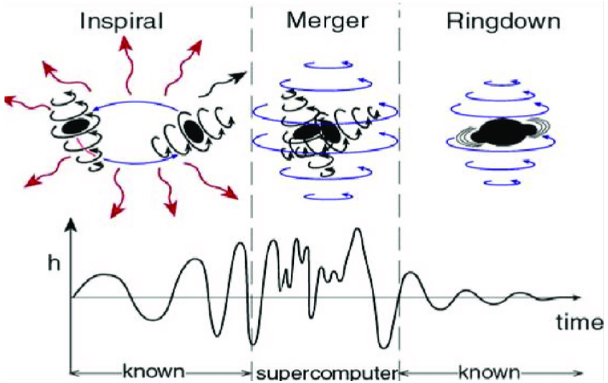
Some new developments in black hole spectroscopy

Emanuele Berti, Johns Hopkins University

IBS CTPU-CGA 2025 Workshop on quasinormal modes and black hole perturbations
Daejeon, Korea
May 26 2025



Talk based largely on:

Cotesta+ 2201.00822, Cheung+ 2208.07374, Baibhav+ 2302.03050, Redondo-Yuste+ 2308.14796, Cheung+ 2310.04489, Yi+ 2403.09767, Oshita+ 2503.21276, Yang+ 2504.06072…

JOHNS HOPKINS UNIVERSITY

How black hole spectroscopy can put general relativity to the test FREE

10 April 2025

Einstein's theory makes specific predictions about the nonlinear spacetime oscillations that propagate from merging black holes. Next-generation gravitational-wave detectors should enable researchers to evaluate those predictions.

Emanuele Berti, Mark Ho-Yeuk Cheung, and Sophia Yi

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Black hole spectroscopy: from theory to experiment

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To appear very soon!

Johns Hopkins gravitational wave group

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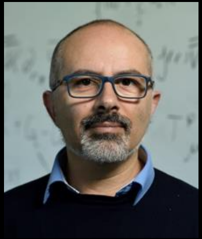

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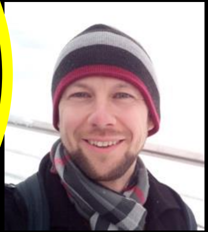
K.Wong

V.Baibhav

V.Strokov



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Black hole spectroscopy: from theory to experiment
Overtones, agnostic spectroscopy, and current observations
The future: tests of strong gravity with nonlinear modes
Spectral instabilities and exceptional points*
The future: tests of theories beyond general relativity**

Black hole spectroscopy: from theory to experiment

The Schwarzschild metric

November 18, 1915:
Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$r = 0$: physical curvature singularity

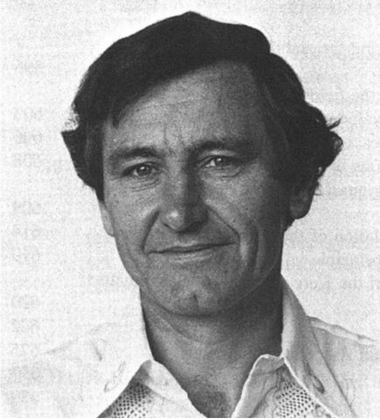
$r = \dfrac{2GM}{c^2}$: "Schwarzschild radius"

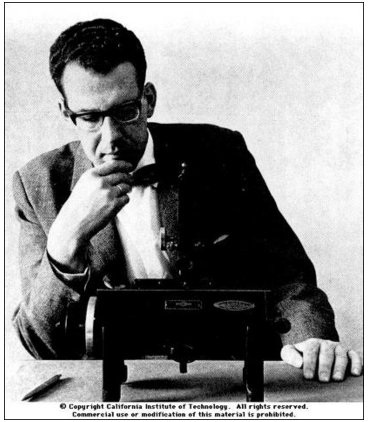
Key questions:

1) **Is the Schwarzschild "singularity" the end point of gravitational collapse?**
   1939: Oppenheimer-Snyder, yes (for dust, in spherical symmetry)
   1963: Lifshitz-Khalatnikov, not generically
   Wheeler (following Schmidt's discovery of a quasar): do nuclear forces halt collapse?
   Answer: Penrose-Hawking singularity theorems

2) **If so, is the Schwarzschild solution stable?**
   Answer: black hole perturbation theory, quasinormal modes (QNMs) and black hole spectroscopy

Are black holes stable? The Golden Age (1963-1970s)





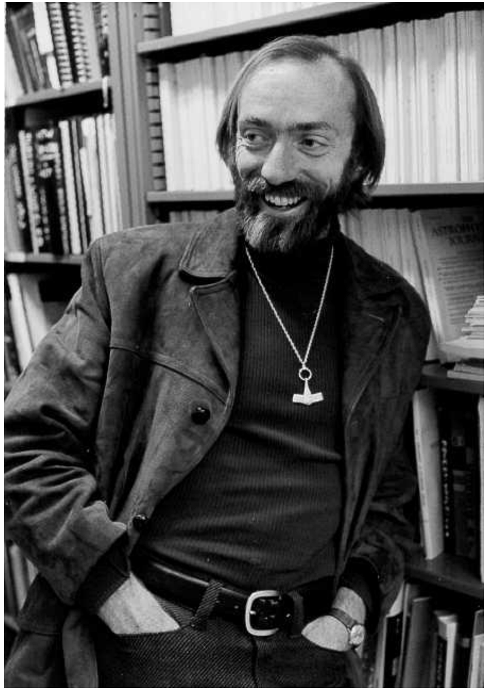
1963:

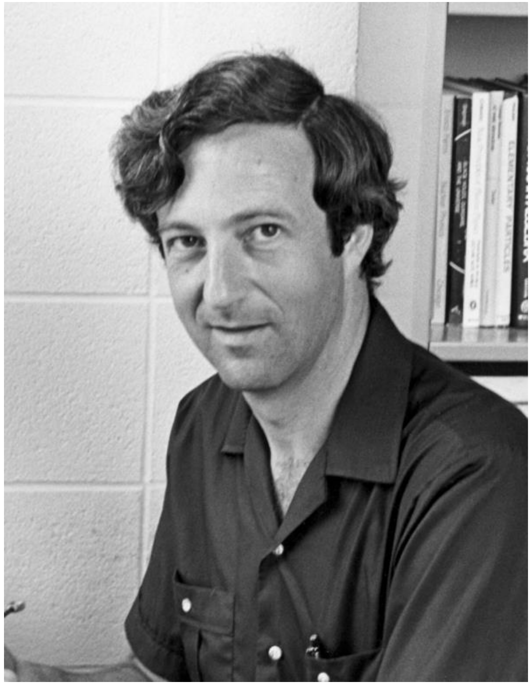
- ✓ Roy Kerr: **rotating black holes**

- ✓ Maarten Schmidt at Caltech discovers the first quasar, 3C273 at z=0.15 – extragalactic!
- ✓ Must be compact and outshines the brightest galaxies: first supermassive black hole
- ✓ Giacconi-Gursky propose orbital satellite to study X-ray sources
  1964: Cygnus X-1, first stellar-mass black hole

Late 1960s and 1970s:

- ✓ “Golden Age” of black hole physics
- ✓ Misner-Thorne-Wheeler, “Gravitation”
- ✓ Kip Thorne and students (including Saul Teukolsky) lay the foundations to understand black hole stability and dynamics





QNMs and overtones: some milestones. Phase 1 – theory development

1957 – Regge-Wheeler axial (odd-parity) perturbations as a scattering problem, boundary conditions not understood
1970 – Zerilli polar (even-parity) perturbations, much harder!

Scalar, electromagnetic and gravitational perturbations of a Schwarzschild BH: Regge-Wheeler/Zerilli equations

$$f\frac{d}{dr}\left(f\frac{d\Phi}{dr}\right)+\left[\omega^2-fV_\pm\right]\Phi=0$$

$$V_s=\frac{\ell(\ell+1)}{r^2}+\left(1-s^2\right)\frac{r_H}{r^3}$$

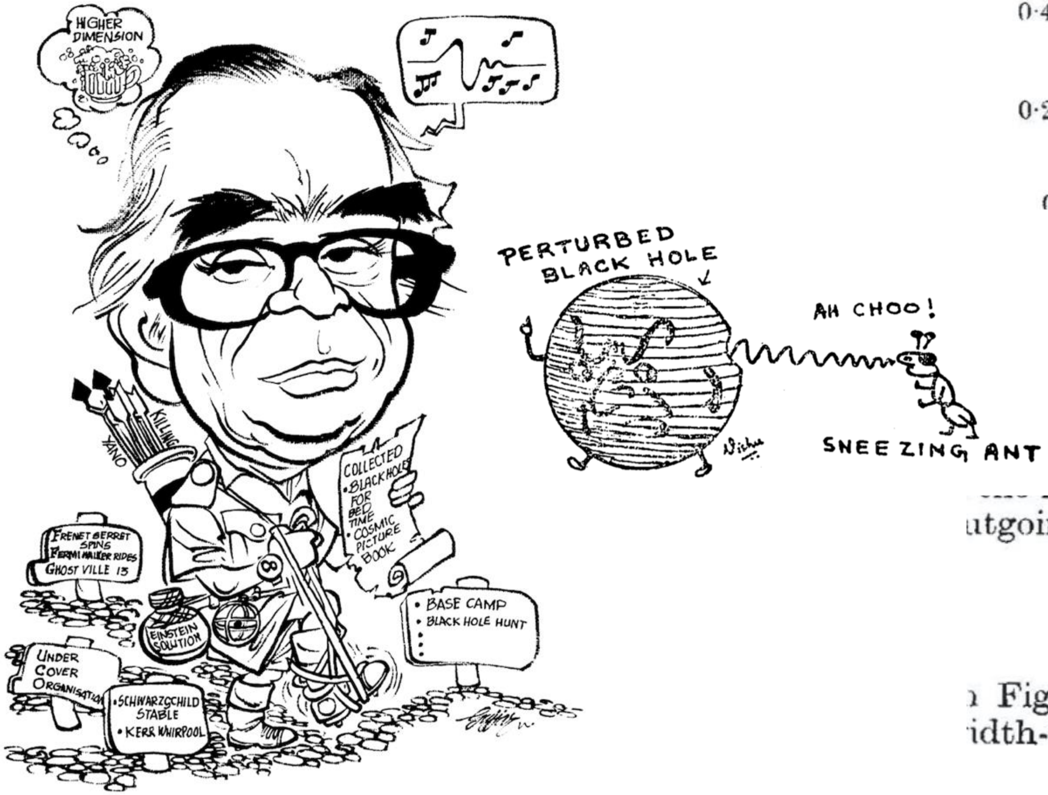
$$V_-=\frac{\ell(\ell+1)}{r^2}-\frac{3r_H}{r^3}$$

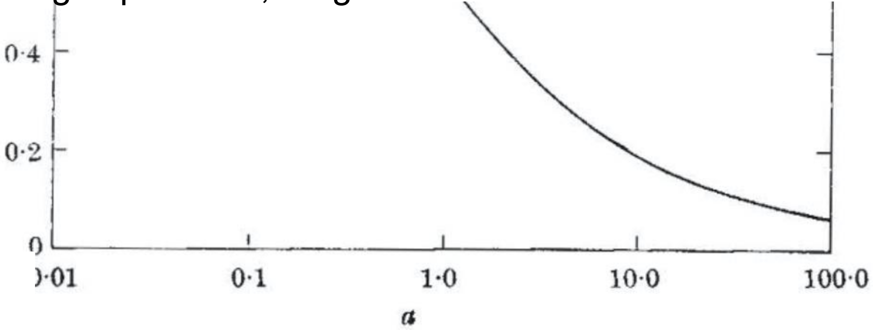
$$V_+=\frac{9\lambda r_H^2 r+3\lambda^2 r_H r^2+\lambda^2(\lambda+2)r^3+9r_H^3}{r^3\left(\lambda r+3r_H\right)^2}$$

[for reviews: Berti-Cardoso-Starinets, gr-qc/0905.2975; EB-Cardoso-Carullo+, to appear]

QNMs and overtones: some milestones. Phase 1 – theory development

1957 – Regge-Wheeler axial (odd-parity) perturbations as a scattering problem, boundary conditions not understood
1970 – Zerilli polar (even-parity) perturbations, much harder!
1970 – Vishveshwara now boundary conditions are clear: scattering experiment, "ringdown waves"





. The fraction F of the incident energy carried by the scattered ng wave packet at spatial infinity plotted as a function of the parameter a.

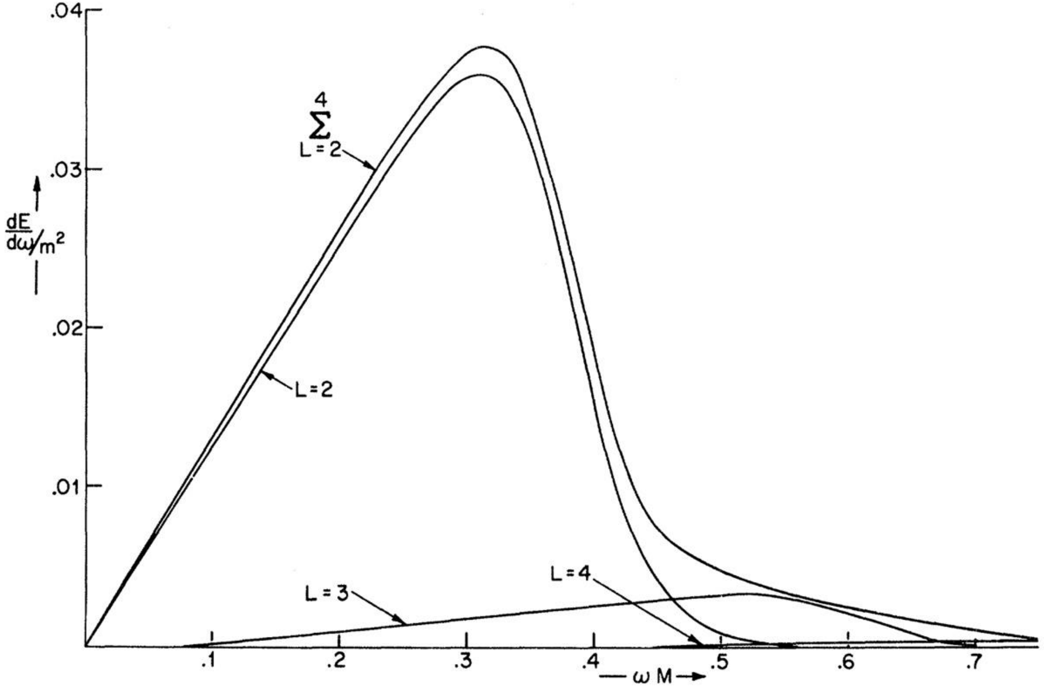
... ratio of the energies carried by the incoming and th utgoing wave packets and is readily computed as

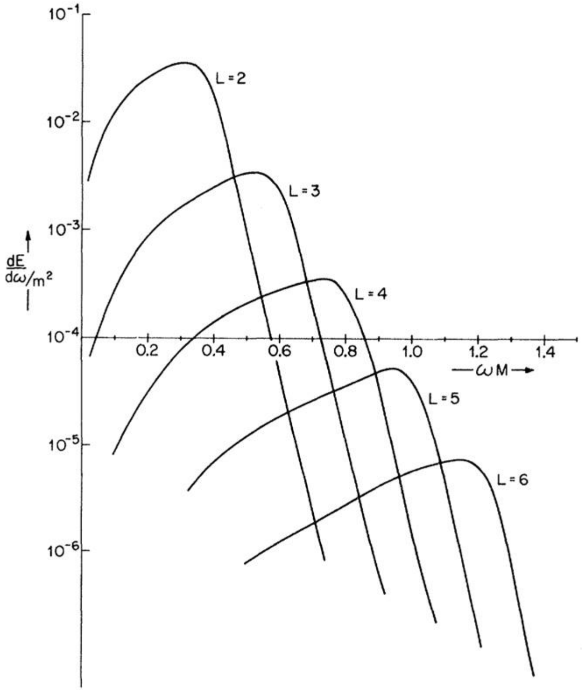
$$F = \frac{\int(\psi_{out})^2\,dx}{\int(\psi_{in})^2\,dx} = \left(\frac{2a}{\pi}\right)^{1/2} \int(\psi_{out})^2\,dx$$

ı Fig. 4 the fraction F is plotted as a function of th idth-parameter a. For an incident wave packet, th

QNMs and overtones: some milestones. Phase 1 – theory development

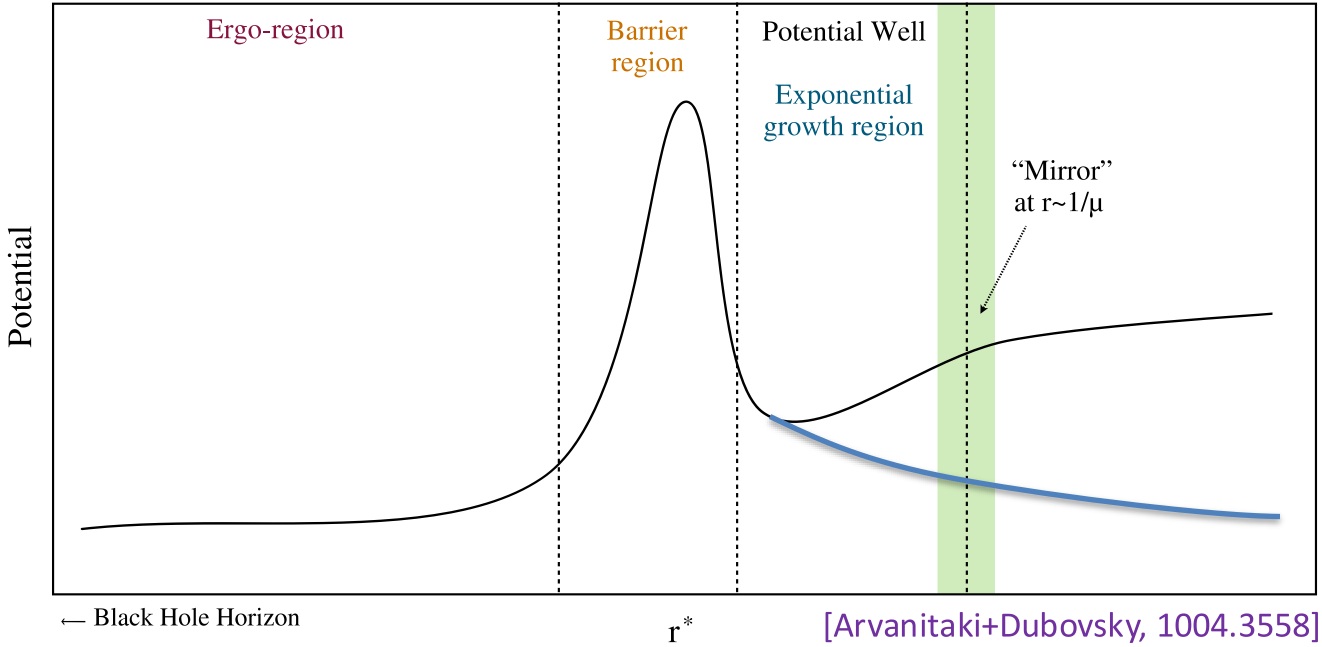
1957 – Regge-Wheeler axial (odd-parity) perturbations as a scattering problem, boundary conditions not understood
1970 – Zerilli polar (even-parity) perturbations, much harder!
1970 – Vishveshwara now boundary conditions are clear: scattering experiment, “ringdown waves”
1971 – Press ringdown waves are free oscillation modes of the black hole
1971 – Davis-Ruffini-Press-Price these modes are excited when radially falling particles cross the light ring
1973 – Teukolsky formalism for Kerr perturbations





QNMs and overtones: some milestones. Phase 2 – overtones and spectroscopy

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Phi = V\Phi$$



[Arvanitaki+Dubovsky, 1004.3558]

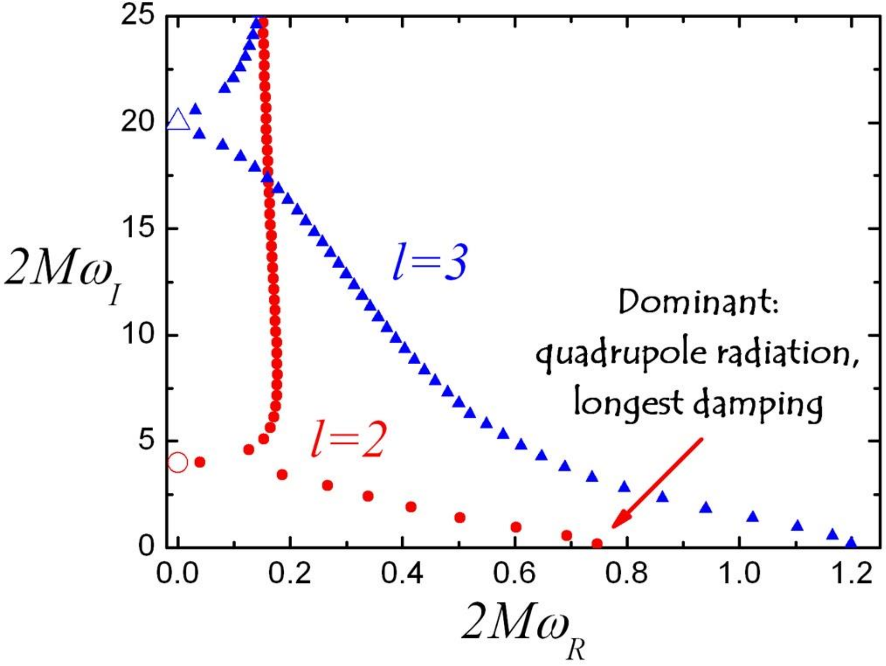
Quasinormal modes:

