IBS-CTPU-CGA QNM & BH Perturbation Workshop, May 27th (Tue), 2025

Excitability, convergence and (in)completeness of quasi-normal modes

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Quasi-normal modes (QNMs)

Black hole's "free oscillation" → excitation of BH QNMs

wave equation (Regge-Wheeler, Teukolsky, …)

$$\left[\partial_{r^*}^2 + \omega^2 - V(r^*)\right] \psi_\omega(r^*) = S_\omega(r^*)$$

e.g., for Schwarzschild

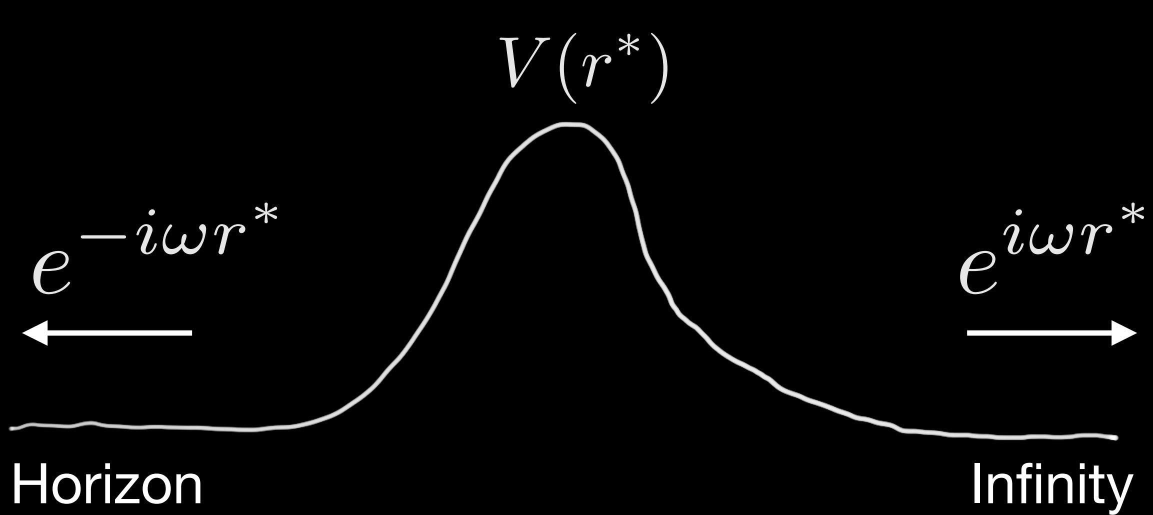
$$r^* \equiv r + r_g \log(r/r_g - 1)$$

horizon $r \to r_+, \; r^* \to -\infty$

infinity $r \to +\infty, \; r^* \to +\infty$

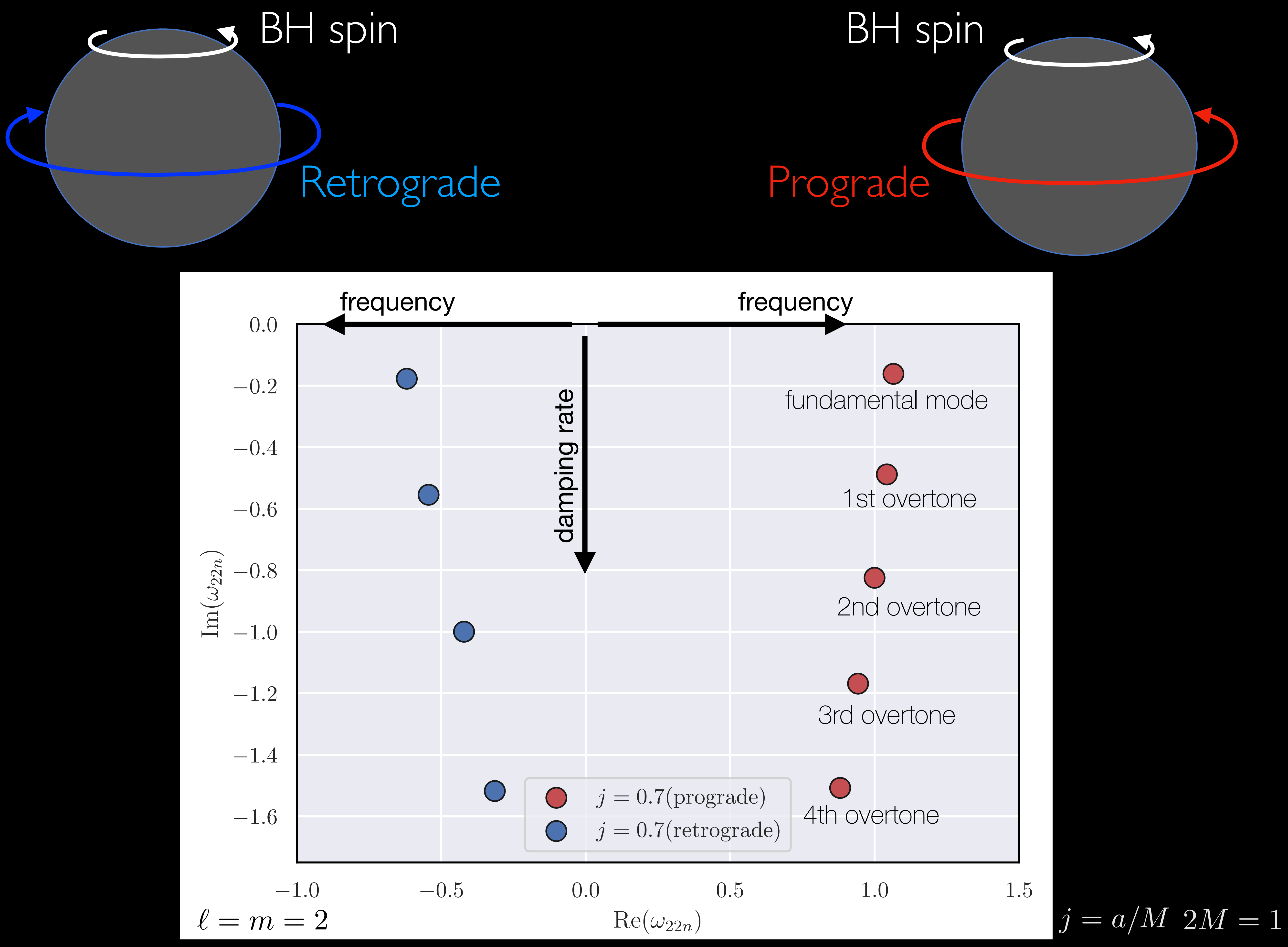
Regge, Wheeler (1957)
Zerilli (1970)
Teukolsky (1972)
Chandrasekhar, Detweiler (1976)
Detweiler (1977)
Sasaki, Nakamura (1982)

boundary condition



This B. C. is satisfied
only at complex frequencies:

