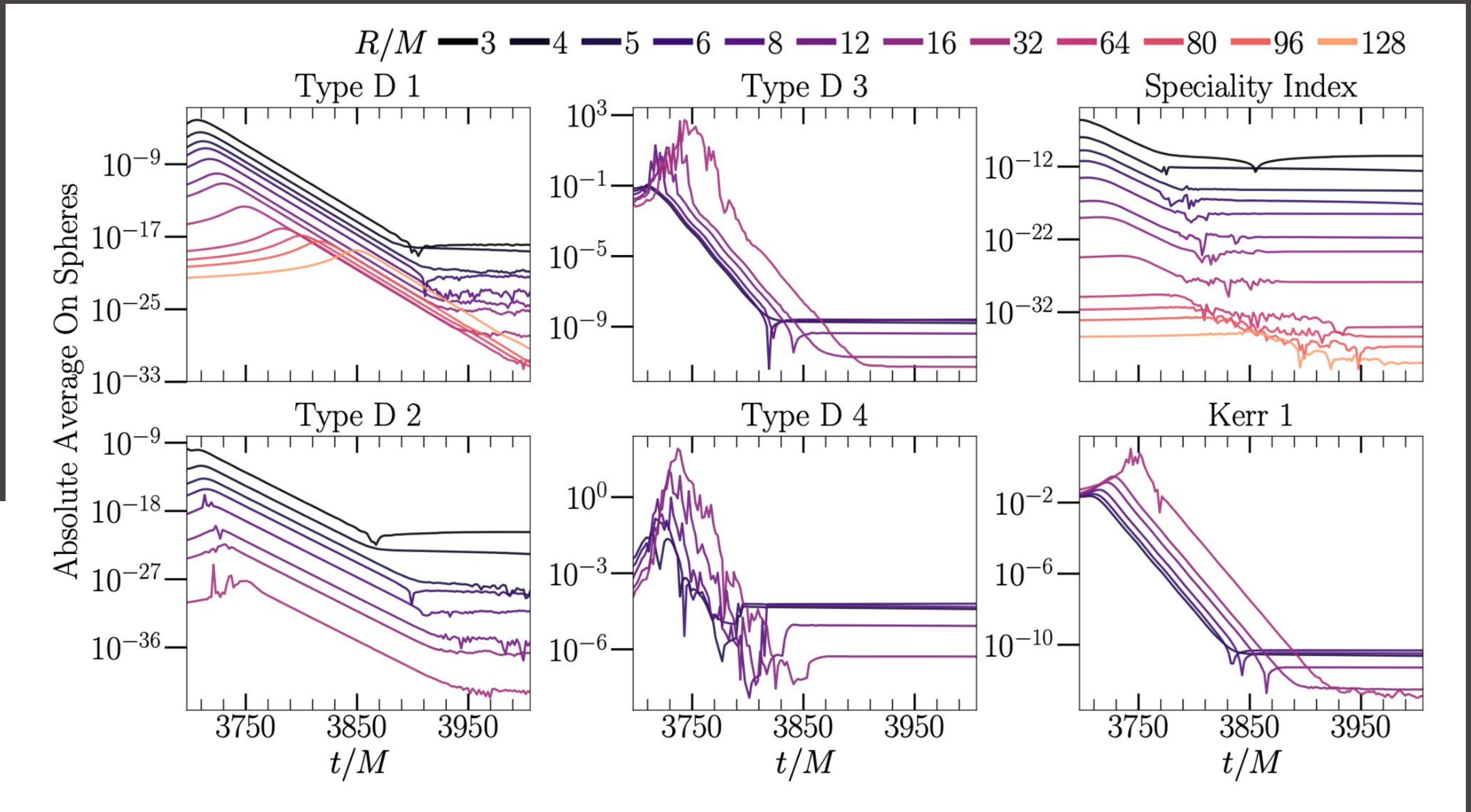
How good are the fits for using in data analysis?