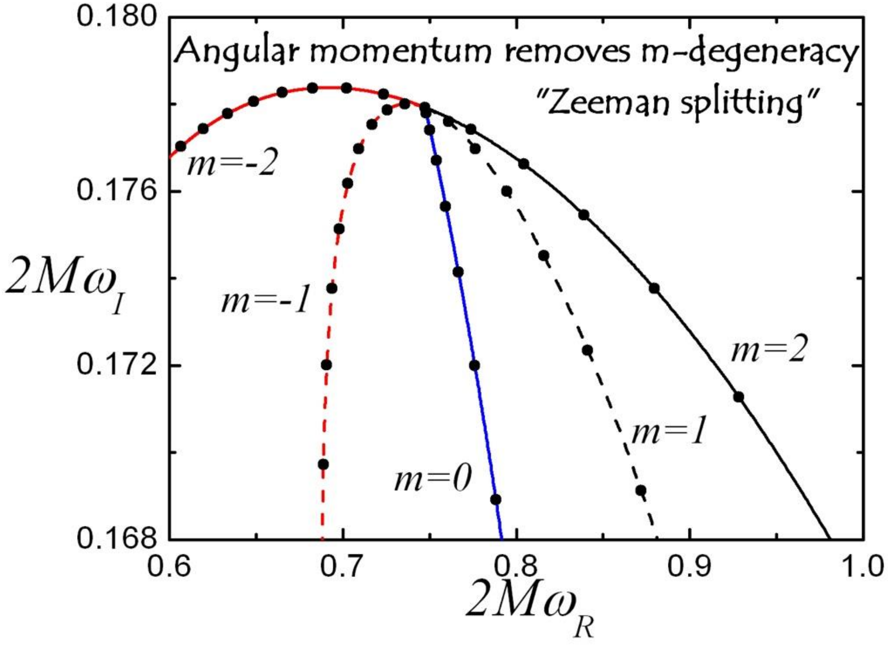
- Ingoing waves at the horizon, outgoing waves at infinity
- Spectrum of damped modes ("ringdown")

Massive scalar field:

- Superradiance: black hole bomb when $0 < \omega < m\Omega_H$ [Press-Teukolsky 1972]
- Hydrogen-like, unstable bound states [Detweiler 1980, Zouros+Eardley, Dolan…]

QNMs and overtones: some milestones. Phase 2 – overtones and spectroscopy



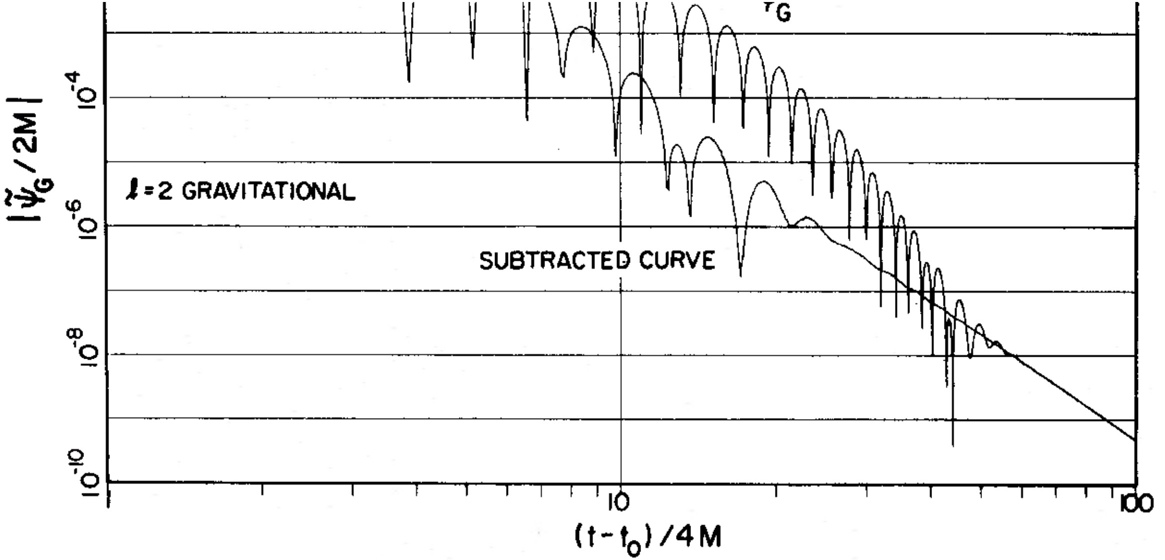


- **One mode** fixes mass and spin – and the whole spectrum!
- **N modes:** N tests of GR dynamics…**if** they can be measured
- **Measurement requires understanding of QNM excitation** (as in atomic physics!)
- **Retrograde modes, nonlinear modes** (not negligible)

[Berti-Cardoso-Will, gr-qc/0512160; EB+, gr-qc/0707.1202]

QNMs and overtones: some milestones. Phase 2 – overtones and spectroscopy

1975 – Chandrasekhar-Detweiler first numerical calculation of overtones in Schwarzschild, with limited accuracy
1978 – Cunningham-Price-Moncrief observe overtones in perturbative calculation of collapse to Schwarzschild
1979 – Detweiler first complete calculation of the Kerr spectrum, **“black hole spectroscopy”**
“After the advent of gravitational wave astronomy, the observation of [the black hole’s] resonant frequencies might finally provide direct evidence of black holes with the same certainty as, say, the 21 cm line identifies interstellar hydrogen.”

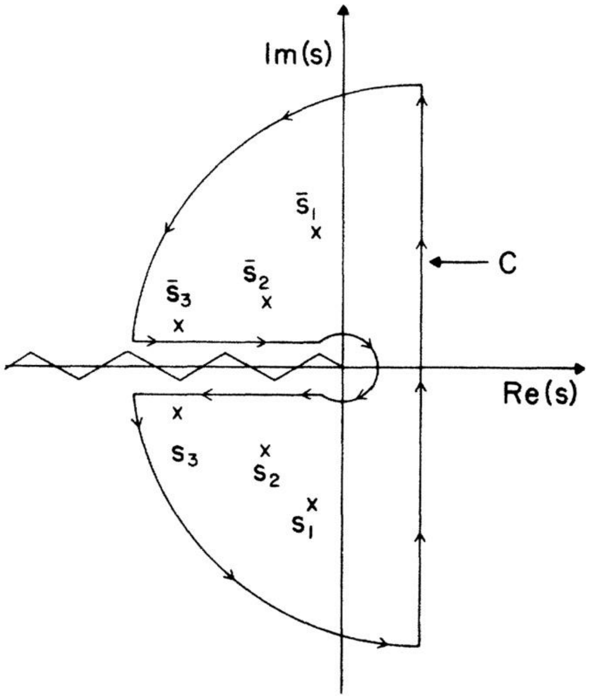


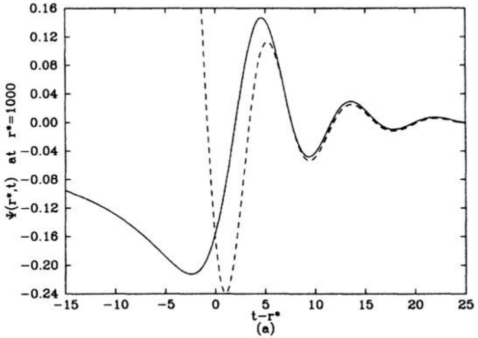
. 12.—Fundamental quasi-normal ringing subtracted from $\tilde{\psi}_G$ for quadrupole gravitational radiation. The subtrac
ients $\tilde{\psi}_G - A\exp(-\omega_I t)\sin(\omega_R t - \delta)$ where ω_R, ω_I are the real and imaginary parts of the least damped quas
ncy. The parameters A and δ are chosen to minimize oscillations at late times in the subtracted curve.

hase adjusted to minimize oscillations at late times. The result is plotted in Figure 12. The signatur
d quasi-normal mode $\omega_2 = (0.69687 + 0.54938i)(2M)^{-1}$ is a ratio between the magnitudes of adjacen
imum and minimum) of

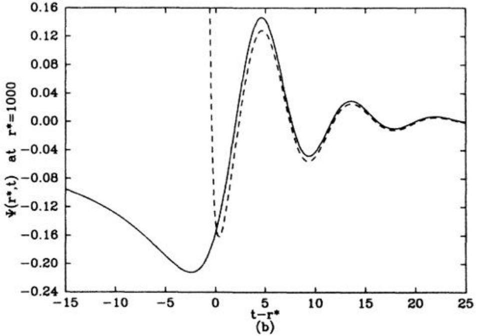
QNMs and overtones: some milestones. Phase 3 – excitation, pre-NR

1986 – Leaver Green's function, continued fractions, excitation factors (also **Andersson**) – **by analogy with H_2^+ ion!**
1989 – Echeverria quantifies how well you can measure mass and spin from a single mode
1998 – Flanagan-Hughes ringdown may have as much SNR as inspiral
2002 – Hod-Dreyer are QNMs related with Bekenstein's ideas on area quantization and LQG?
2003 – Dreyer+ revive/rebrand Detweiler's idea of "black hole spectroscopy"
2005 – Berti-Cardoso-Will SNRs, measurability, QNM frequencies+fits, **overtones vs. higher multipoles**





Radial infall vs. one mode



Radial infall vs. six modes

QNMs and overtones: some milestones. Phase 4 – excitation, post-NR

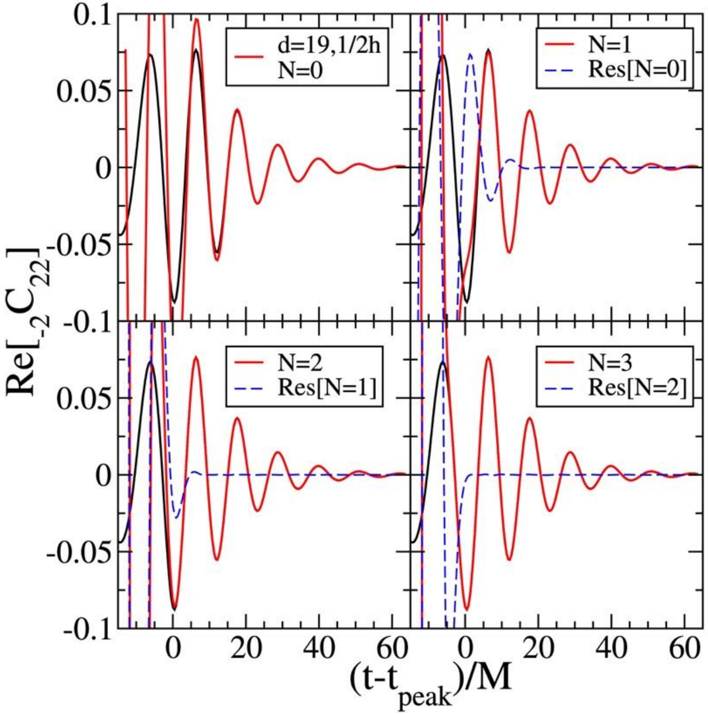
2005 – Pretorius numerical relativity breakthrough: merger simulations. Soon after Brownsville/RIT, Goddard…

2006 – Berti-Cardoso systematic calculation of Kerr excitation factors

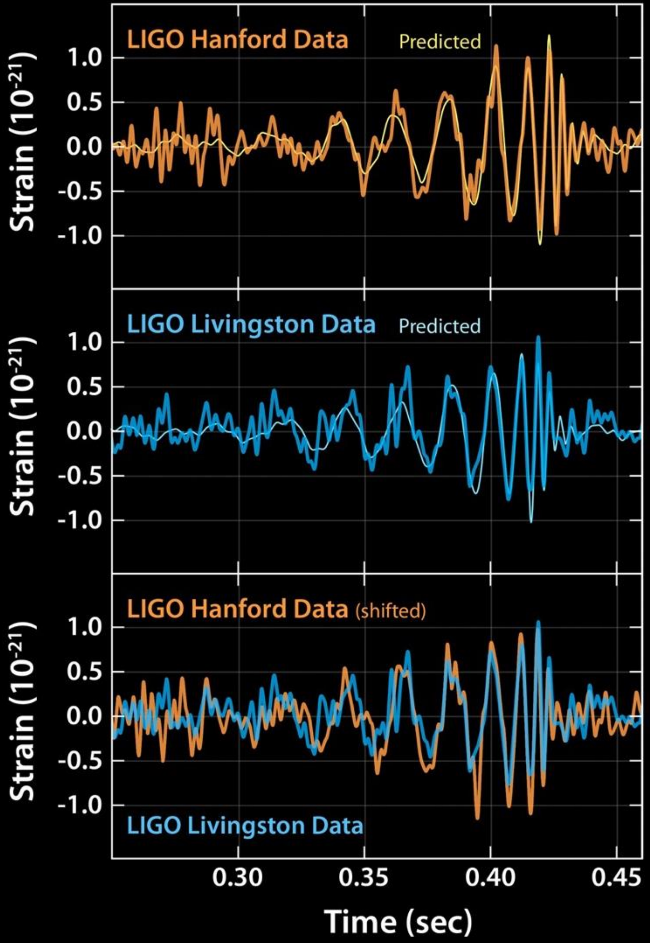
2006 – Buonanno-Cook-Pretorius fit overtones to Pretorius' equal-mass simulations – but are they physical?
Spherical-spheroidal mixing: numerical simulations use the "wrong" basis (Berti-Cardoso-Casals 2005)

2007 – Berti+ quantify excitation of higher multipole QNMs in unequal-mass, nonspinning mergers

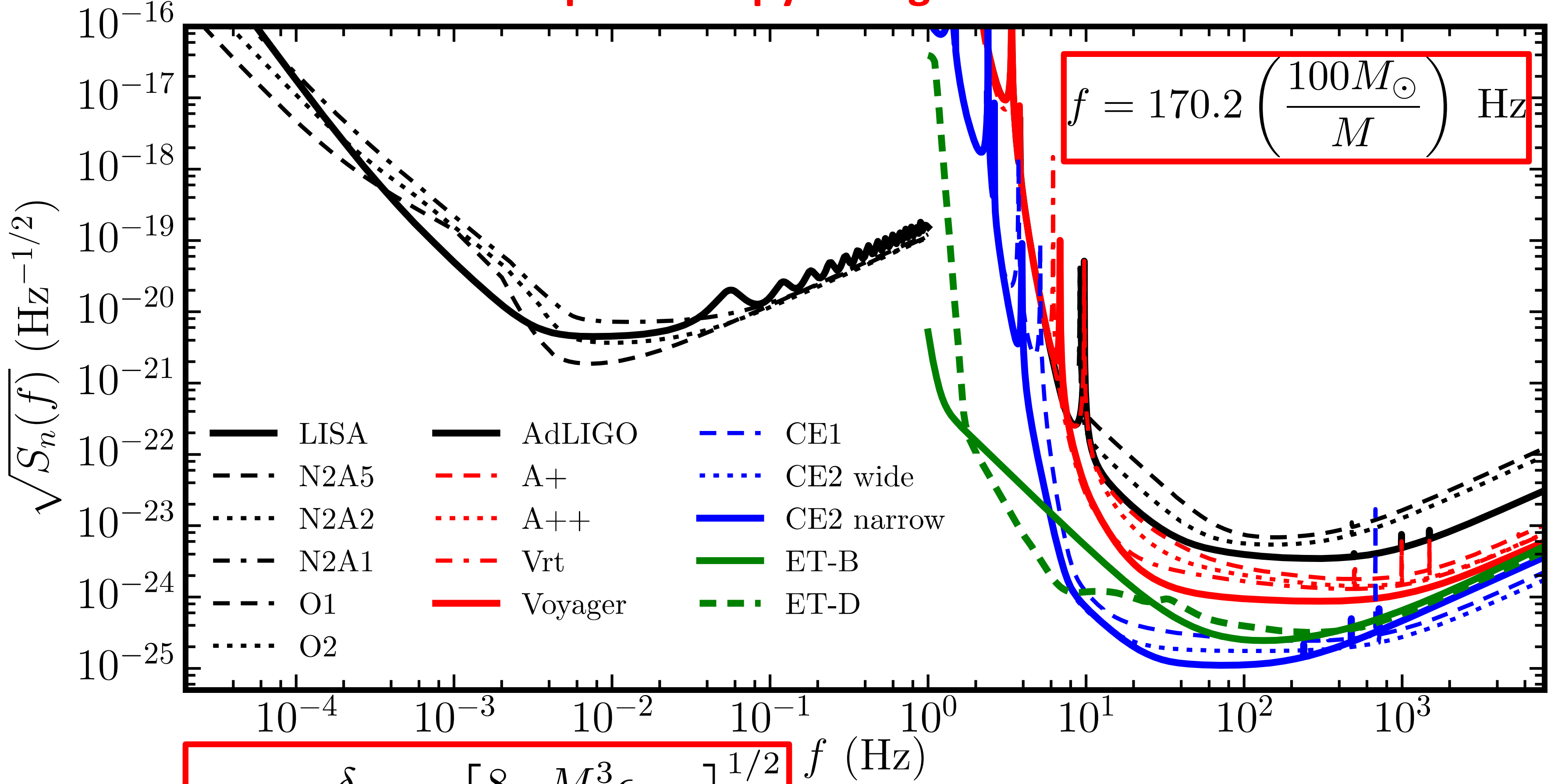
2012, 2014 – Gossan+, Meidam+ first Bayesian study of ringdown



A new Golden age: GW150914 – SNR~7 in ringdown



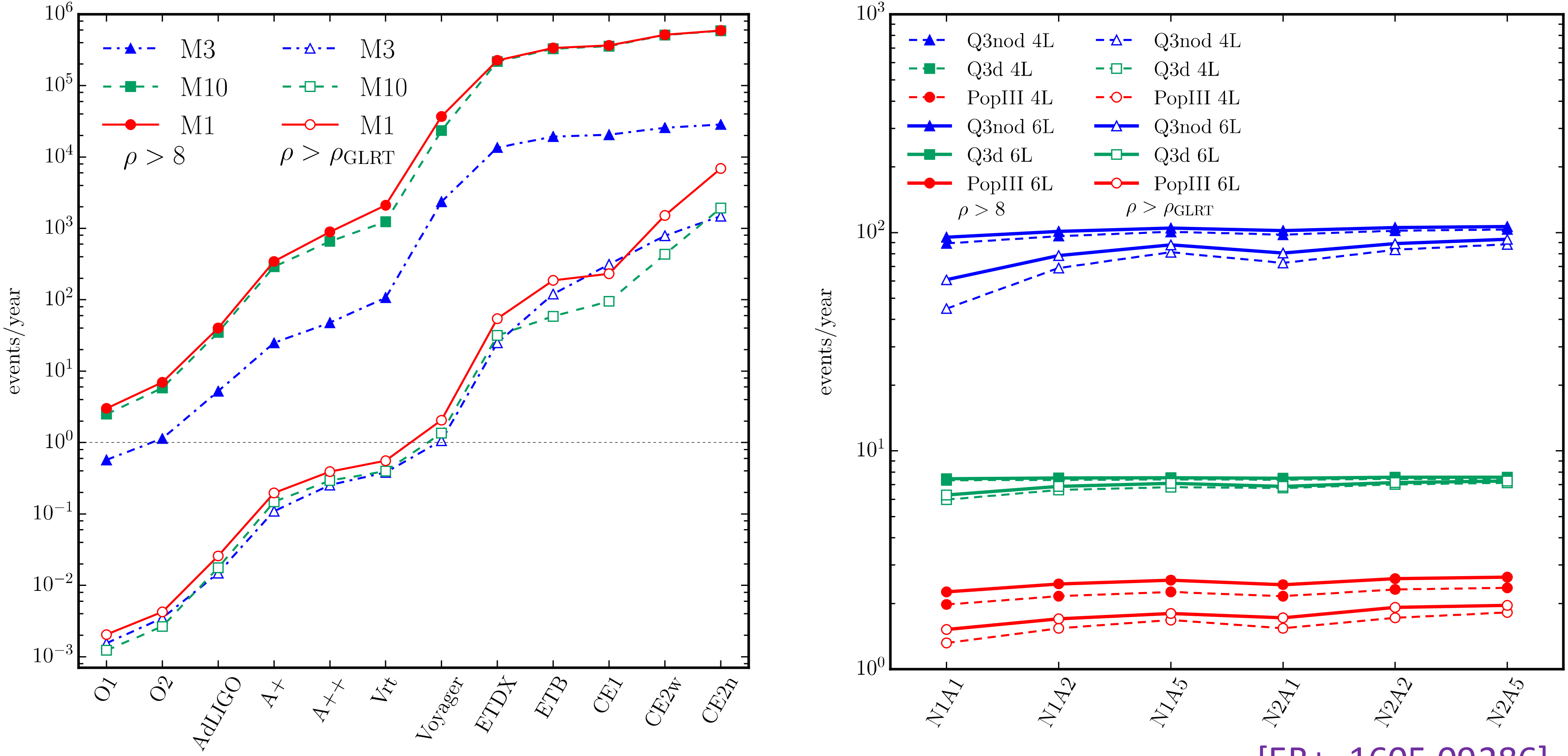
Black hole spectroscopy with gravitational waves



$$\rho = \frac{\delta_{\rm eq}}{D_{\rm L}\mathcal{F}_{lmn}}\left[\frac{8}{5}\frac{M_z^3\epsilon_{\rm rd}}{S_n(f_{lmn})}\right]^{1/2}$$

[EB+, 1605.09286]

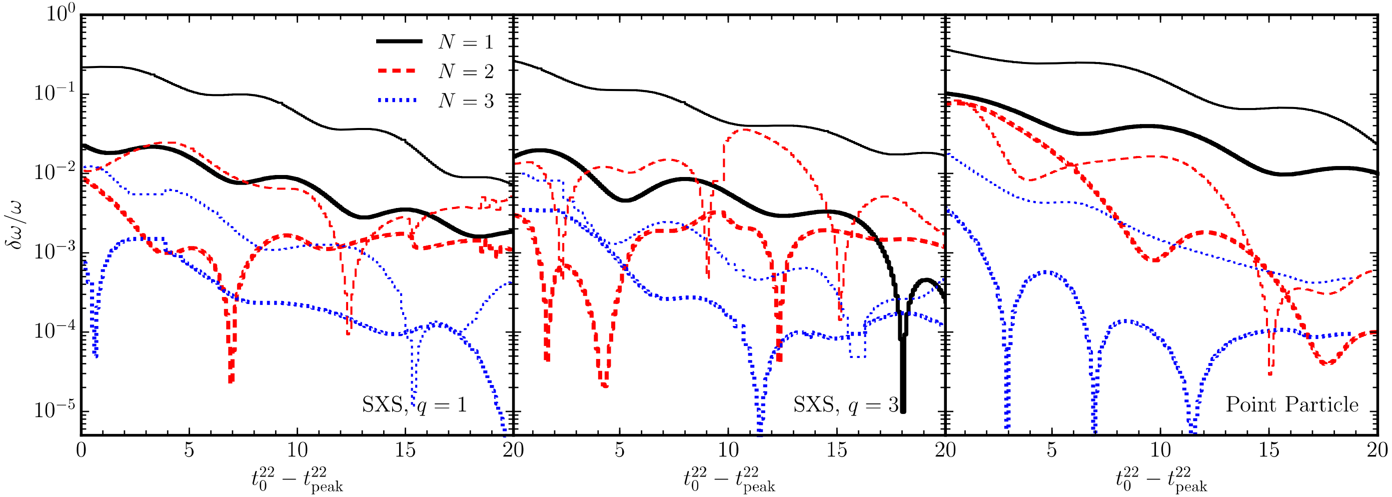
Earth vs. space-based: ringdown detections and black hole spectroscopy