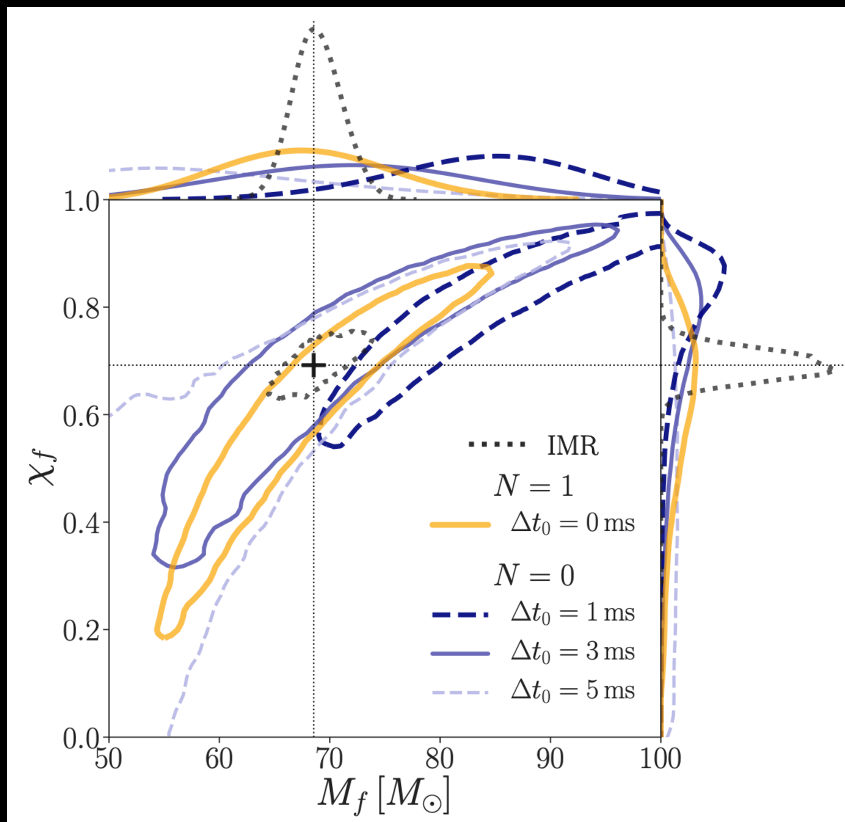
$$\omega = \omega_n \in \mathbb{C}$$



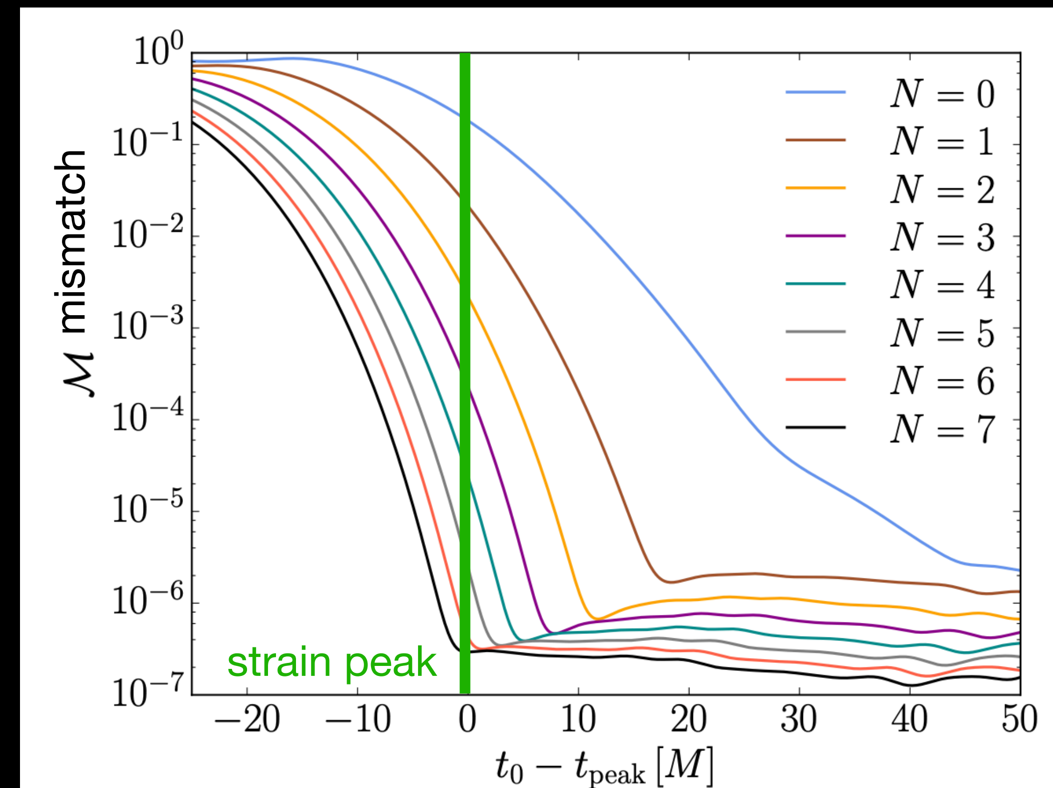
BH spin
Retrograde
BH spin
Prograde
frequency
frequency
damping rate
fundamental mode
1st overtone
2nd overtone
3rd overtone
4th overtone
$j = 0.7$(prograde)
$j = 0.7$(retrograde)
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
−1.4
−1.6
$\mathrm{Im}(\omega_{22n})$
−1.0
−0.5
0.0
0.5
1.0
1.5
$\mathrm{Re}(\omega_{22n})$
$\ell = m = 2$
$j = a/M \;\; 2M = 1$