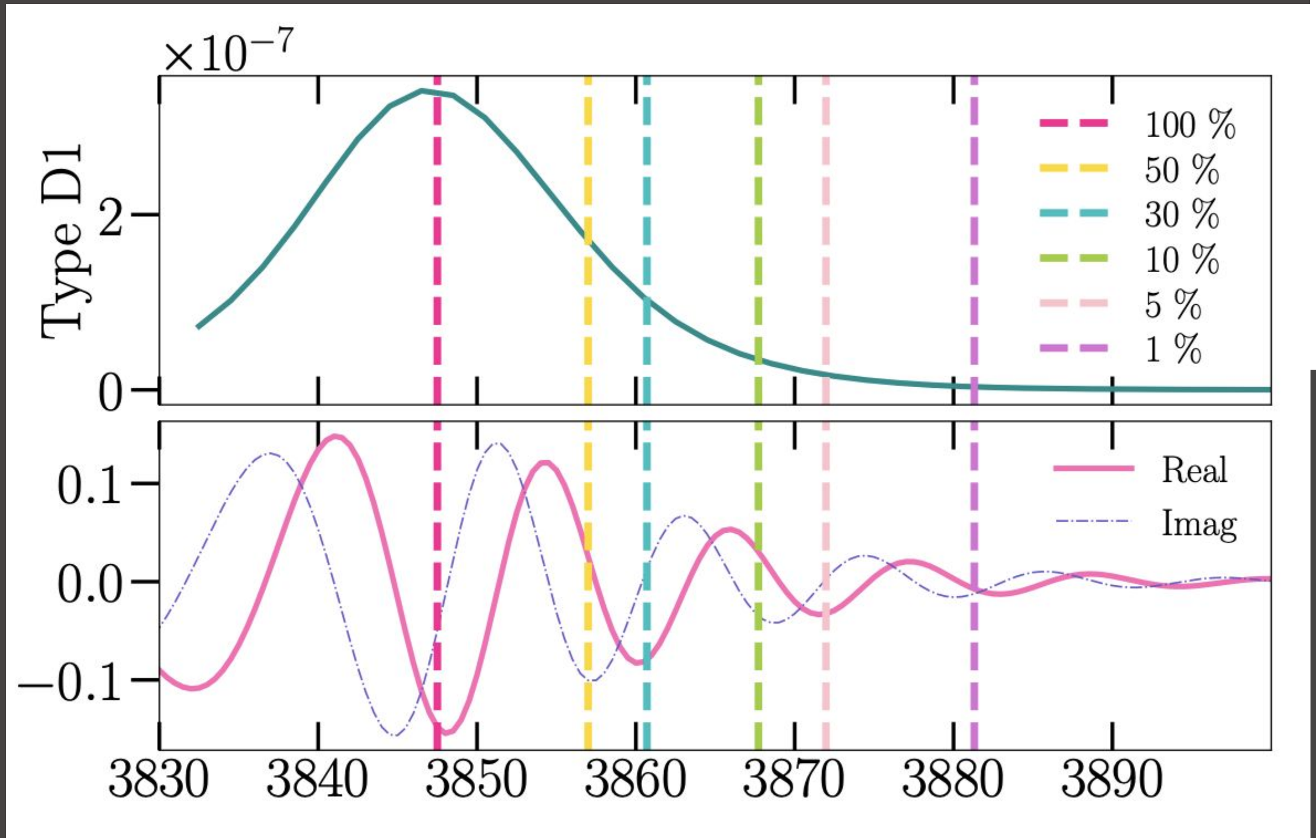
		Absolute error		Relative error	
		Precessing fit	Aligned fit	Precessing fit	Aligned fit
Precessing sources	A_{22}	$0.0056^{+0.0147}_{-0.0052}$	$0.020^{+0.062}_{-0.019}$	$0.049^{+0.164}_{-0.046}$	$0.18^{+0.64}_{-0.17}$
	A_{21}	$0.0045^{+0.0114}_{-0.0041}$	$0.020^{+0.026}_{-0.019}$	$0.15^{+0.97}_{-0.14}$	$0.69^{+0.28}_{-0.62}$
	A_{33}	$0.0015^{+0.0045}_{-0.0013}$	$0.0063^{+0.0273}_{-0.0059}$	$0.086^{+0.420}_{-0.079}$	$0.47^{+1.50}_{-0.45}$
Aligned sources	A_{22}	$0.0008^{+0.0027}_{-0.0007}$	$0.0022^{+0.0021}_{-0.0011}$	$0.0067^{+0.0279}_{-0.0062}$	$0.018^{+0.017}_{-0.008}$
	A_{21}	$0.0010^{+0.0029}_{-0.0009}$	$0.0004^{+0.0030}_{-0.0004}$	$0.060^{+1.006}_{-0.055}$	$0.023^{+0.685}_{-0.020}$
	A_{33}	$0.0004^{+0.0019}_{-0.0004}$	$0.0006^{+0.0013}_{-0.0005}$	$0.021^{+0.577}_{-0.019}$	$0.030^{+0.285}_{-0.023}$

Start of the linear regime and non-linearity

The transition from non-linear to linear regime is tricky

Spatial Localisation Challenge: In a dynamic spacetime, identifying which region approximates the Kerr geometry is complex and depends on coordinate choices and slicing.





- **Quantifying 'Closeness':** Multiple analytical measures exist, but each evaluates different geometric properties.
- **Example – Speciality Index:** One such measure is the speciality index, which assesses how nearly the spacetime's Weyl tensor satisfies the conditions for Petrov type D—characteristic of the Kerr solution.

Bhagwat, Okunkova, et al, 2018