[EB+, 1605.09286]

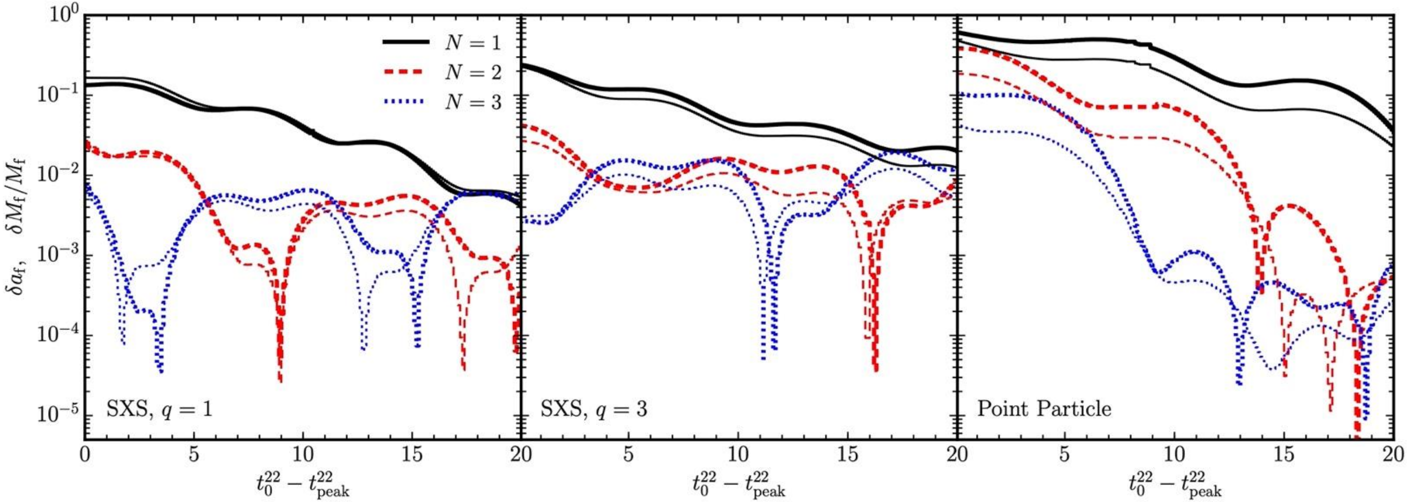
Overtones and current observations

Overtones are needed to reduce mass/spin errors



Top:
real part (thick)
imaginary part (thin)

1% determination of ω_{220} needs one overtone (better if two or three)



Bottom:
spin (thick)
mass (thin)

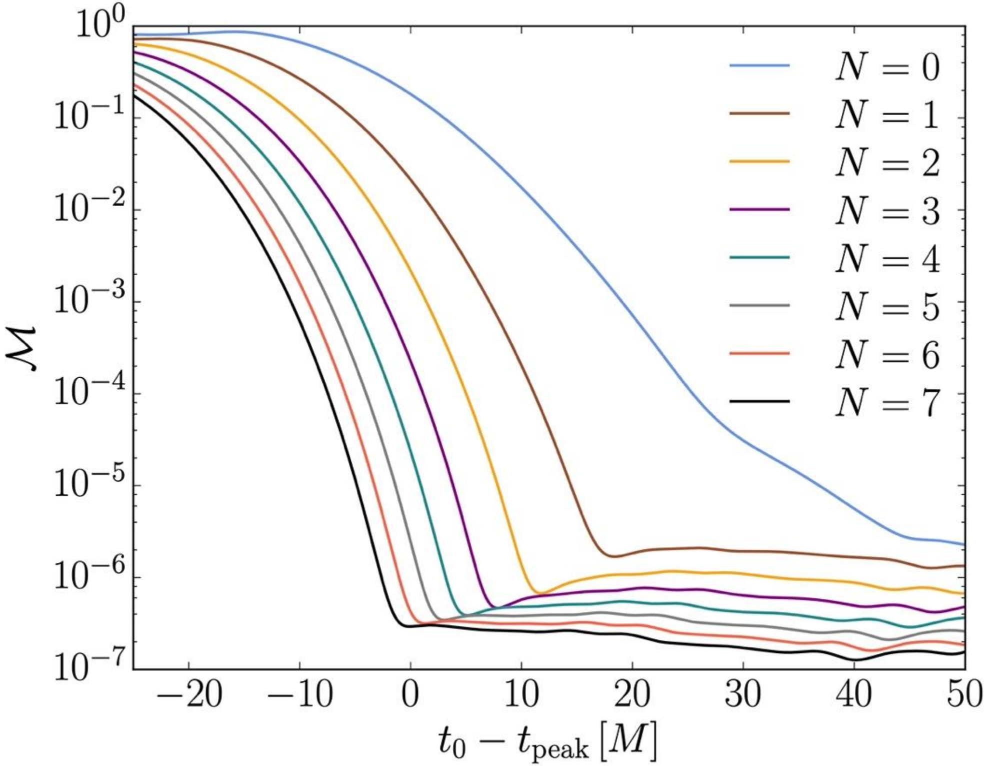
1% determination of mass and spin needs at least two modes

[Baibhav+, 1710.02156]

Nonlinear merger: is it just a superposition of linear QNMs?

"Including overtones allows for the modeling of the ringdown signal for all times beyond the peak strain amplitude, indicating that **the linear quasinormal regime starts much sooner than previously expected**. This implies that the spacetime is **well described as a linearly perturbed black hole with a fixed mass and spin as early as the peak**"

$$\mathcal{M} = 1 - \frac{\langle h_{22}^{\rm NR}, h_{22}^{N}\rangle}{\sqrt{\langle h_{22}^{\rm NR}, h_{22}^{\rm NR}\rangle \langle h_{22}^{N}, h_{22}^{N}\rangle}}$$

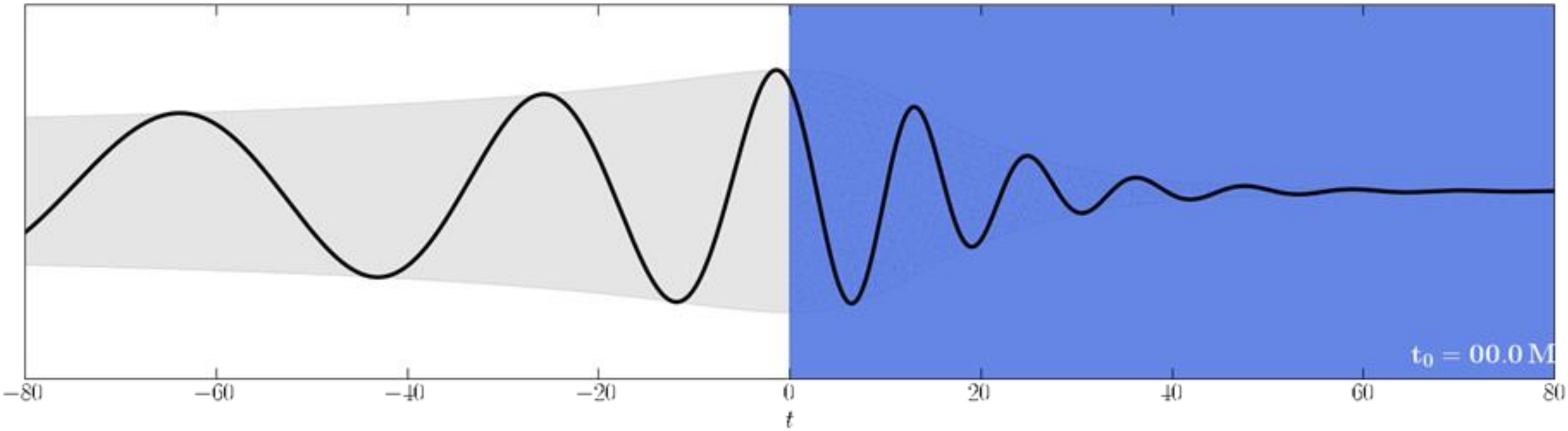


Does it?

[Giesler+, 1903.08284]

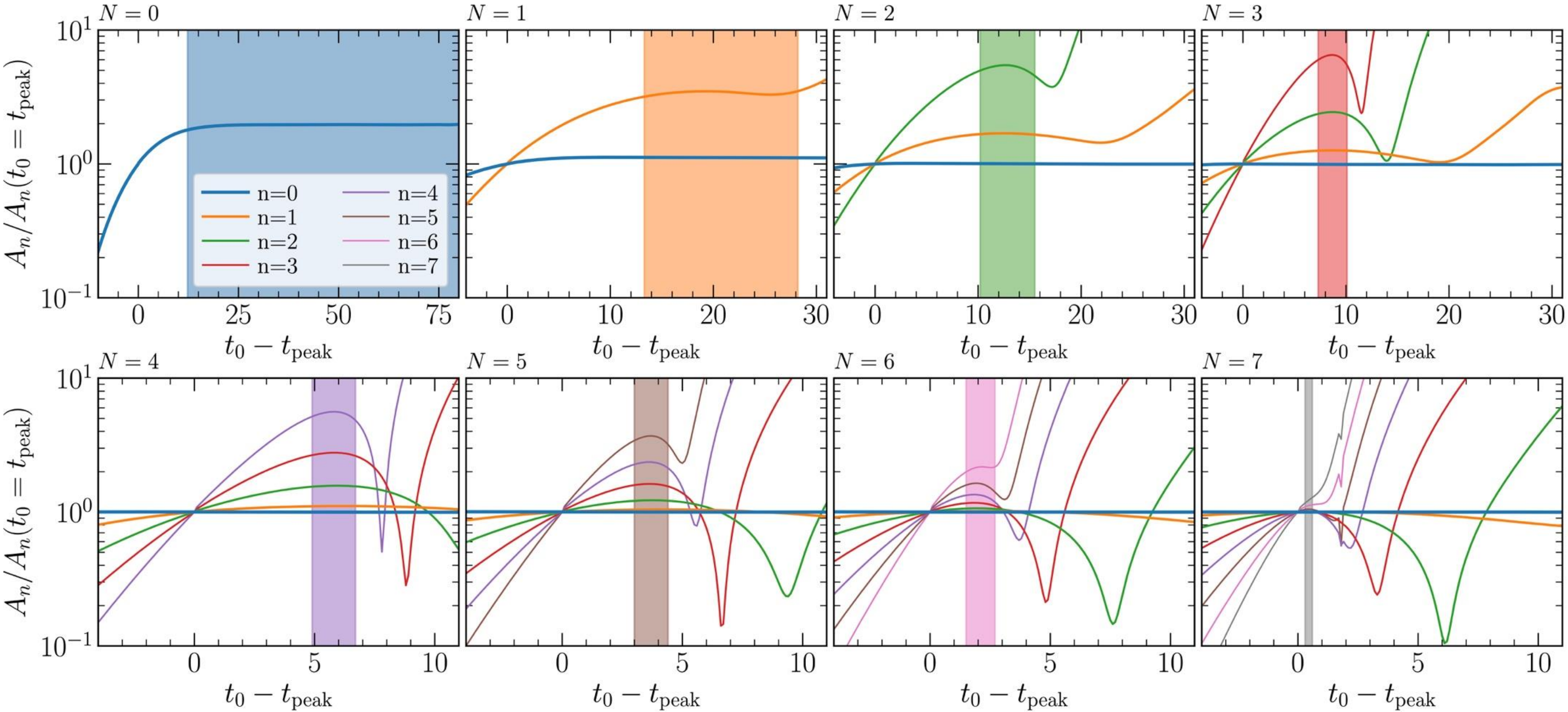
Is the linear model consistent when we change the fitting window?

$$h = \sum A e^{-\omega_i(\chi,M)t} \cos(\omega_r(\chi, M)t + \phi)$$



QNM amplitudes are not constant near the peak

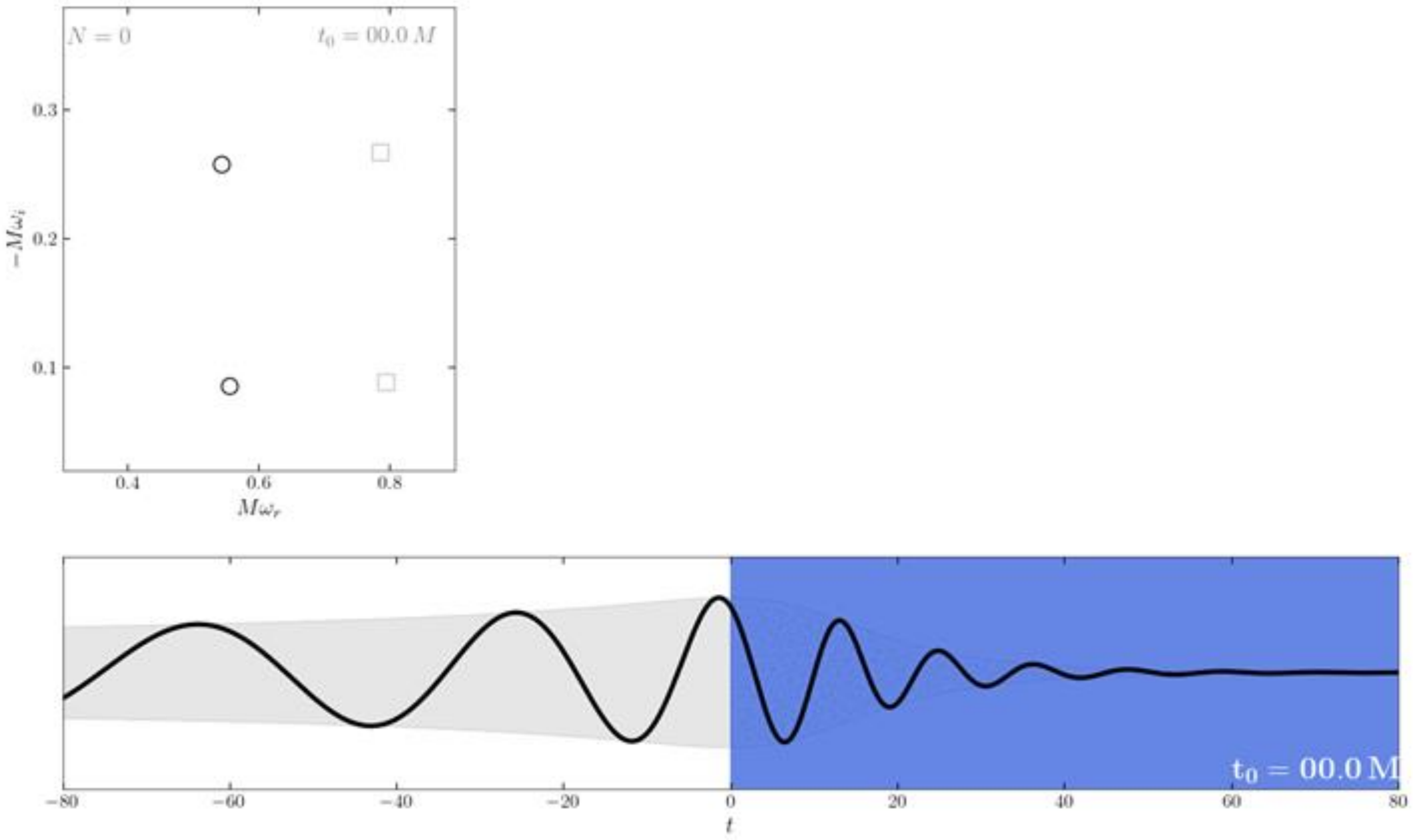
“Fixed-frequency” fits as in Giesler (weaker test). Bands show regions where amplitudes are constant within 10%



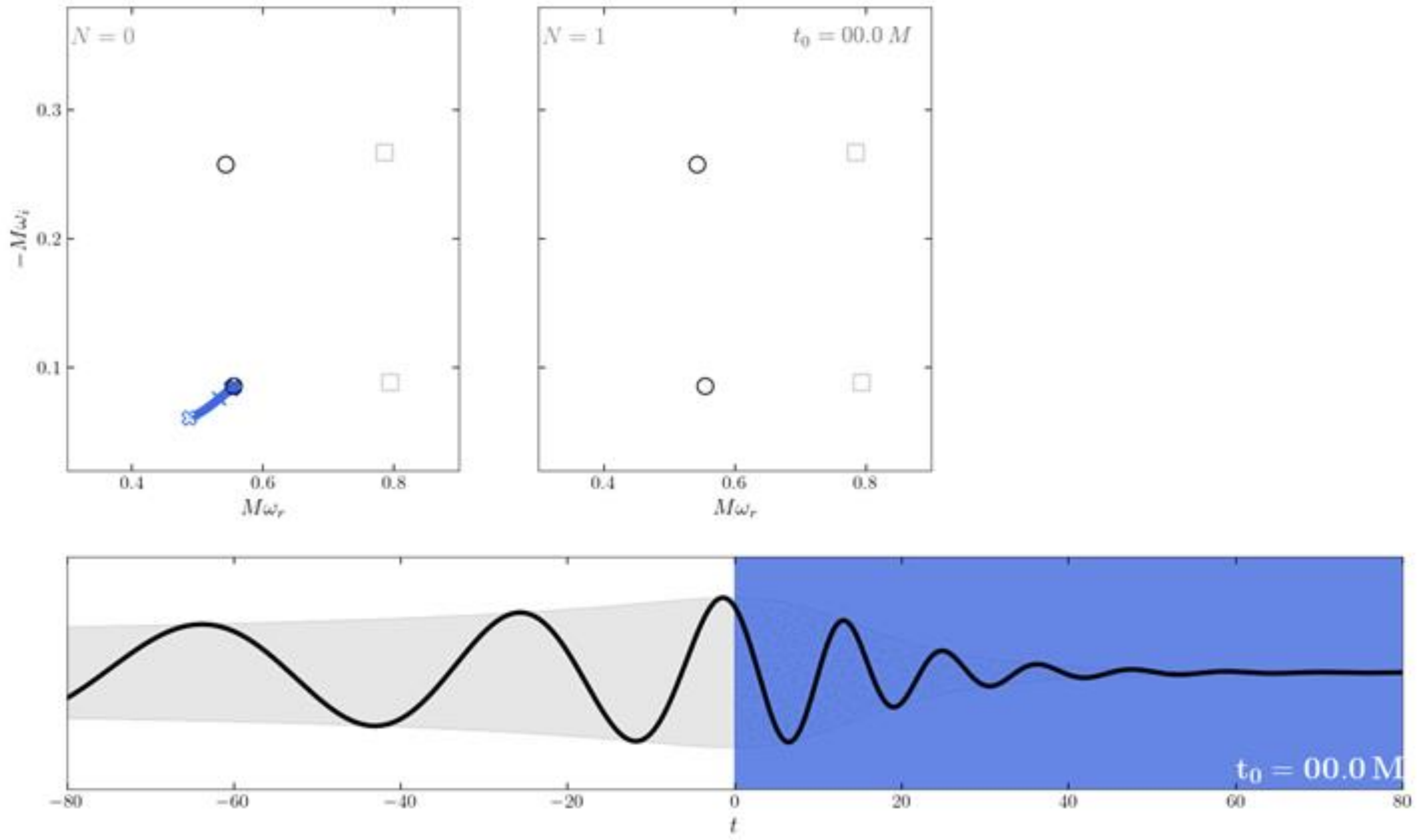
[Bhagwat+, 1910.08708; Baibhav+, 2302.03050]

My mom always said life was like a box of chocolates.
You never know what you're gonna get.