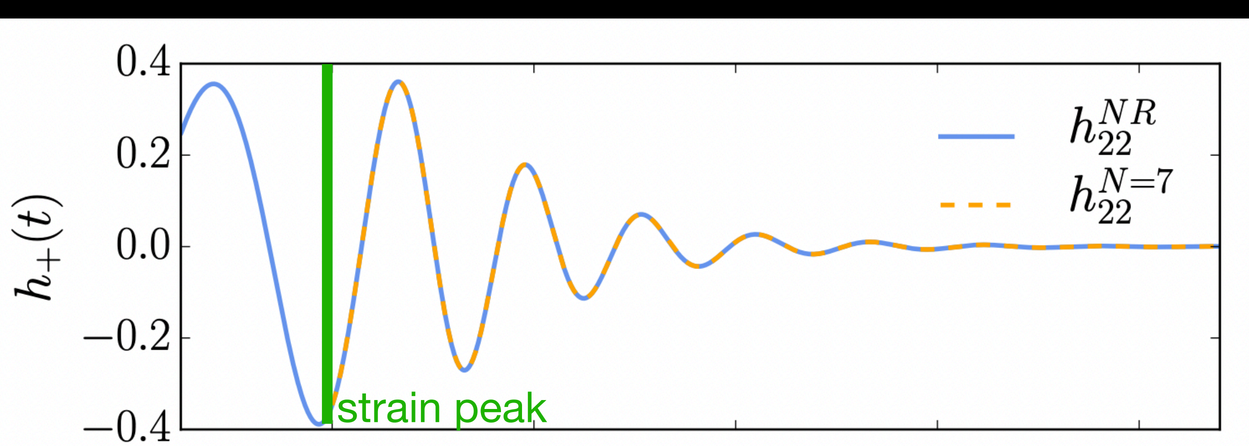
Overtones? Onset of QNM excitations?

Isi+ (2019)



Giesler+ (2019)





Fundamental mode ⟷ Higher overtones

N	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	$t_{\mathrm{fit}} - t_{\mathrm{peak}}$
0	0.971	-	-	-	-	-	-	-	47.00
1	0.974	3.89	-	-	-	-	-	-	18.48
2	0.973	4.14	8.1	-	-	-	-	-	11.85
3	0.972	4.19	9.9	11.4	-	-	-	-	8.05
4	0.972	4.20	10.6	16.6	11.6	-	-	-	5.04
5	0.972	4.21	11.0	19.8	21.4	10.1	-	-	3.01
6	0.971	4.22	11.2	21.8	28	21	6.6	-	1.50
7	0.971	4.22	11.3	23.0	33	29	14	2.9	0.00

Excitation of QNMs

Leaver (1986) Zhang, Berti, Cardoso (2013)

$$h = \frac{e^{im\phi}}{r} \int d\omega dr' \sum_{lm} e^{i\omega(r^* - t + t_0)} {}_{-2}S_{lm}(\omega, \theta) G^{(\mathrm{BH})}_{lm}(r, r') \tilde{T}_{lm}(r', \omega)$$

(spin-weighted) spheroidal harmonic function — **Green's function** — **source term**

$$= \frac{1}{r} \sum_{lmn} E_{lmn} T_{lmn} S_{lmn} e^{-i\omega_{lmn}(t - r^*)} \qquad S_{lmn} \equiv {}_{-2}S_{lm}(\omega_{lmn}, \theta)$$

Source factor:
Initial data of a distorted BH

Excitation factor:
Intrinsic quantity of BHs
Quantify the "ease-of-excitaton" of QNMs
Residues of Green's function

$$E_{lmn} \equiv \frac{A^{(\mathrm{out})}_{lm}(\omega_{lmn})}{2i\omega^3_{lmn}} \left(\frac{dA^{(\mathrm{in})}_{lm}}{d\omega} \right)^{-1}_{\omega = \omega_{lmn}}$$

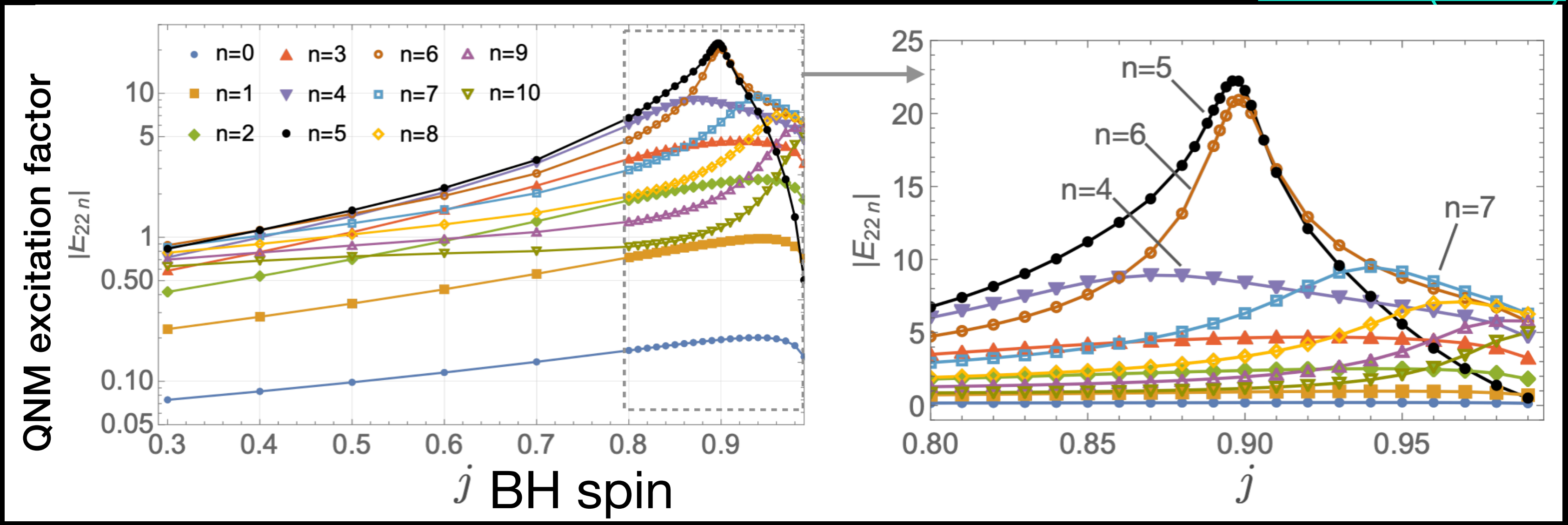
$$R^{(\mathrm{H})}_{lm}(\omega, r) = \begin{cases} A^{(\mathrm{trans})}_{lm}(\omega) \Delta^2 e^{-ikr^*} & \text{for } r^* \to -\infty, \\ r^{-1} A^{(\mathrm{in})}_{lm}(\omega) e^{-i\omega r^*} + r^3 A^{(\mathrm{out})}_{lm}(\omega) e^{i\omega r^*} & \text{for } r^* \to +\infty, \end{cases}$$

$$R^{(\infty)}_{lm}(\omega, r) = \begin{cases} B^{(\mathrm{in})}_{lm}(\omega) \Delta^2 e^{-ikr^*} + B^{(\mathrm{out})}_{lm}(\omega) e^{+ikr^*} & \text{for } r^* \to -\infty, \\ r^3 B^{(\mathrm{trans})}_{lm}(\omega) e^{i\omega r^*} & \text{for } r^* \to +\infty. \end{cases}$$