Agnostic spectroscopy: fundamental mode from numerical simulations

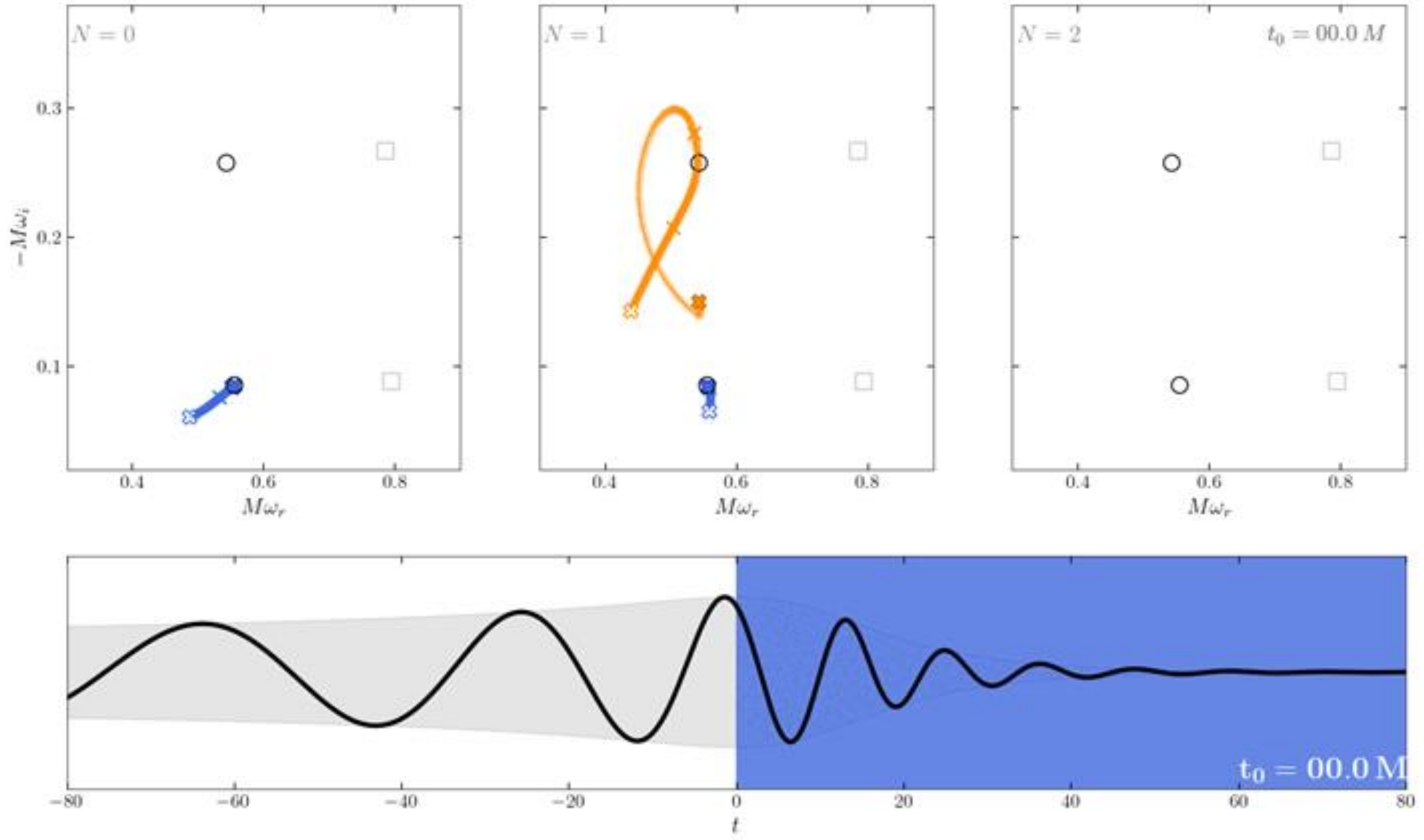


Agnostic spectroscopy: first overtone from numerical simulations



Why wrong? Spherical-spheroidal mode mixing

$$_{-2}S_{lm} = {}_{-2}Y_{lm} + j_f\,\tilde{\omega}_{lmn} \sum_{l\neq l'} {}_{-2}Y_{l'm}\, c_{l'lm}$$



Search for nonlinearities and nonlinear modes

Two stages

Before the 2005 NR breakthrough: perturbation theory to the rescue

Close limit approximation [e.g. Gleiser+ gr-qc/9609022…]

“Lazarus project”, second-order Kerr [e.g. Campanelli-Lousto gr-qc/9811019]

After the 2005 NR breakthrough:

Where are all the nonlinearities?

[Zlochower+, gr-qc/0306098; Ioka-Nakano, 0704.3467 + 0708.0450; Brizuela+, 0903.1134; Pazos+, 1009.4665]

$$\psi = \epsilon\psi_{(1)} + \epsilon^2\psi_{(2)}$$

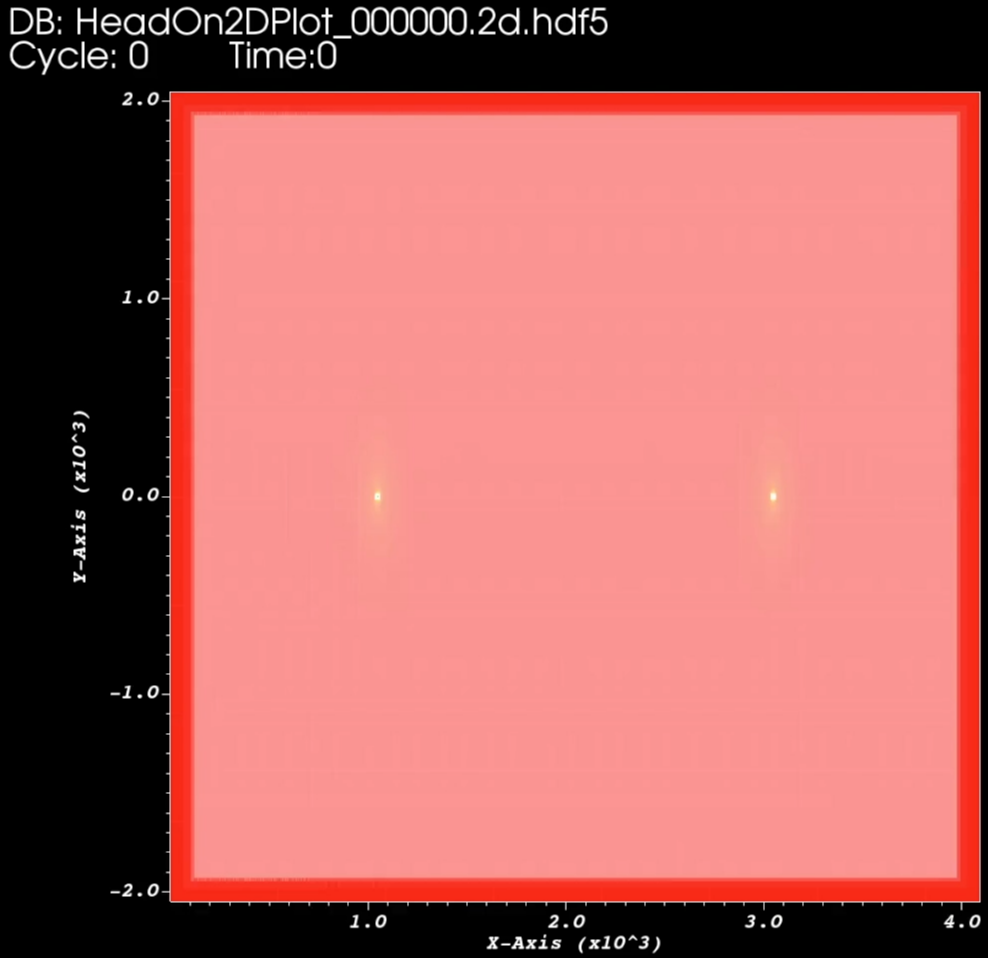
$$\mathcal{L}\psi_{(2)} \propto \psi_{(1)}^2 \sim A_1 A_2 e^{i(\phi_1+\phi_2)}$$

Pioneering search for nonlinearities in the Georgia Tech NR catalog

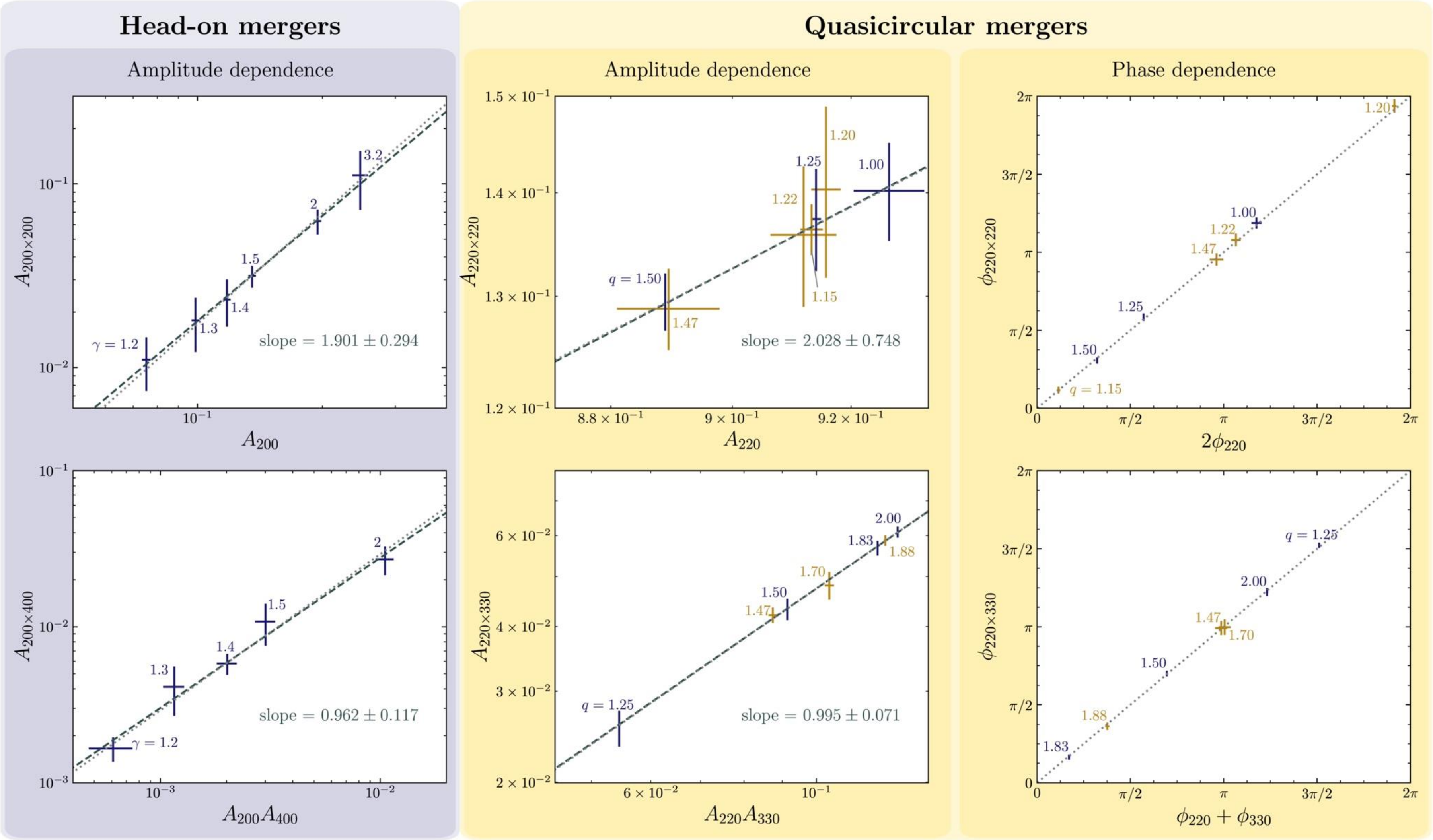
[London+, 1404.3197]

Recent explosion of activity – analytical and numerical

[Loutrel+, 2008.11770; Ripley+, 2010.00162; Magana-Zertuche+, 2110.15922; Sberna+, 2112.11168; Ma+, 2207.10870; Lagos-Hui, 2208.07379; Cheung+, 2208.07374; Mitman+, 2208.07380; Zhu+, 2309.13204; Kehagias+, 2301.09345 + 2302.01240; Nee+, 2302.06634; Perrone+, 2308.15886; Bucciotti+, 2309.08501…]



[Cheung+, 2208.07374]



Head-on mergers
Quasicircular mergers
Amplitude dependence
Amplitude dependence
Phase dependence
$A_{200\times200}$
A_{200}
slope = 1.901 ± 0.294
$\gamma = 1.2$
1.3
1.4
1.5
2
3.2
$A_{200\times400}$
$A_{200}A_{400}$
slope = 0.962 ± 0.117
$A_{220\times220}$
A_{220}
slope = 2.028 ± 0.748
$q = 1.50$
1.47
1.22
1.25
1.15
1.20
1.00
$A_{220\times330}$
$A_{220}A_{330}$
slope = 0.995 ± 0.071
$q = 1.25$
1.50
1.70
1.83
1.88
2.00
$\phi_{220\times220}$
$2\phi_{220}$
$q = 1.15$
$\phi_{220\times330}$
$\phi_{220} + \phi_{330}$

For all your quasinormal mode fitting needs…

https://mhycheung.github.io/jaxqualin/
pronounced “Jacqueline”

[Cheung+, arXiv:2310.04489]

2,2,0 mode

Fit expressions

The hyperfit expressions should be used with caution, especially in regions of low amplitude or outside of the convex hull of the data points. The hyperfit function for the amplitude could go to negative values in these regions, which is unphysical. The phase data has been unwrapped before fitting to the best of our ability, but there may still be some jumps of 2π in the data, which could be seen in the error plot. Please consult the fit error plot on the bottom of this page before using the fits.

Amplitude

$$A_{2,2,0} = 4.004 + 1.349\chi_+ + 0.333\chi_- - 1.325\eta^2 - 1.369\eta\chi_- + 2.622\chi_+\chi_- - 32.74\eta^2\chi_+ + 4.313\eta\chi_+^2 - 25.18\eta\chi_+\chi_- + 83.37\eta^3\chi_+ - 13.39\eta^2\chi_+^2 + 58.01\eta^2\chi_+\chi_- - 0.3837\eta\chi_+^3 - 0.2075\chi_+^4$$

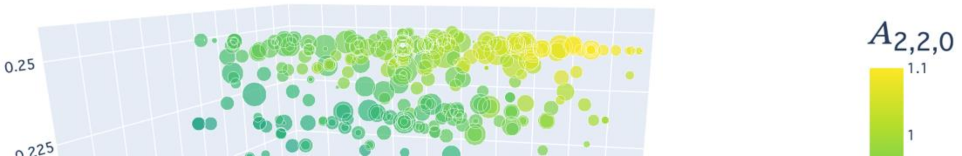
Phase

$$\phi_{2,2,0} = 0$$

Interactive plot

Click on the buttons below to switch between the amplitude, phase and starting time plots.

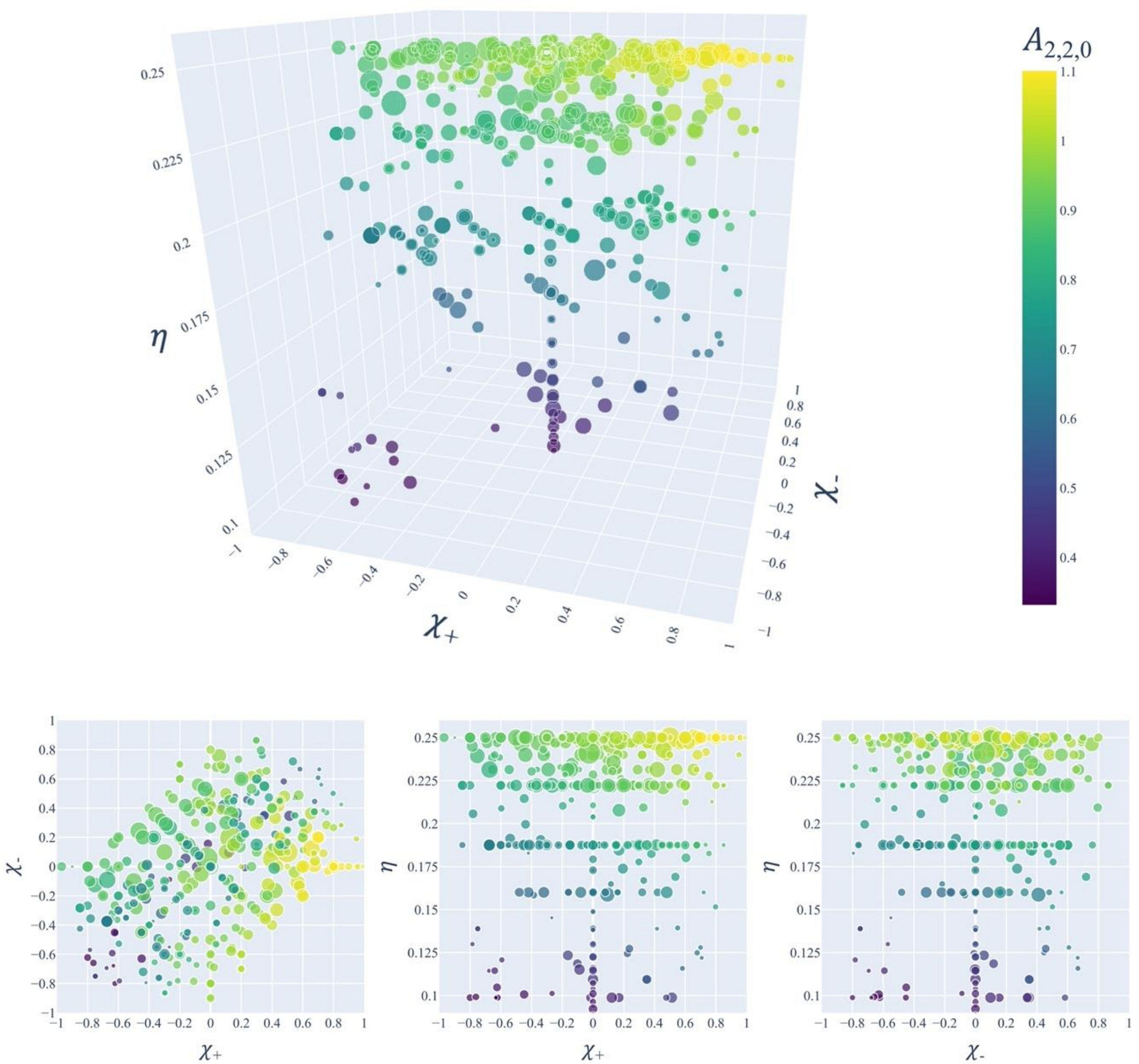
$A_{2,2,0}$ $\phi_{2,2,0}$ t_{start}



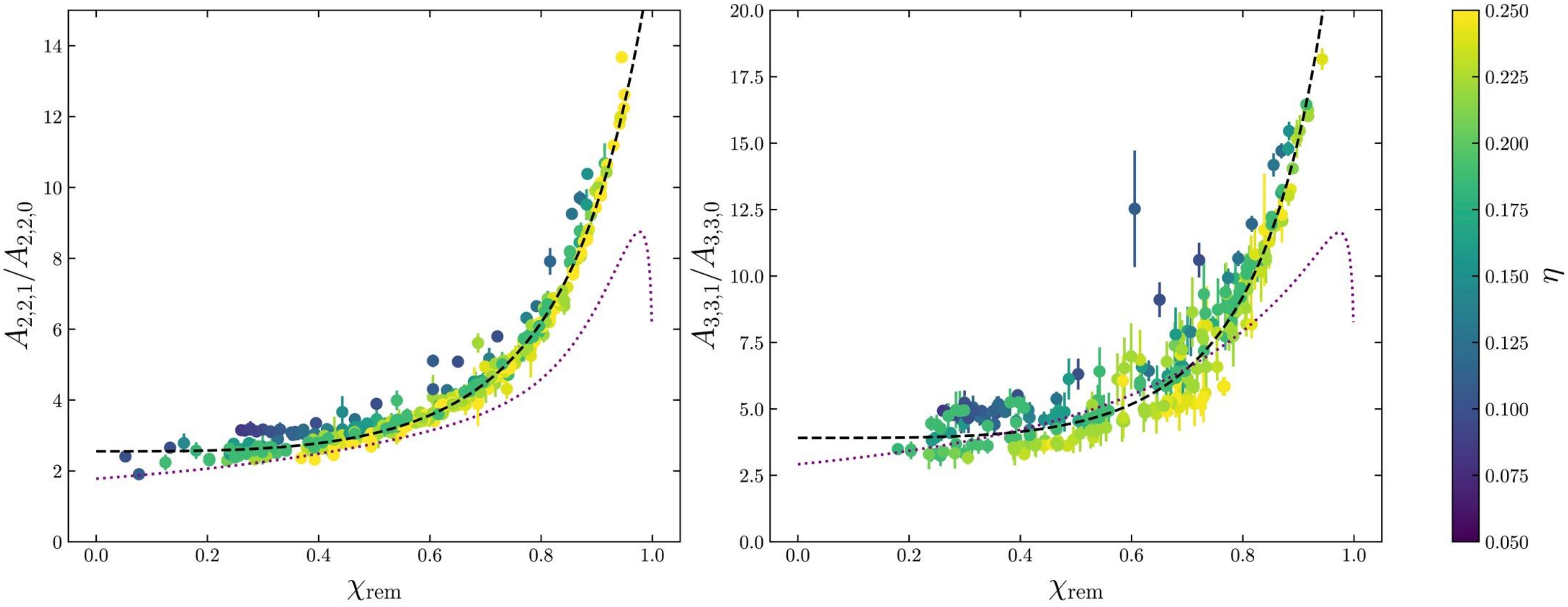
$$\eta = \frac{q}{(1+q)^2}$$

$$\chi_+ = \frac{q\chi_1 + \chi_2}{1+q}$$

$$\chi_- = \frac{q\chi_1 - \chi_2}{1+q}$$

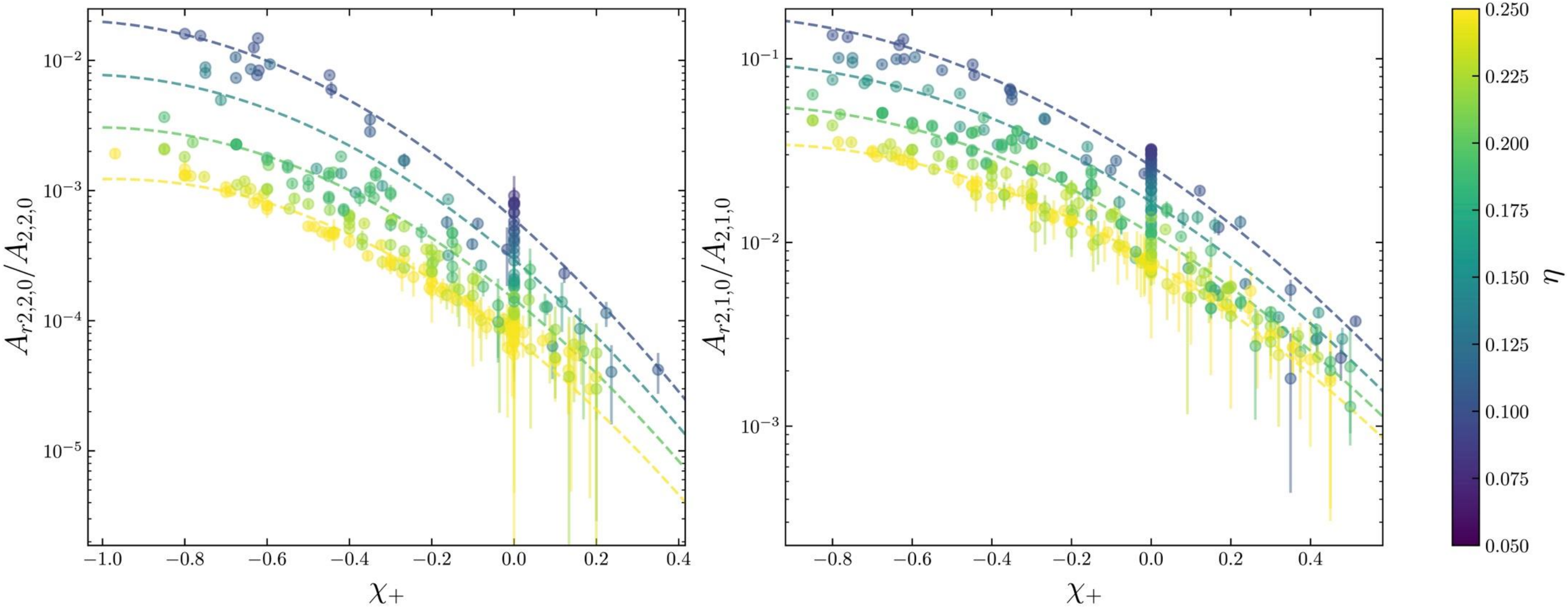


Ratio of first overtone to fundamental mode: excitation coeffs vs. excitation factors



[Cheung+, arXiv:2310.04489]

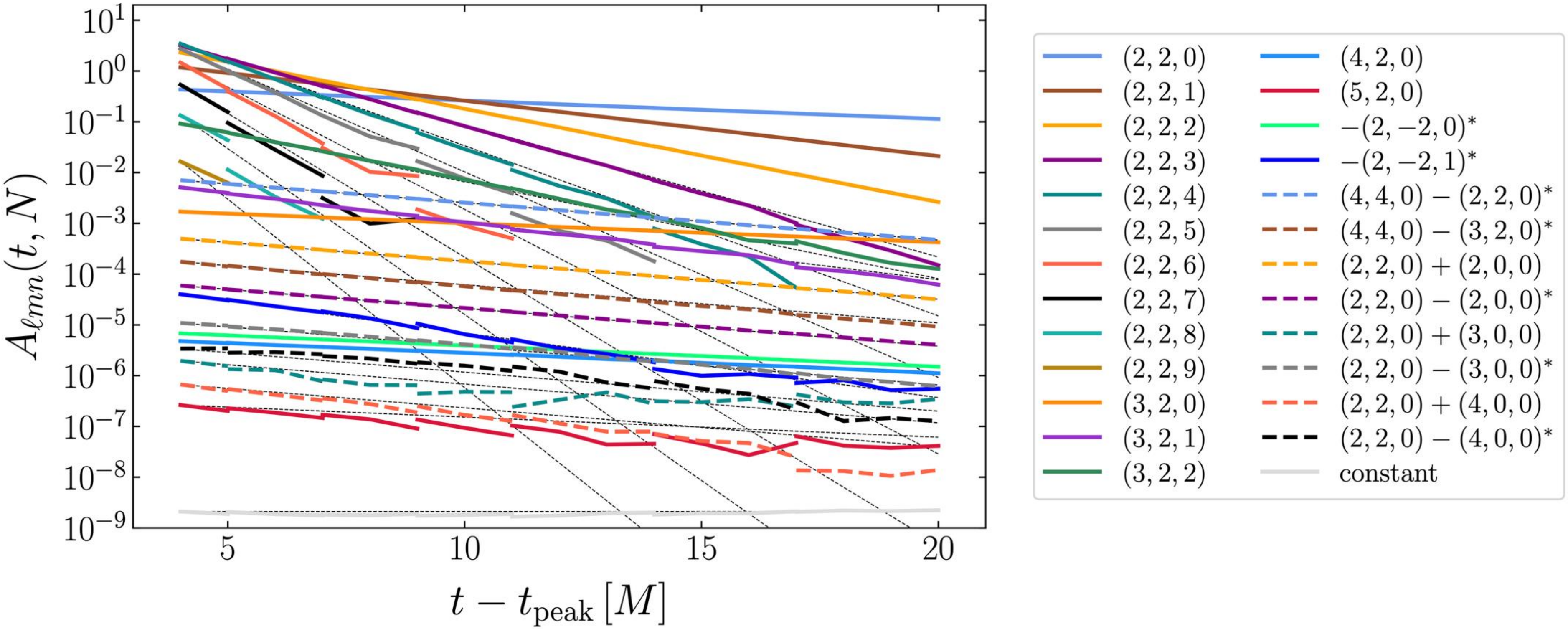
Ratio of retrograde modes to prograde modes



[Cheung+, arXiv:2310.04489]

Agnostic, nonlinear spectroscopy reloaded: nonlinear modes in the (2,2)

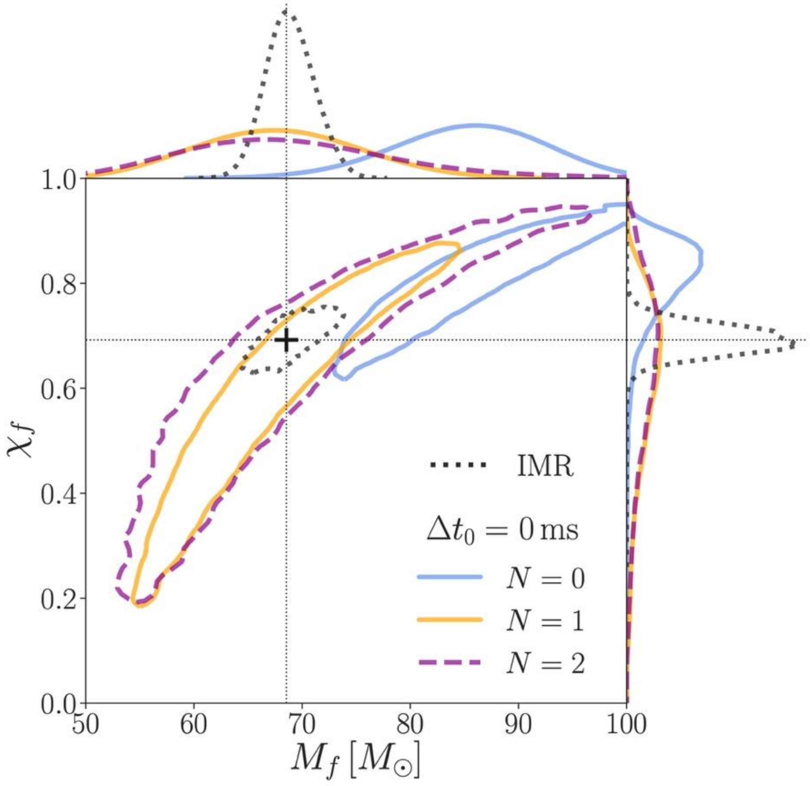
More accurate simulations (CCE crucial), agnostic fits, includes nonlinear modes, variable projection
Finds more overtones as long as nonlinear modes are included. "Stable amplitudes are not achievable until ~4M after the peak in a moderately spinning case and until ~8M post-peak in a high-spin case"



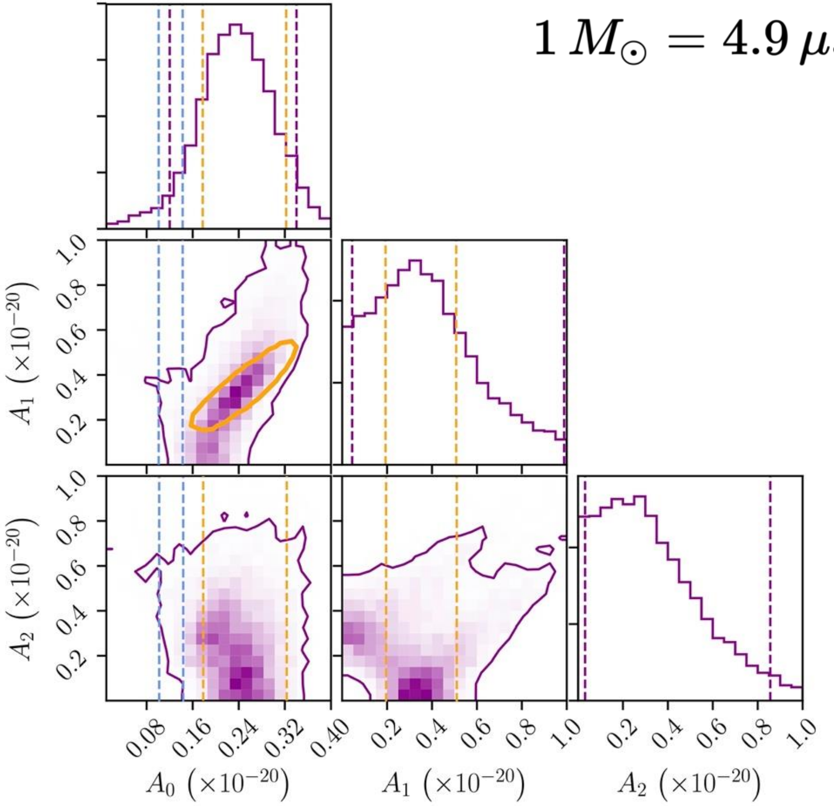
[Giesler+, 2411.11269; see also Gao+, 2502.15921; Nobili+, 2504.17021]

GW150914 tests of the no-hair theorem with the first overtone?

Overtones improve quality of consistency tests for GW150914
Is the overtone detection robust? Assumes t_{start}=1126259462.423 s



$1\,M_\odot = 4.9\,\mu s$



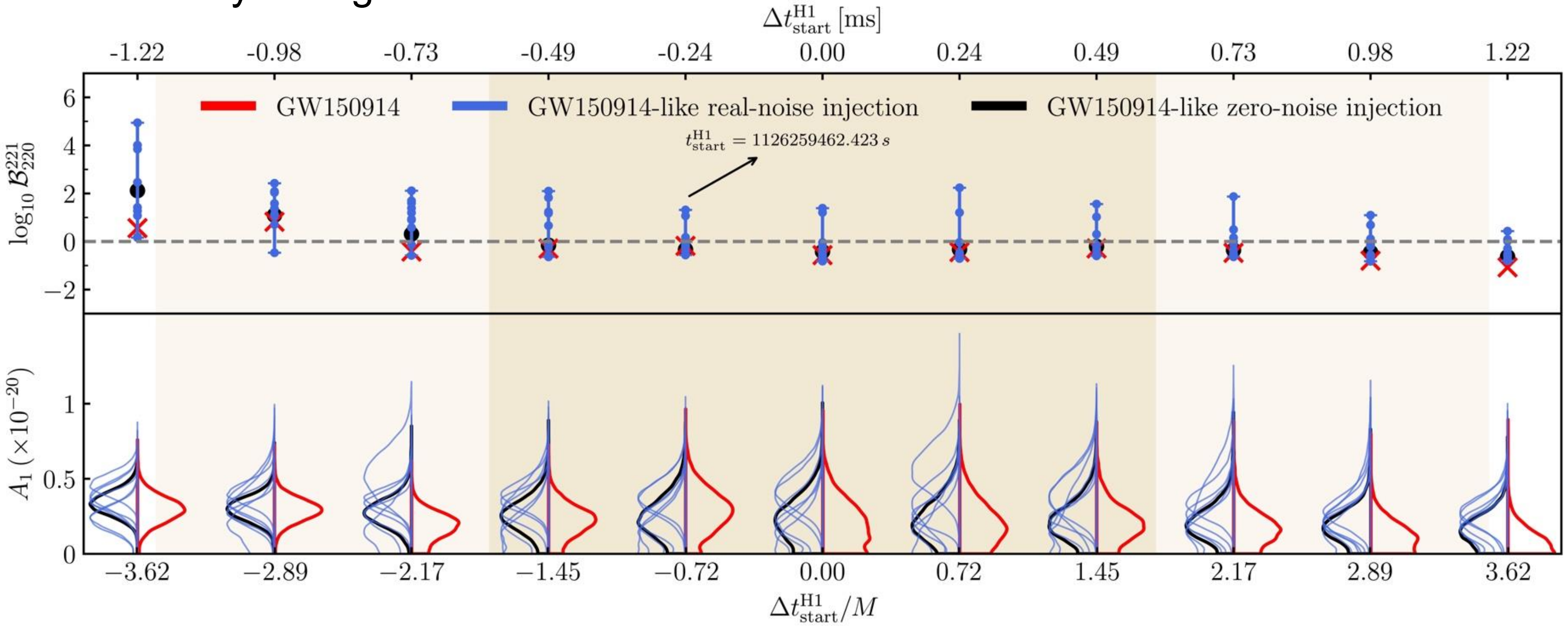
[Isi+, 1905.00869]

The first overtone: amplitude and Bayes factor

Cotesta+, Isi-Farr criticism and reanalysis. Summary of criticism:

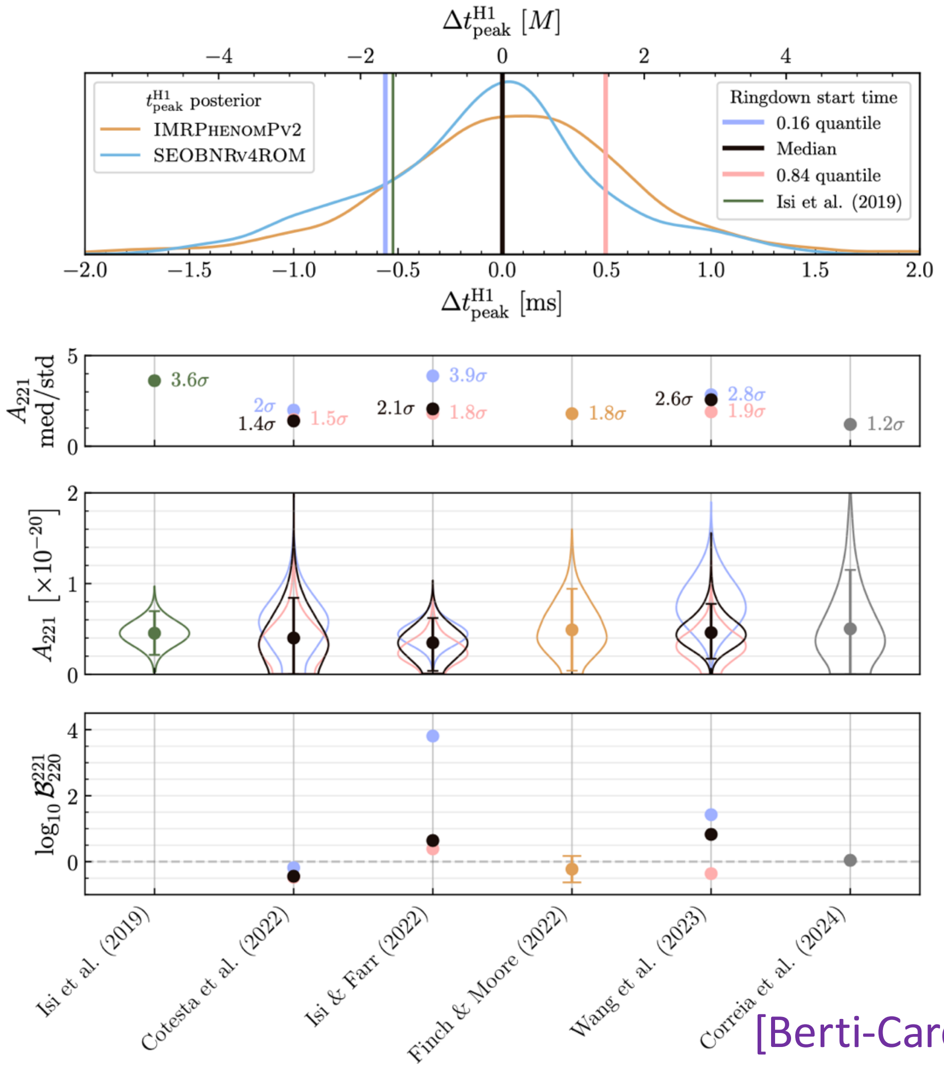
1) Shift in pyRing discretized time axis: 0.06ms (compare to Δt_{peak} ~2.5ms)
2) Analysis segment T=0.2s instead of 0.1s slightly increases amplitudes

Does not really change results.



$$M_{\rm GW150914} \simeq 62 M_\odot = 0.3\,\rm ms$$

[Isi-Farr, Cotesta+, Carullo+, PRL]



[Berti-Cardoso-Carullo+, to appear]

Black hole spectroscopy: are we there yet? Not so fast

Theory: need agnostic analysis of NR waveforms including all physics, not just linear modes

Cannot just *assume* that the second mode is an overtone: must include
BMS effects (memory), tails, transients, mode mixing, counterrotating + nonlinear modes
Low-frequency QNMs (overtones) are good at fitting the inspiral: “pseudo-QNMs” in EOB
QNMs physically present only ~10M after the peak, where SNR is low
High overtones do not contribute to mass/spin estimates, can be unstable
[Baibhav+, 2302.03050; Cheung+, 2111.05415 and 2208.07374; Mitman+, 2208.07380…]

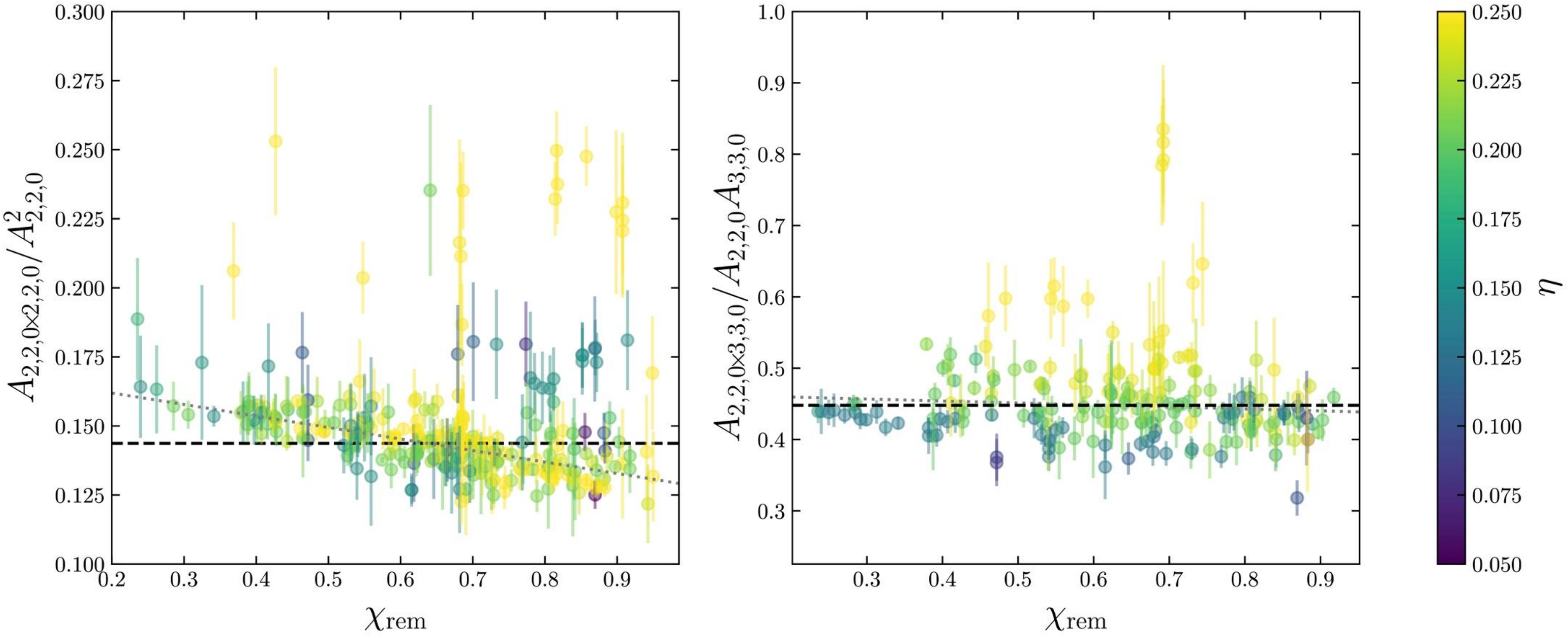
Data analysis: second mode evidence depends on assumptions

Time or frequency domain?
Ringdown only vs. modeled (e.g. pSEOBNR)/unmodeled (wavelets) pre-ringdown
Must take into account uncertainty in starting time, sampling rate, noise modeling…
Weak Bayesian evidence (if any) for a second mode in GW150914, GW190521
What does “mode detection” even mean? [Isi+, Capano+, Cotesta+, Finch-Moore, Wang+…]
XG: “golden” events with SNR~300 for CE/ET, SNR~1000 for LISA – but Ockham penalties
Amplitude/phase tests; population-based tests

[*Ringdown Inside & Out*]

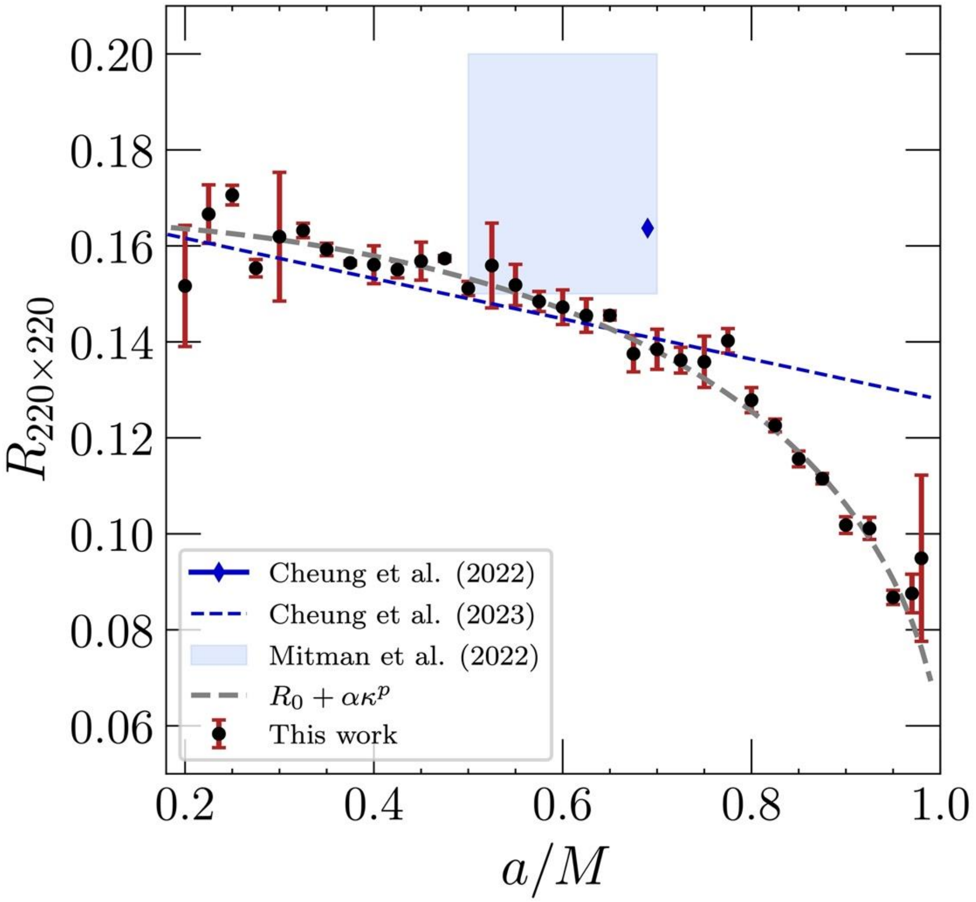
Tests of strong gravity with nonlinear modes

Ratio of quadratic modes to linear modes: remnant spin and mass ratio dependence



[Cheung+, arXiv:2310.04489]

Gaussian scattering of second-order perturbations: good agreement!