QNM excitation

Leaver (1986) Zhang, Berti, Cardoso (2013) Oshita (2021)

Oshita (2021)



QNM excitation factor

$$h = \sum_n (E_{\ell mn} \times T_{\ell mn}) e^{-i\omega_{\ell mn} t}$$

Green's function source

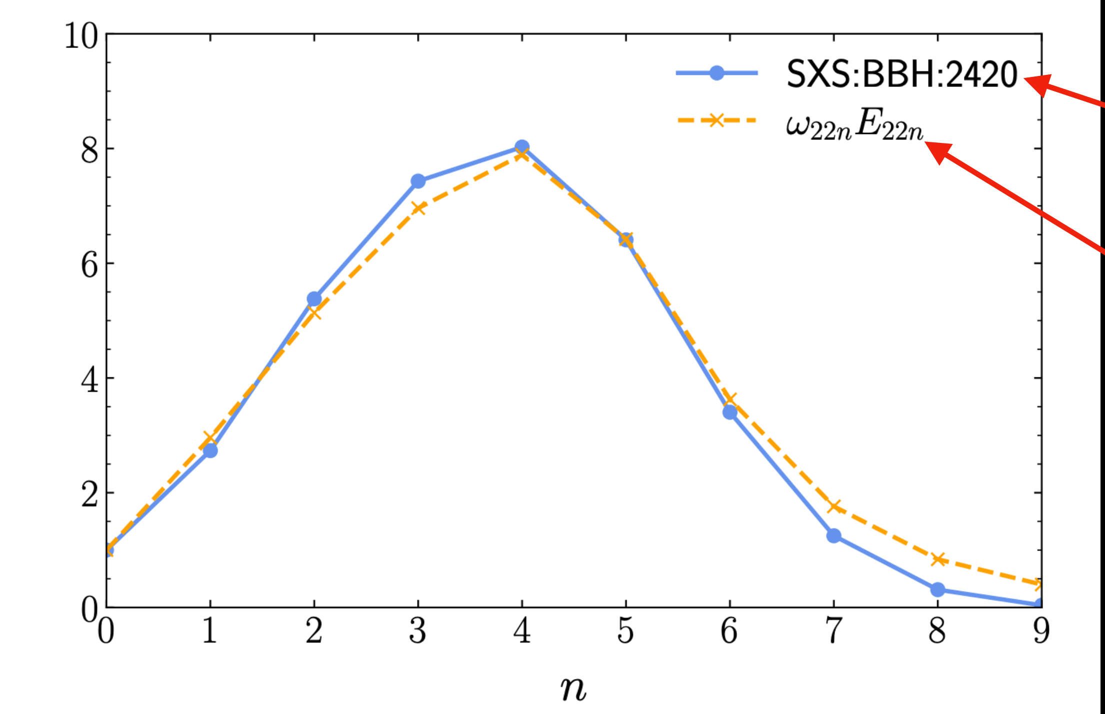


FIG. 12. The $(2, 2, n)$ NR amplitudes as measured in the moderate-spin case of SXS:BBH:2420 and the excitation factors $\omega_{22n}E_{22n}$ for $\chi = 0.75$. The excitation factors are computed using $t_0 = 2.5$ and both curves are normalized such that $C_{220} = \omega_{220}E_{220} = 1$.

QNM&QQNM fit (NR + data analysis)

Theory (excitation factor)

Giesler, Ma, Mitman, Oshita, Teukolsky, et al. (2024)

see also Mitman + (2025)

Extraction of multiple QNMs?
Controversial …

Recent studies claim that the inclusion of the fundamental mode and 7 overtones provides a very accurate description of the ringdown up to the peak strain amplitude, and significantly reduces the uncertainty in the extracted remnant properties [106]. However, we show that the higher overtones lead to very small mismatches by merely overfitting the waveforms. Furthermore, we argue that these higher overtones try to fit other physics (such as time variation in the QNM amplitudes due to initial data, an evolving spacetime background, and non-

Baibhav+ (2023)

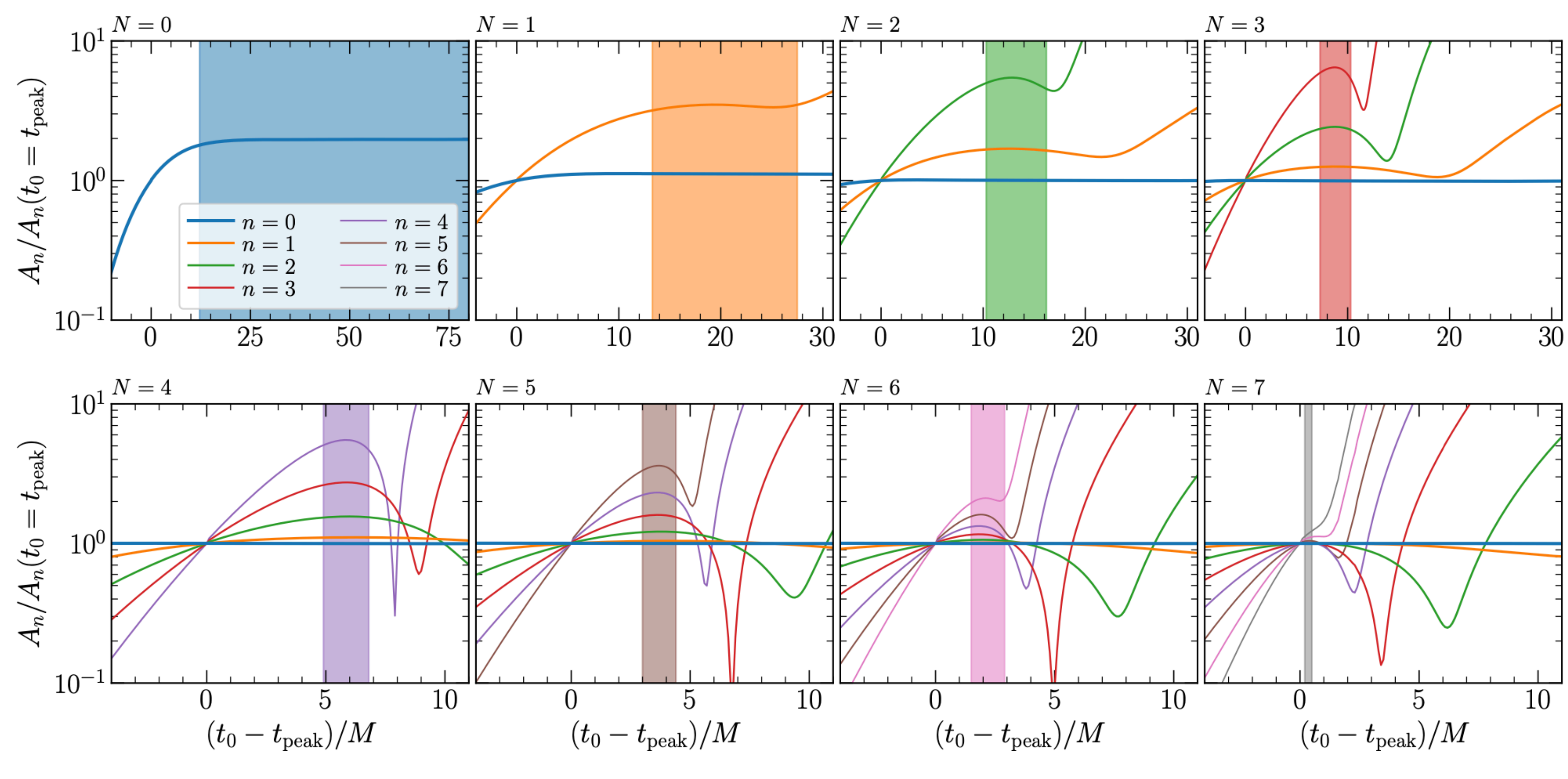
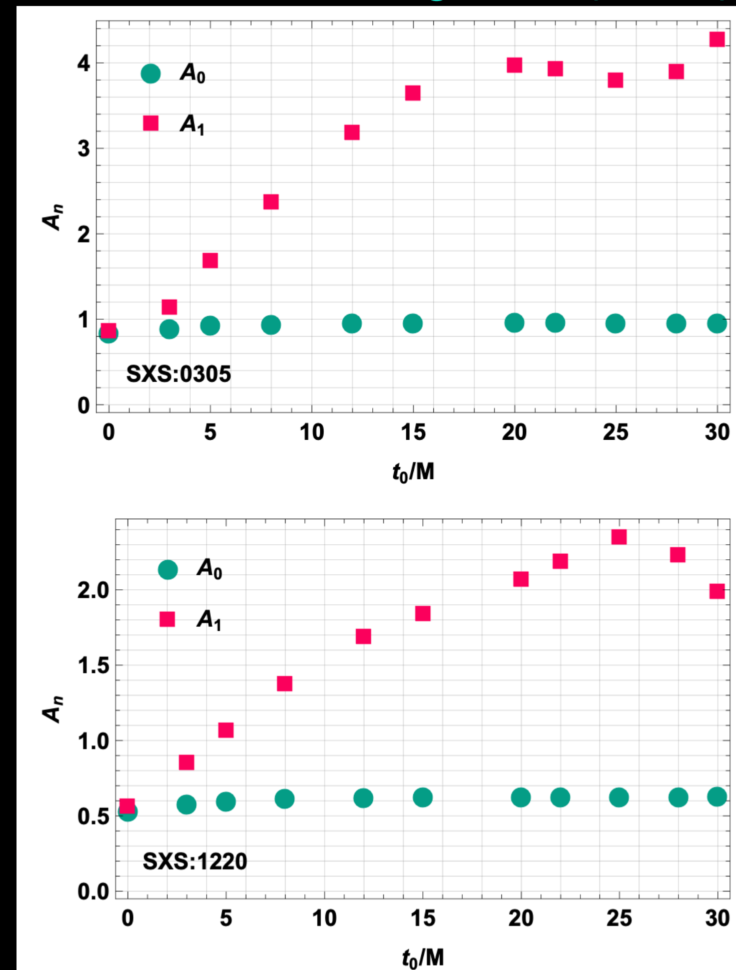


FIG. 6. Amplitude $A_n^N(t_0)/A_n^N(t_0 = t_{\rm peak})$ of QNMs as a function of the starting time t_0 for the SXS:BBH:0305 simulation. The shaded regions show the largest time range such that the amplitude of the highest overtone ($n = N$) is constant within 10%.

Bhagwat+ (2019)

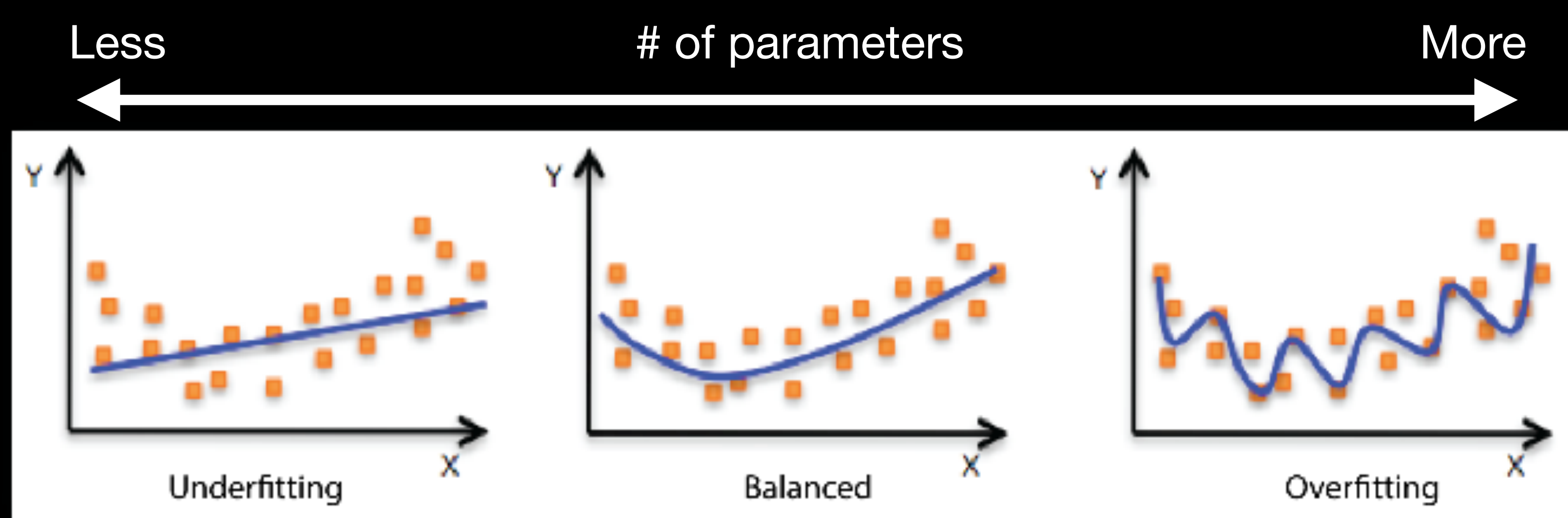


A growing cast — but at what cost?