[Redondo-Yuste+, arXiv:2308.14796; see also Zhu+, 2401.00805; Ma-Yang, 2401.15516]

Ongoing work and a new test: ratio of nonlinear and linear mode amplitudes

Numerical extraction of modes

Takahashi-Motohashi 2311.12762: iterative extraction of overtones
Clarke+, 2402.02819: “striking the right tone” (up to N=3)
Zhu+ 2312.08588: precessing binaries
Carullo 2406.19442: eccentric binaries
Carullo-De Amicis 2310.12968 , Islam+ 2407.04682 : tails for eccentric binaries
Carullo-De Amicis 2406.17018: perturbation theory arguments

Systematic calculation of nonlinear / linear mode amplitudes in Schwarzschild

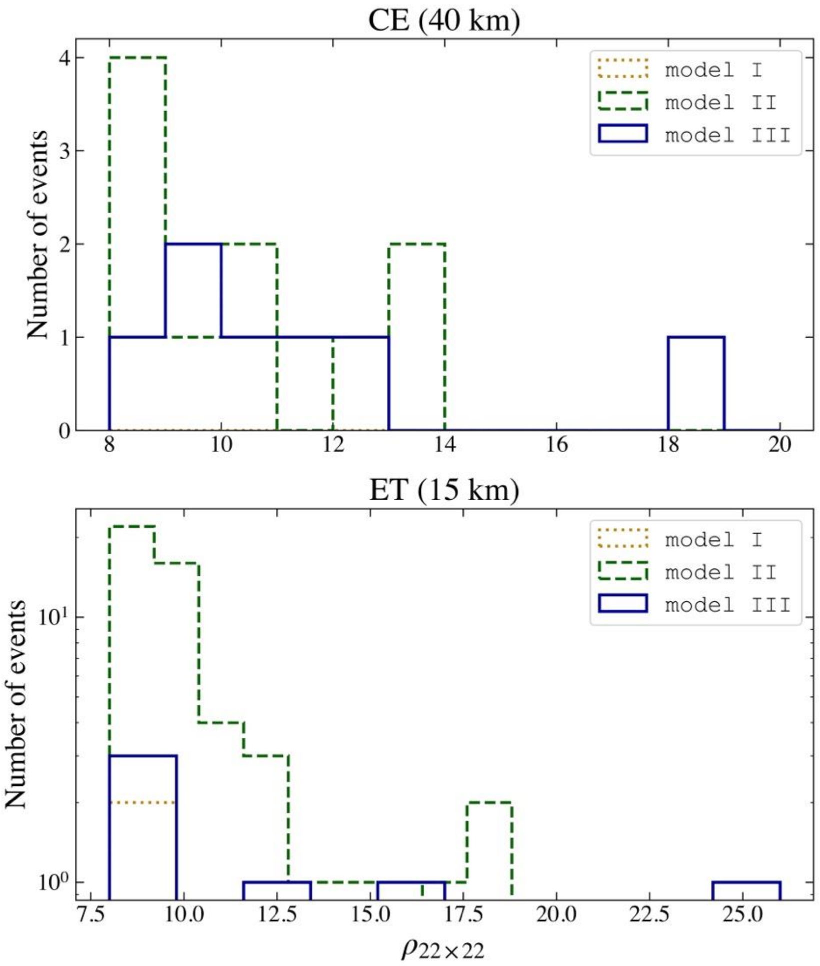
Ioka-Nakano 0704.3467, 0708.0450: first estimate
Lagos-Hui 2208.07379: Green’s function
Kehagias-Riotto 2301.09345 + 2302.01240, Perrone+ 2308.15886: Kerr/CFT, gauge invariance, light ring
Bucciotti+ 2309.08501, 2405.06012, 2406.14611: generic quadratic/linear mode ratios
Bourg+ 2405.10270: dependence on parity

Calculation of nonlinear / linear mode amplitudes in Kerr

Redondo-Yuste+ 2308.14796, Zhu+ 2309.13204 + 2401.00805: time-domain fits with quadratic code
Redondo-Yuste+ 2312.04633, Zhu+ 2404.12424: changing mass and spin (Vaidya/nonlinear evolutions)
May+ 2405.18303: absorption
Ma-Yang 2401.15516, Khera+ 2410.14529: quadratic/linear mode ratios for dominant modes

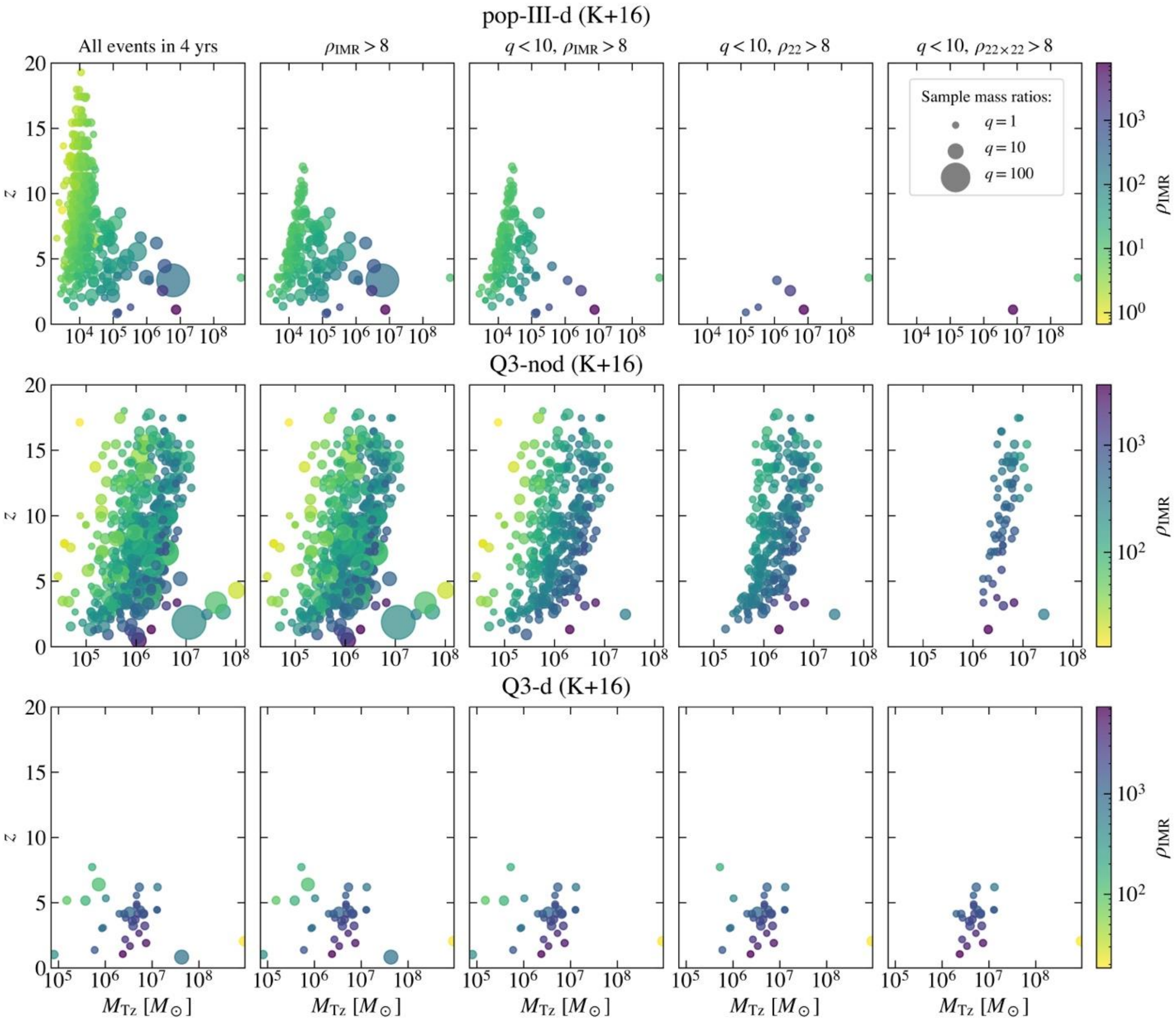
…plus work on greybody factors, spectral stability, etcetera

XG ground-based detectors, quadratic $(220)^2$ mode in the (44) multipole



[Yi+, 2403.09767]

LISA: much better prospects!



[Yi+, 2403.09767]

LISA rates for quadratic mode detectability

TABLE II. Averaged statistics on the MBH binaries observed by LISA. The first and second columns show the total number of mergers and the number of events with observable IMR expected in a 4-year mission lifetime for each catalog. The third and fourth columns show the same quantities when we implement the mass ratio cutoff ($q < 10$). The fifth and sixth columns list the number of events having SNR above threshold for the dominant (22) linear QNM and for the (22×22) quadratic QNM (for the $q < 10$ events only). In the last two columns we list the average and maximum SNRs of the (22×22) mode (again, for $q < 10$ events). Numbers without and with parentheses represent values for the finite-resolution and extrapolated models, respectively.

	Events in 4 yrs	Num. with $\rho_{\rm IMR} > 8$	Events in 4 yrs ($q < 10$)	Num. with $\rho_{\rm IMR} > 8$ ($q < 10$)	Num. with $\rho_{22} > 8$	Num. with $\rho_{22\times22} > 8$	Mean $\rho_{22\times22}$	Max $\rho_{22\times22}$
HS-nod-noSN (B+20)	16288(39785)	16284(39764)	11978(29383)	11977(29380)	6704(20951)	1098(5623)	3(5)	905(2211)
LS-nod-noSN (B+20)	1313(1672)	224(271)	1193(1529)	132(163)	11(13)	3(4)	0.3(0.3)	1149(1152)
LS-nod-SN (B+20)	1279(1626)	6(7)	1276(1622)	5(6)	0(6)	0(0)	0(0)	94(418)
pop-III-d(K+16)	689(1430)	206(382)	662(1376)	180(334)	5(15)	2(7)	0.6(0.7)	1725(1024)
Q3-nod (K+16)	470(660)	470(659)	359(516)	359(516)	277(427)	77(139)	8(14)	964(1744)
Q3-d (K+16)	33(74)	33(74)	31(70)	31(70)	28(66)	22(55)	74(93)	2194(3870)

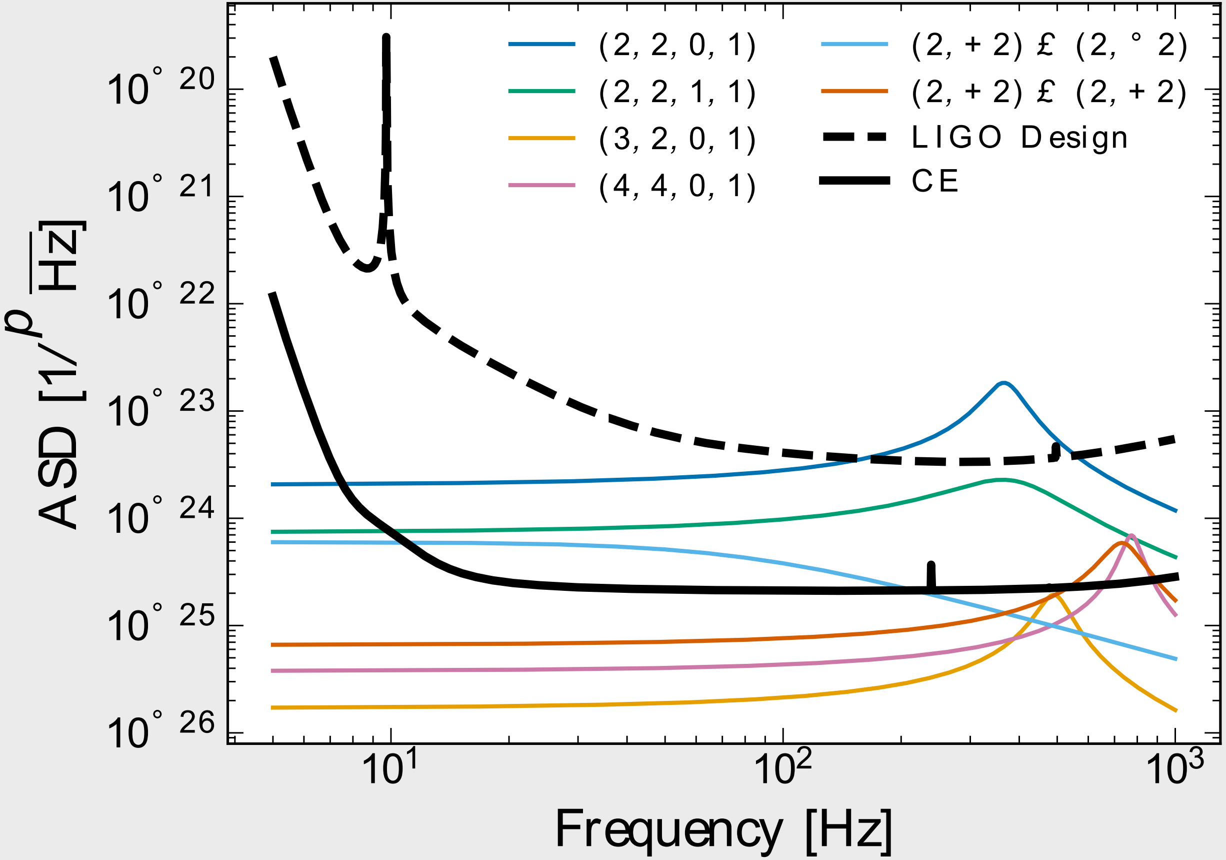
[Yi+, 2403.09767]

Observing Memory as a 2nd-order BHPT Excitation

- **NR simulation:**
  - $q = 1$
  - $\chi^{\vec{}}_{1,2} = \chi^{(z)}_{1,2} = 0.6$
  - $M_f = 60 M_\odot$
  - $R = 400\,\mathrm{Mpc}$
- **QNM Lorentzian:**

$$h_{\mathrm{QNM}} = A_{(\ell,m,n,p)} e^{-i\omega_{(\ell,m,n,p)} t}$$

$$\tilde{h}_{\mathrm{QNM}} = \frac{i}{\sqrt{2\pi}} \frac{A_{(\ell,m,n,p)}}{\omega - \omega_{(\ell,m,n,p)}}$$

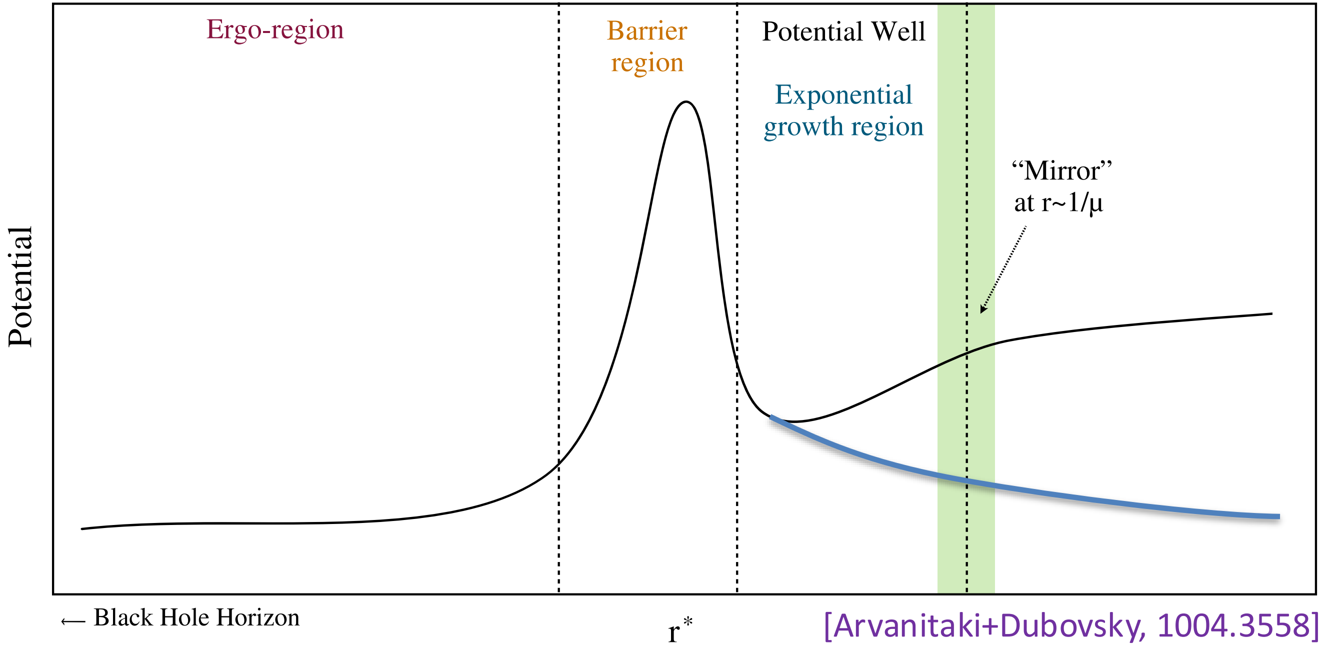


[Mitman, *Ringdown Inside & Out*]

Spectral instabilities and exceptional points

QNMs and overtones: some milestones. Phase 2 – overtones and spectroscopy

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Phi = V\Phi$$



[Arvanitaki+Dubovsky, 1004.3558]

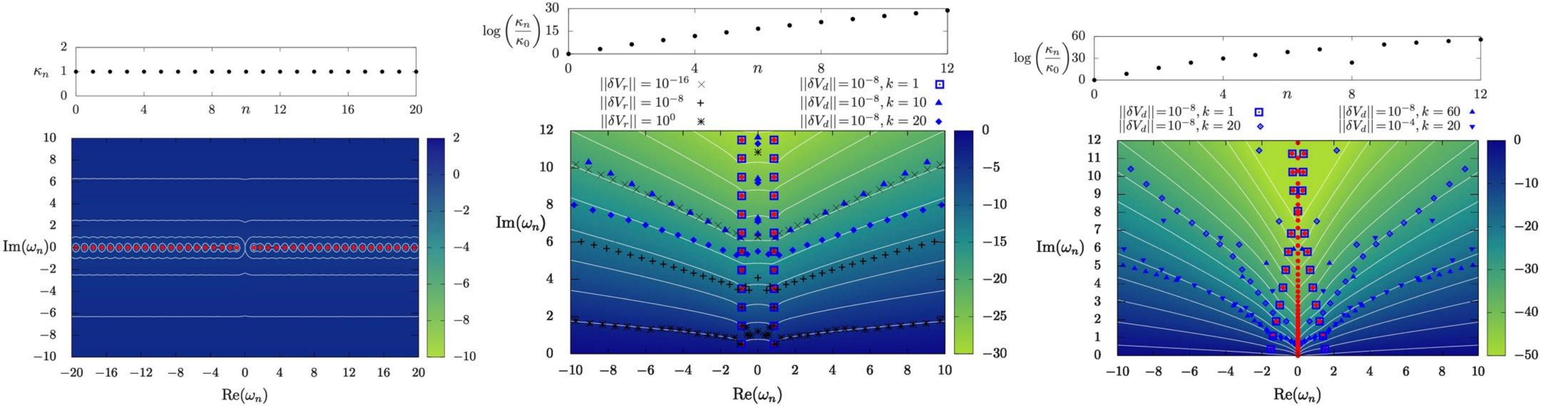
Quasinormal modes:

- Ingoing waves at the horizon, outgoing waves at infinity
- Spectrum of damped modes ("ringdown")

Massive scalar field:

- Superradiance: black hole bomb when $0 < \omega < m\Omega_H$ [Press-Teukolsky 1972]
- Hydrogen-like, unstable bound states [Detweiler 1980, Zouros+Eardley, Dolan…]

Pseudospectra: is the spectrum itself stable?



$$A v_i = \lambda_i v_i$$
$$A(\epsilon) = A + \epsilon \delta A$$

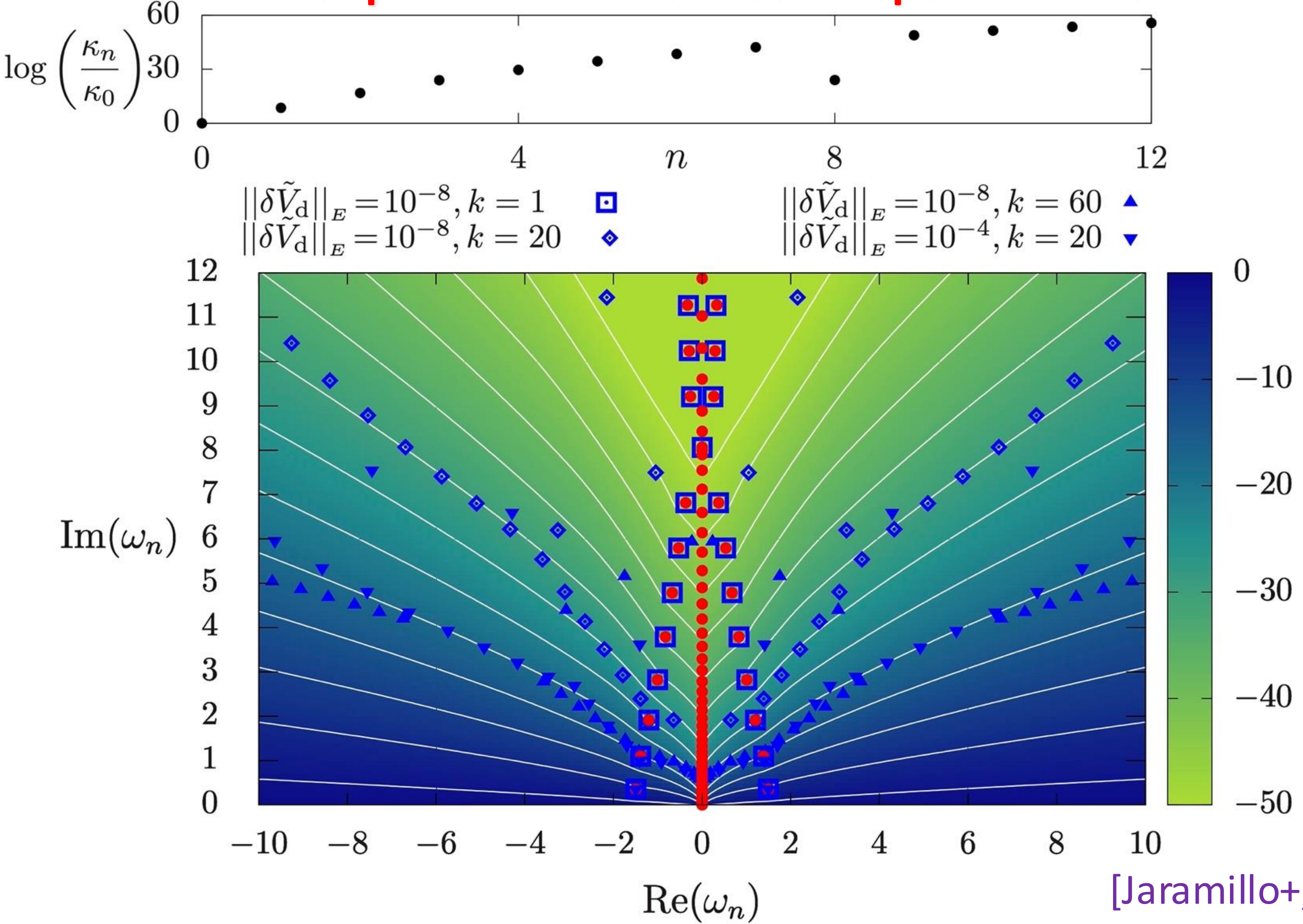
$$|\lambda_i(\epsilon) - \lambda_i| \le \epsilon \kappa_i, \quad \kappa_i = \frac{\|u_i\| \|v_i\|}{|\langle u_i, v_i \rangle|}$$

$$\sigma^{\epsilon}(A) = \{\lambda \in \mathbb{C} : \|(\lambda \mathrm{Id} - A)^{-1}\| > 1/\epsilon\}$$
$$= \{\lambda \in \mathbb{C}, \exists \delta A \in M_n(\mathbb{C}), \|\delta A\| < \epsilon : \lambda \in \sigma(A + \delta A)\}$$

First definition: the resolvent can be very large far from the spectrum
Second definition: points in the pseudospectrum are eigenvalues of the perturbed operator
Under perturbations, the spectrum migrates out to the boundaries of the pseudospectrum

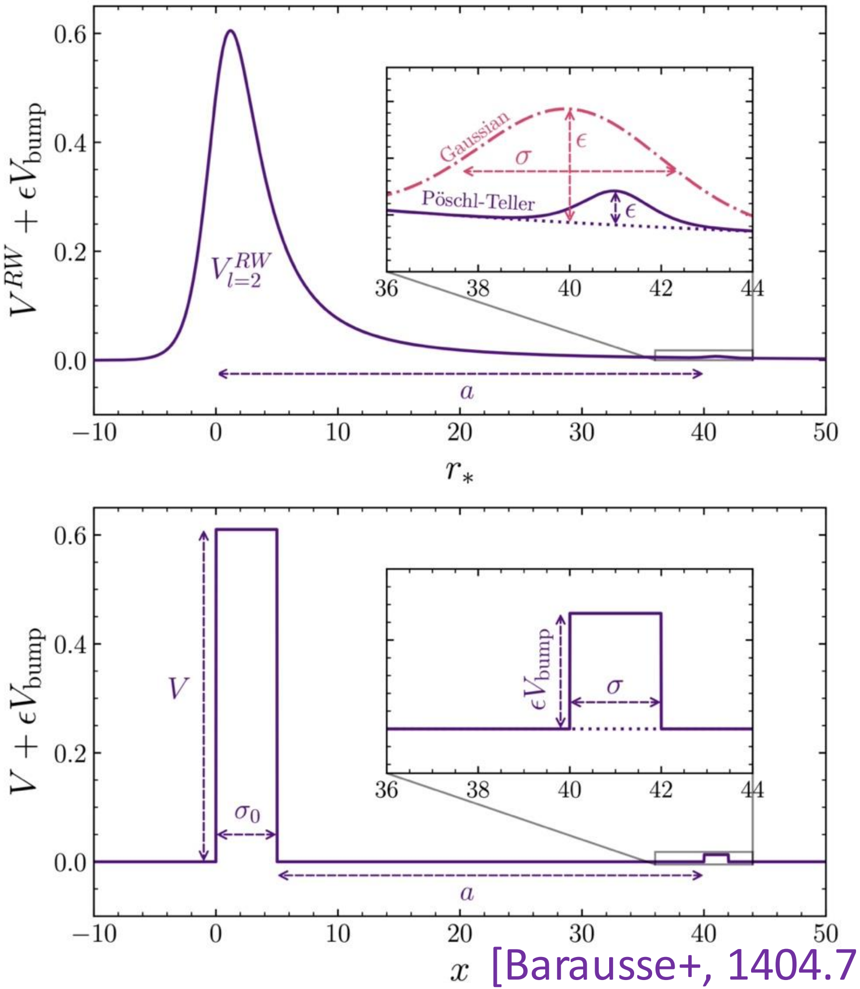
[Jaramillo+, 2004.06434]

Is the spectrum stable? Pseudospectrum: no!



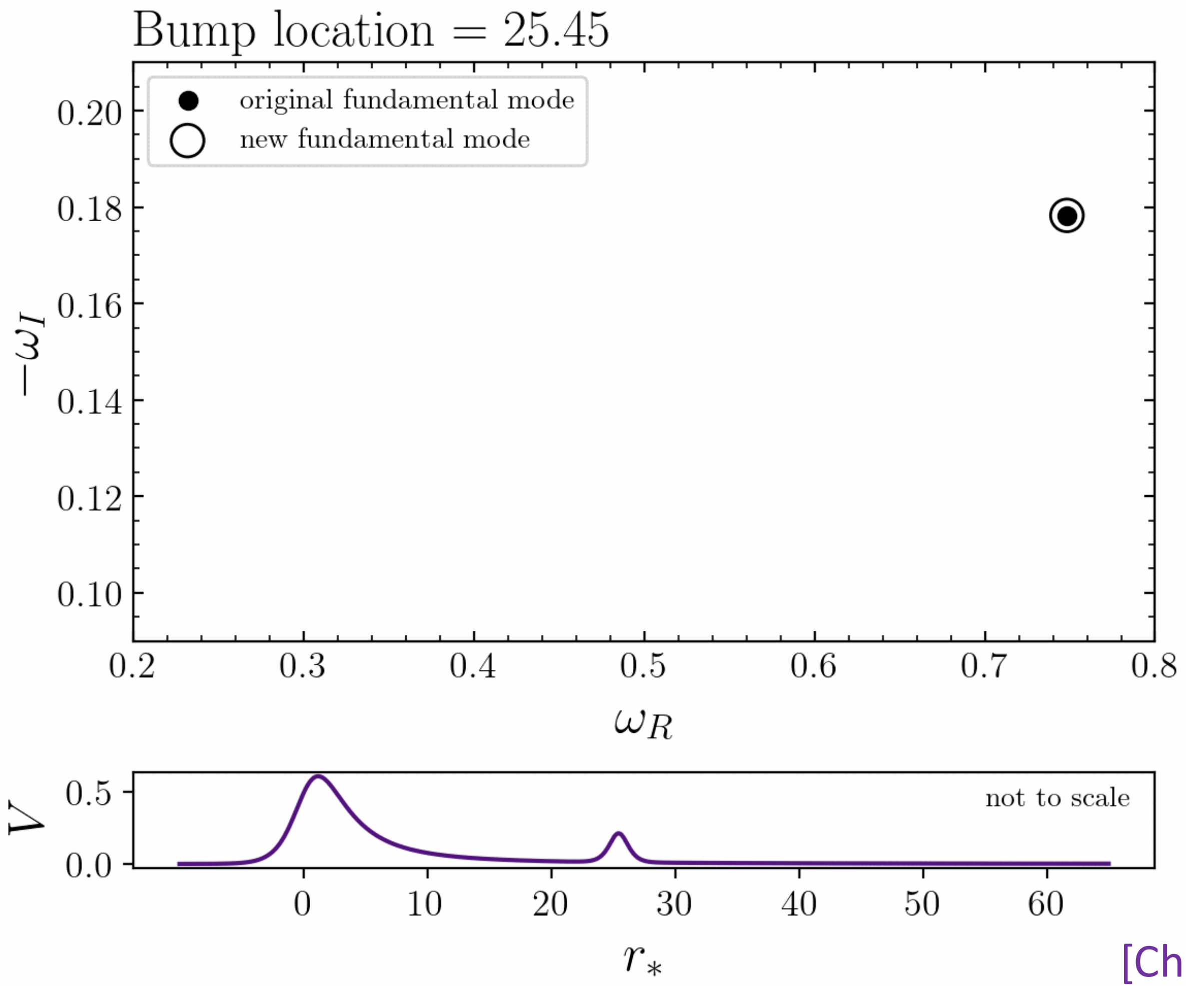
[Jaramillo+, 2004.06434]

Is the fundamental mode stable? The elephant and the flea



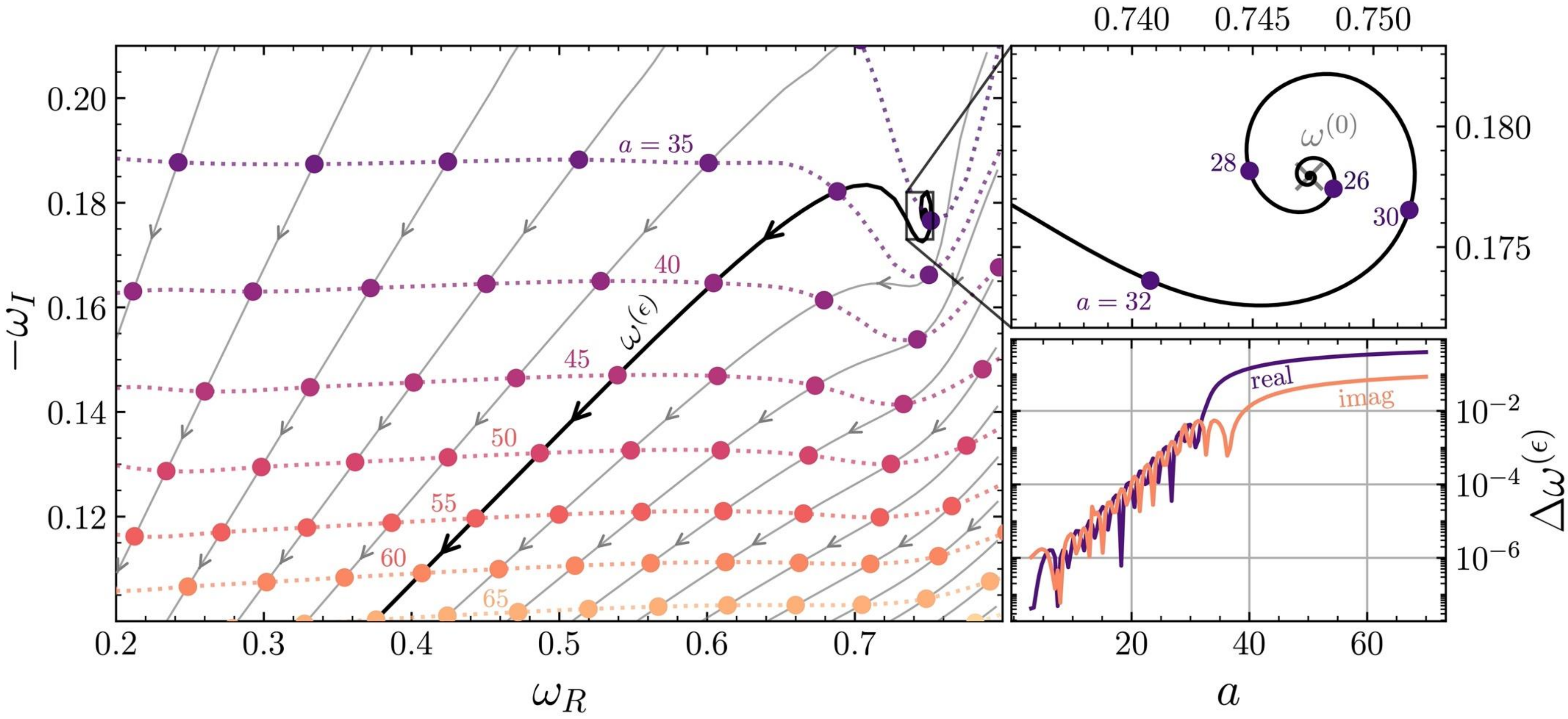
[Barausse+, 1404.7149; Cheung+, 2111.05415]

Is the fundamental mode stable? The elephant and the flea



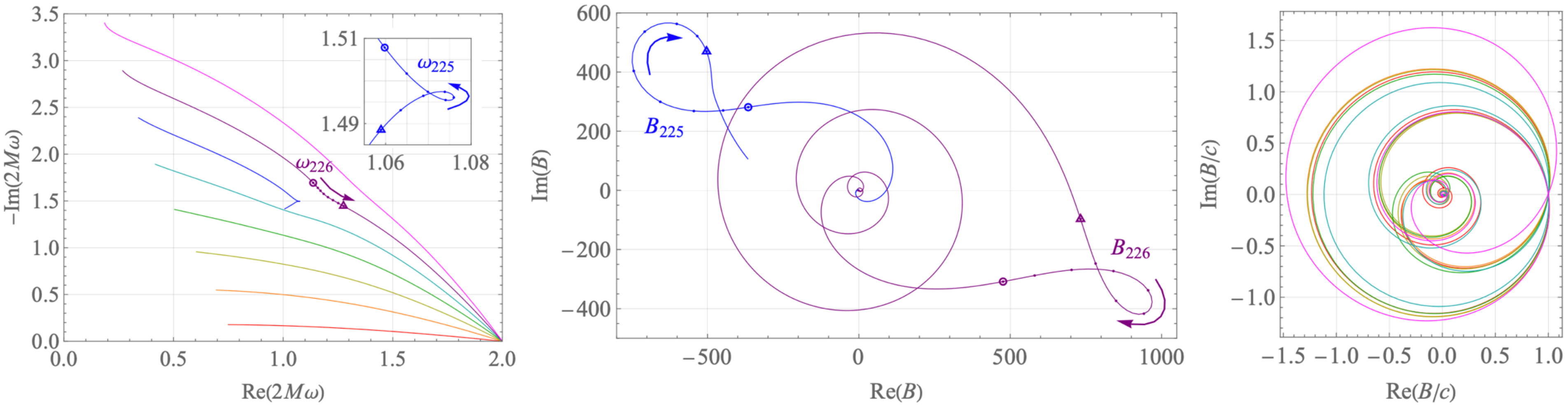
[Cheung+, 2111.05415]

Is the fundamental mode stable? No – but this does not affect spectroscopy



[Cheung+, 2111.05415; see also EB+, 2205.08547; Yang+ 2407.20131; Ianniccari+, 2407.20144]

Avoided crossings and resonant excitation



[Motohashi, 2407.15191]

Are QNMs and excitation factors stable? Change boundary conditions

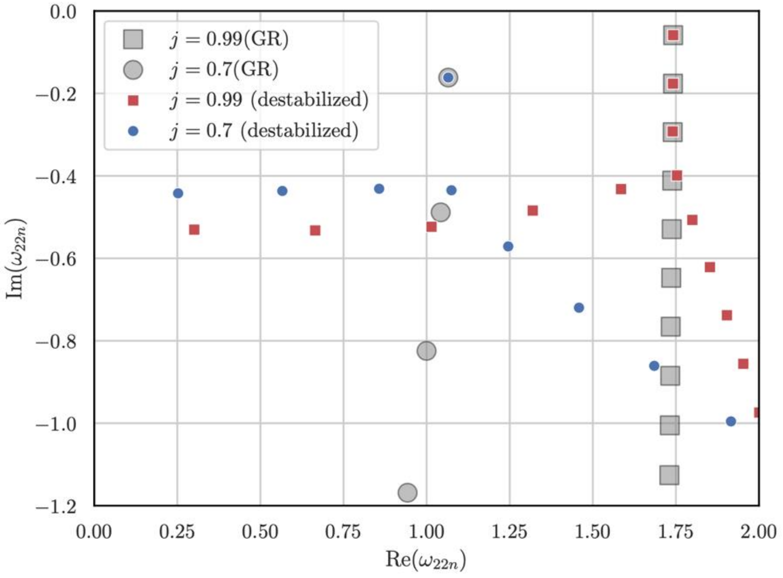


FIG. 1. Gray markers are the $\ell = m = 2$ QNM frequencies of the Sasaki-Nakamura equation in GR for $j = 0.7$ (circles) and 0.99 (squares). Blue circles and red squares correspond to the QNM frequencies for a Pöschl-Teller perturbation with $V_0 = 10^{-4}$, $x_0 = 10$, and $\sigma = 1$ in Eq. (3).

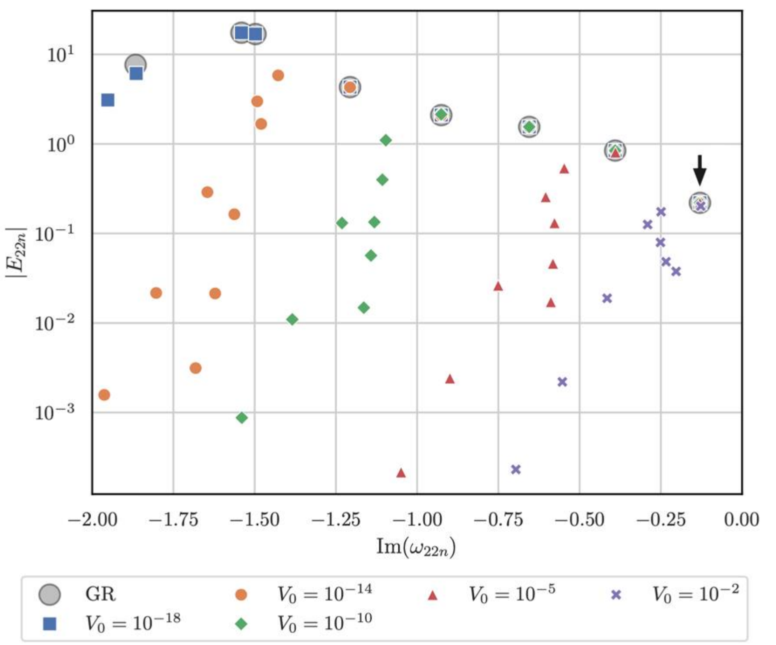


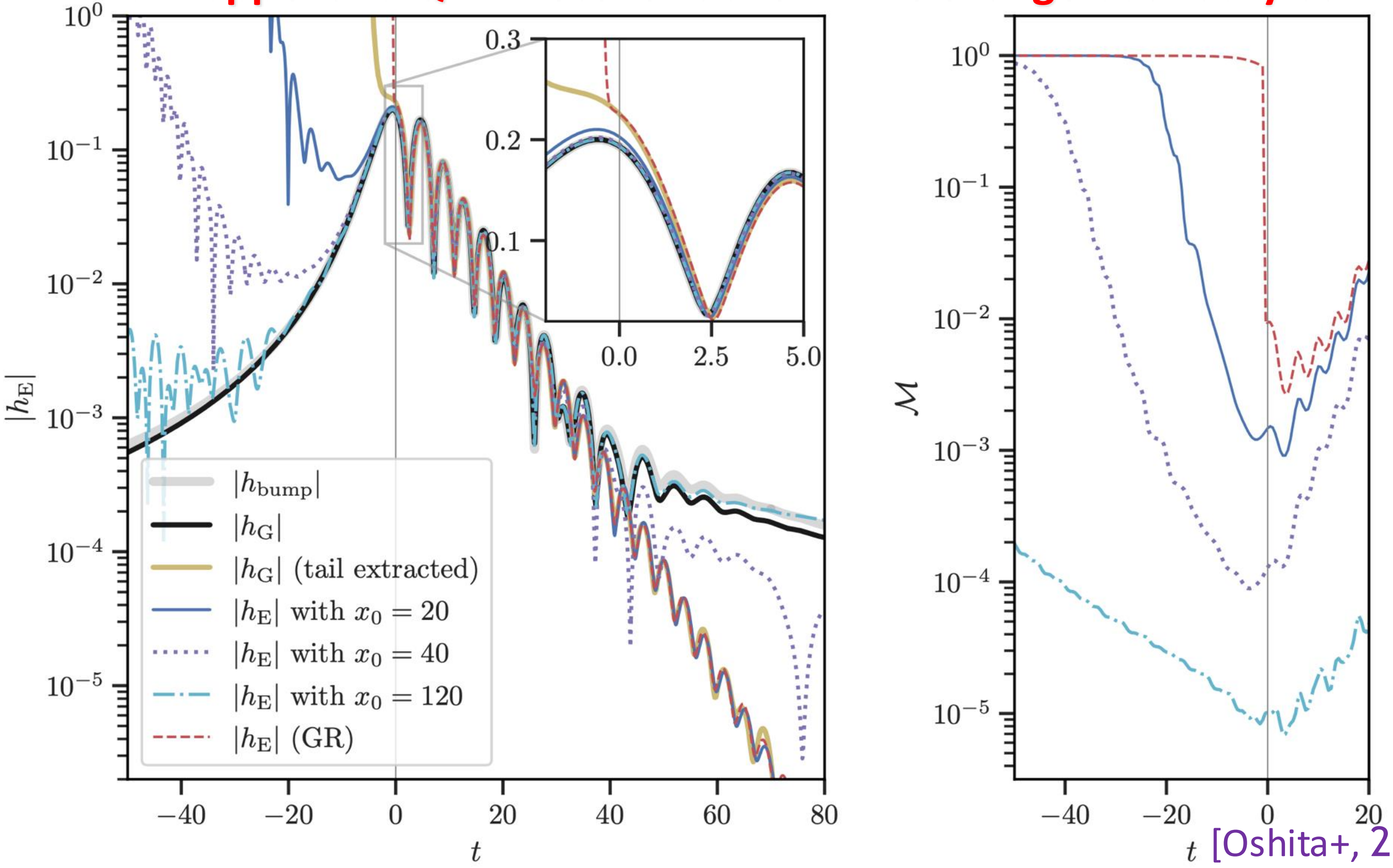
FIG. 2. Absolute value of the QNMEFs $|E_{22n}|$ in GR and for destabilized modes with $j = 0.9$, $x_0 = 10$, $\sigma = 1$ and different values of V_0, plotted as functions of $\mathrm{Im}(\omega_{22n})$. The QNMEF of the fundamental mode $(2, 2, 0)$ is marked by an arrow.