linear QNMs, quadratic QNMs, late-time tails, (memory effects…)

The stage is getting crowded – with parameters

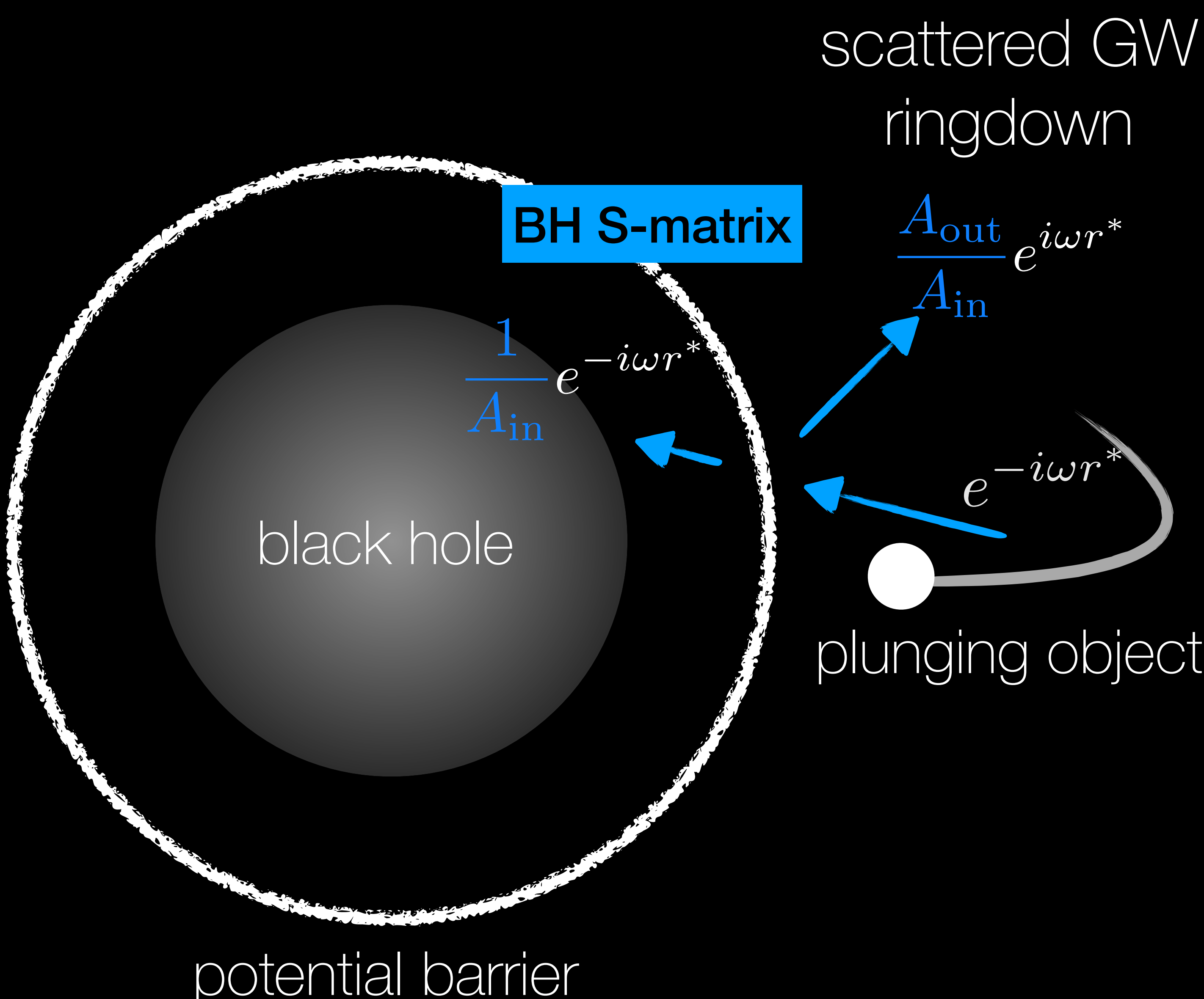
- Extracting QN modes from GW data involves many fitting parameters.
- Sensitive to the choice of the start time of ringdown.

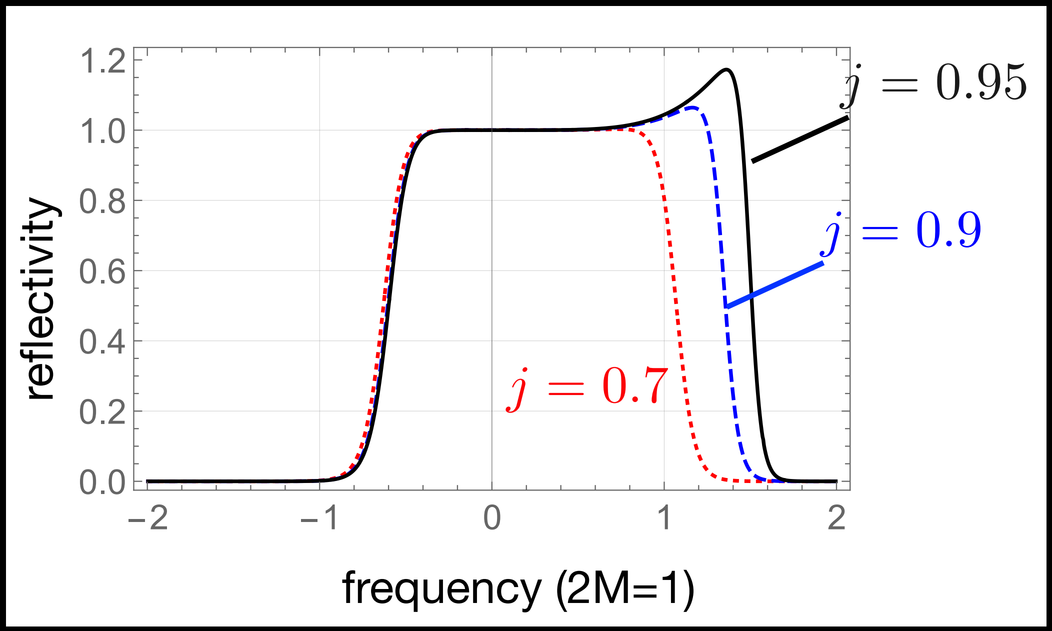


“Amazon Machine Learning”