$$X_{\ell m\omega} = \begin{cases} e^{-ik_{\mathrm{H}} r_*} & (r_* \to -\infty)\,, \\ e^{i\omega r_*} + \epsilon(\omega) e^{-i\omega(r_* - 2x_0)}, & (r_* \to \infty) \end{cases}$$

$$k_{\mathrm{H}} \equiv \omega - m\Omega_{\mathrm{H}}$$

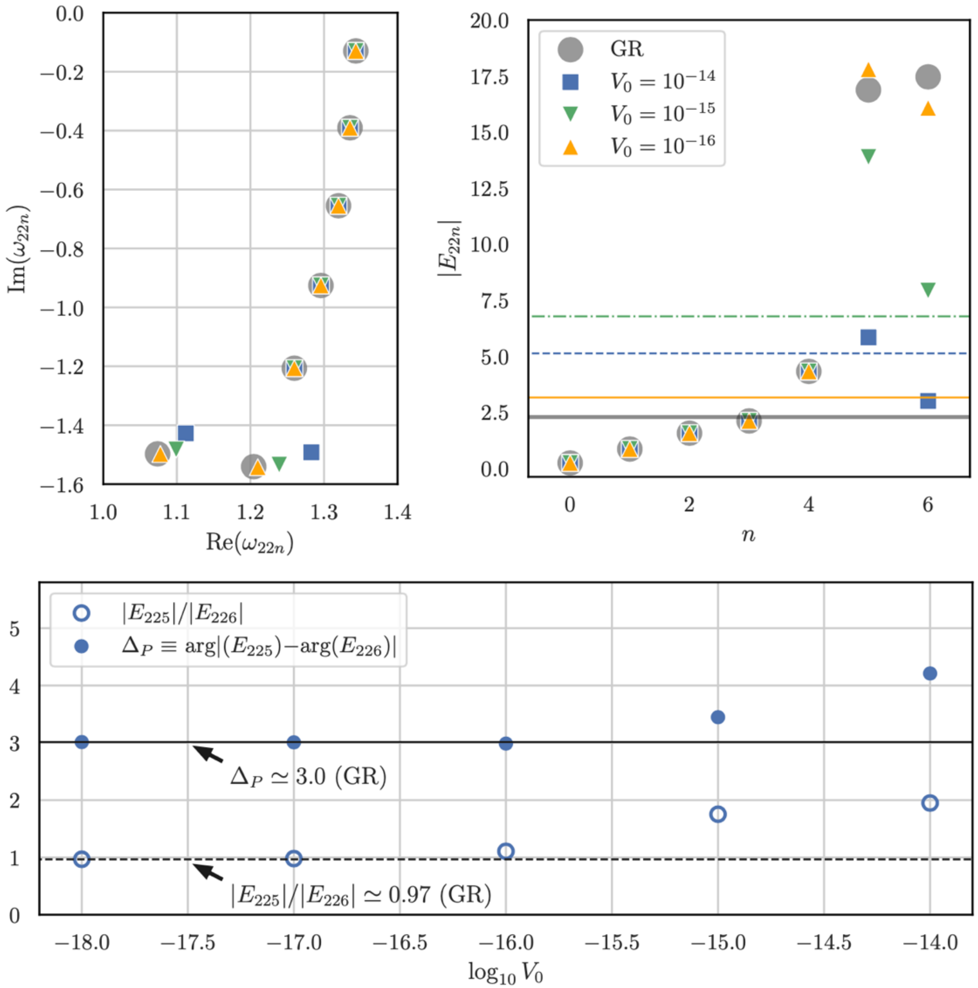
[Oshita+, 2503.21276]

What happens to QNM reconstruction if we change boundary conditions?



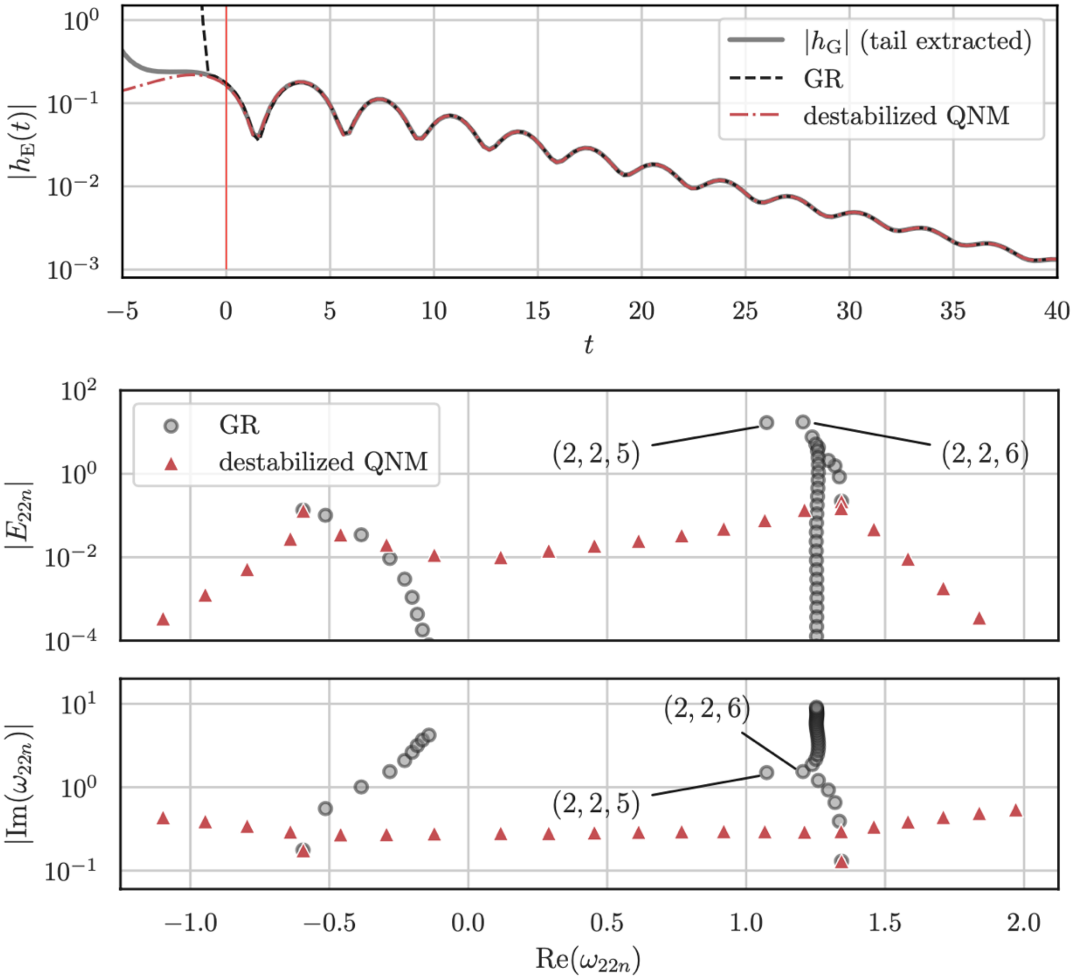
[Oshita+, 2503.21276]

Are the excitation factors stable at avoided crossings?



[Oshita+, 2503.21276]

Is the time-domain waveform stable at avoided crossings?



[Oshita+, 2503.21276]

Exceptional points

Consider now **massive** scalar perturbations of Kerr: **hysteresis**

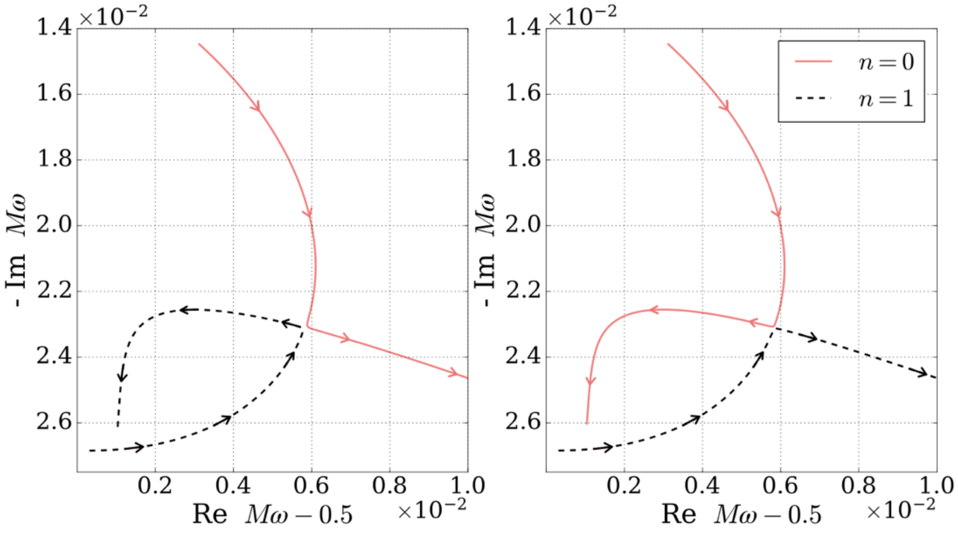
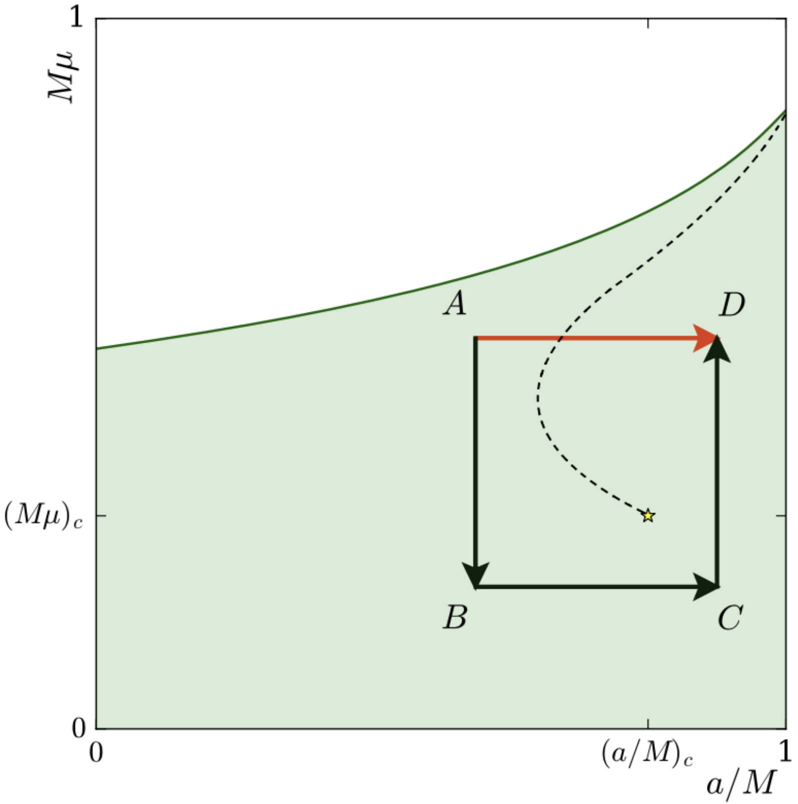


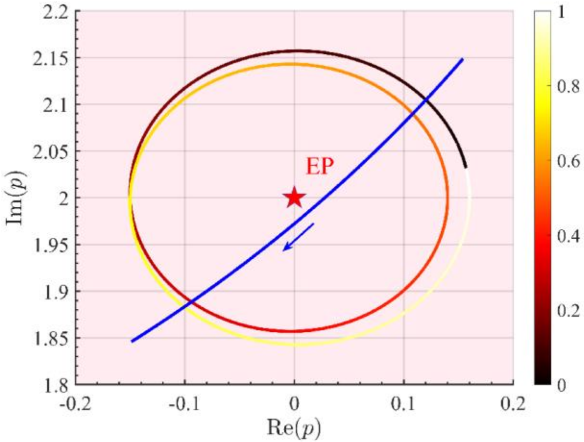
FIG. 3. The $n = 0$ and $n = 1$ QNMs as a function of $M\mu$ for $a/M \lesssim (a/M)_c$ (left) and $a/M \gtrsim (a/M)_c$ (right). The arrows denote the direction of increasing $M\mu \in [0.0, 0.8]$. At $\{a/M, M\mu\} = \{(a/M)_c, (M\mu)_c\}$, the QNMs become degenerate and transform into each other, resembling the level crossing behavior of thermodynamical systems.

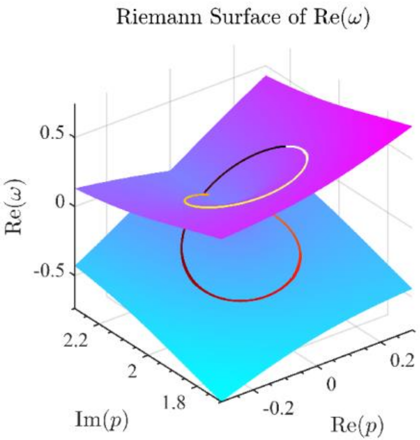


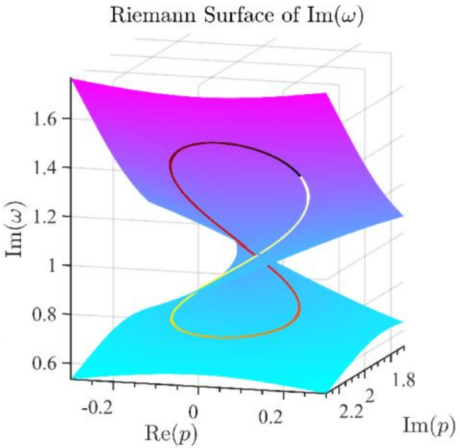
[Cavalcante+, 2407.20850]

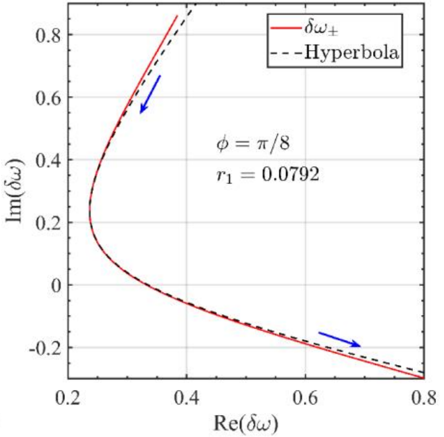
Two Riemann sheets!

$$A_{\text{in}}(\omega_n, p_i) = 0 \qquad \nabla_{\boldsymbol{p}}\omega_n = -\frac{1}{A'_{\text{in}}(\omega_n, p_i)}\nabla_{\boldsymbol{p}}A_{\text{in}}(\omega_n, p_i) \qquad \delta\omega_{nm} = \sqrt{\boldsymbol{A}_{\boldsymbol{nm}}\cdot(\boldsymbol{p}-\boldsymbol{p}_\star)}$$







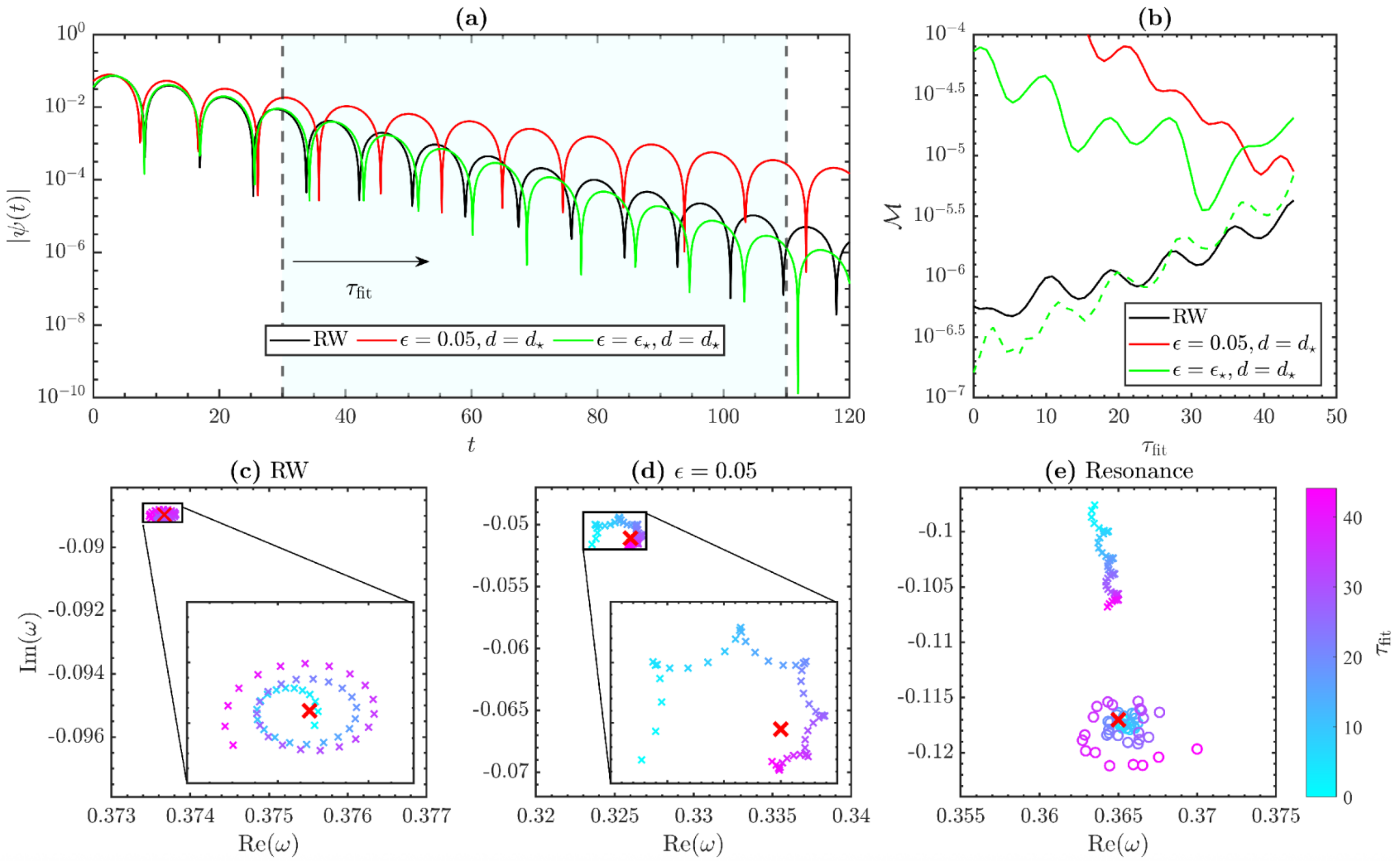


$$E_n e^{-i\omega_n t} + E_m e^{-i(\omega_n-\delta\omega_{nm})t} \approx 2\sin\left(\frac{\delta\omega_{nm}t}{2}\right)E_n e^{-i\omega_n t}$$

$$\approx \left[-\frac{A_{\text{out}}(\omega_n)}{2i\omega_n f(\omega_n)}\right]\times t e^{-i\omega_n t}$$

[Yang+, 2504.06072]

The linear growth in time matters when extracting QNM frequencies



Take-home messages

Addition of overtones long known to provide a better fit to:

point-particle waveforms, nonrotating (1970s) and rotating (1980s) collapse
head-on black hole collisions (1990s), quasicircular mergers (circa 2005)

Can a linear superposition of overtones describe nonlinear mergers up to the peak? No

Clear evidence (now from multiple groups) of nonlinear modes in numerical waveforms
jaxqualin, variable projection: systematic extraction of linear and nonlinear modes from NR simulations
Need more modeling of nonlinear modes (merger simulations, high-order perturbation theory)

Have we observed overtones in GW150914?

At best inconclusive - analysis **at or before** the peak, where the linear model is not applicable;
noise can induce artificial evidence for an overtone
Best bet for O4/O5: higher multipole observation in high mass, unequal-mass events

Spectral instability, avoided crossings and exceptional points

Not problematic for low-n modes; can be used to find a new expansion basis
Time domain waveform stable (but possibly affected) in the presence of exceptional points

Future

LISA (and less likely, XG detectors on the ground) may observe nonlinear modes
Ringdown bounds on theories with mass-dependent scales severely limited by SNR/curvature interplay
Beware of false general relativity violations! Systematics
Complementarity with ngEHT, BHEX: imaging & ringdown probe same physics (light ring)