Black hole greybody factor / S-matrix

NO (2022), NO (2023)
Okabayashi, NO (2024)
Rosato, Destounis, Pani (2024)
NO, Takahashi, Mukohyama (2024)





$l = m = 2$

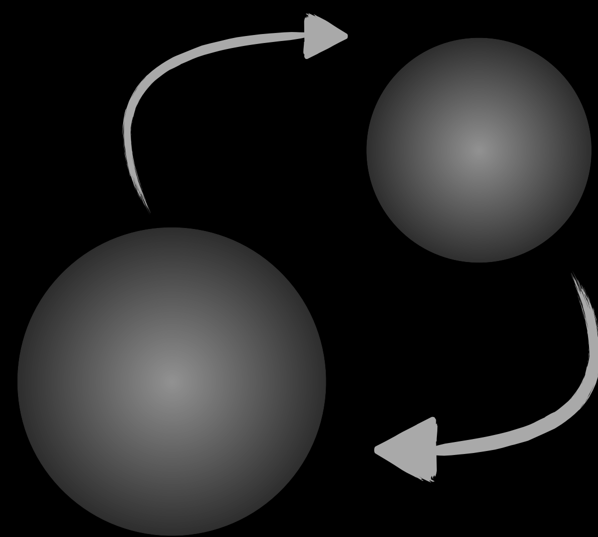
$$\text{reflectivity} = 1 - \Gamma$$

$$\text{reflectivity} = \left|\frac{A_{\rm out}}{A_{\rm in}}\right|^2$$

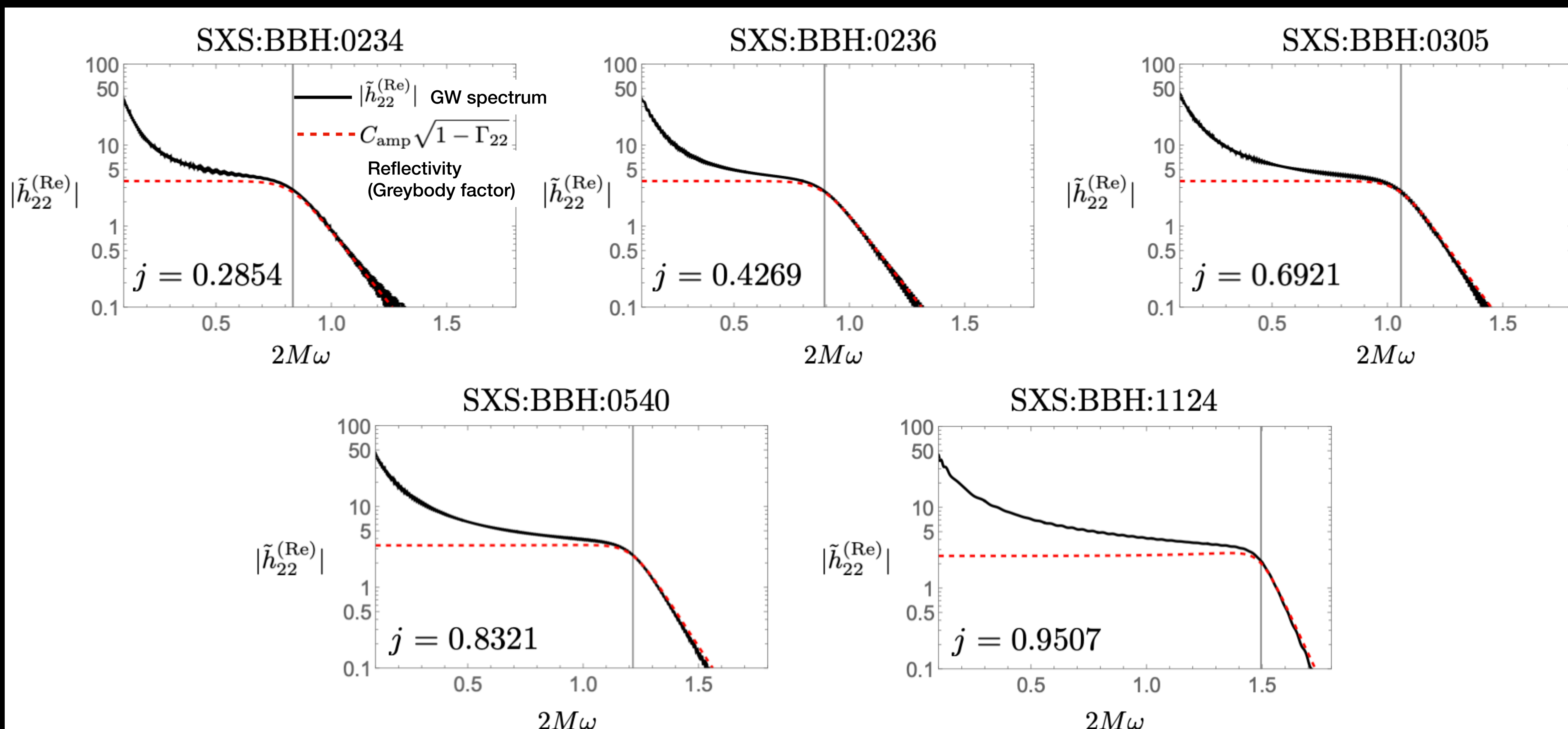
$$\text{greybody factor} \quad \Gamma = \left|\frac{1}{A_{\rm in}}\right|^2$$

Greybody factors imprinted in GW spectrum?

Numerical simulation of a binary black hole merger (SXS collaboration)



Okabayashi, NO (2024)



Why exponential decay? -conjecture-

First law of BH thermodynamics

surface gravity, horizon area, horizon frequency — Bekenstein (1972)

$$\frac{\kappa}{8\pi G}\delta \mathcal{A}_{\rm H} = -(\omega - m\Omega_{\rm H})$$

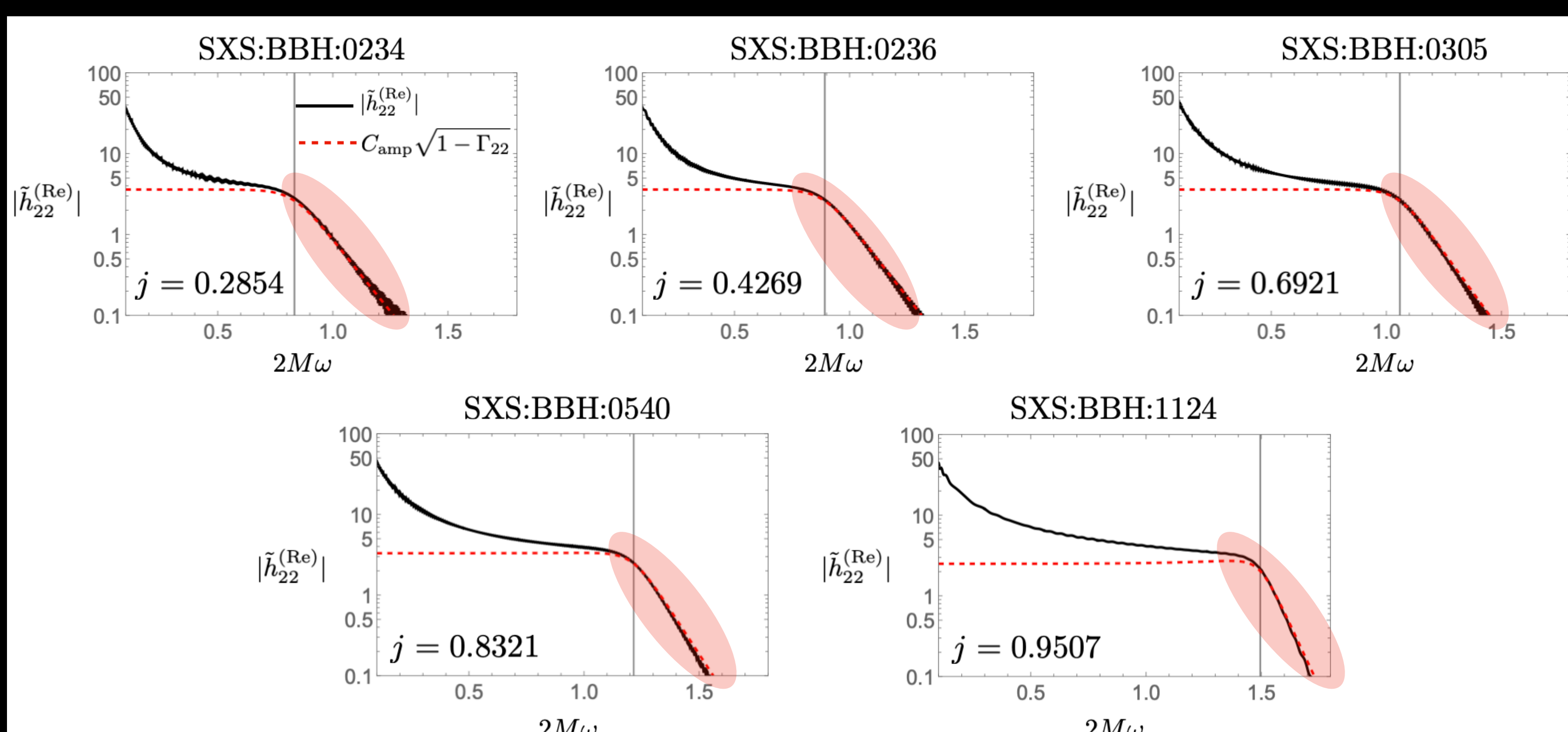
temperature, entropy, energy (emission), chemical potential

$$\omega > m\Omega_{\rm H} \rightarrow \delta S < 0$$

NOT strictly prohibited during a non-equilibrium process.

Dias, Emparan, Maccarrone (2007)

Boltzmann factor $e^{-(\omega - m\Omega_{\rm H})/T_{\rm H}} = e^{\delta S}$



← ~Boltzmann with the Hawking temperature

Oshita (2022)

Why is the greybody factor imprinted? -perturbation manner-

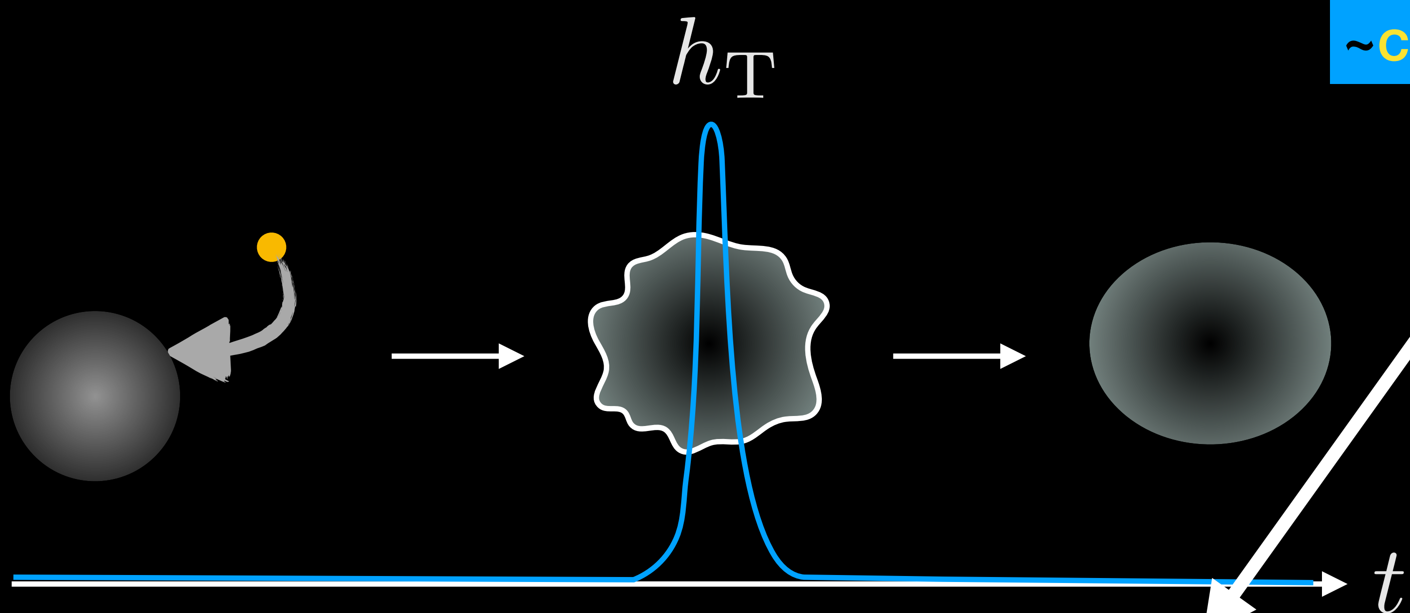
	GW waveform	universality Green's function	Initial data source
time domain			

$$h(t) = h_{\rm G}(t) * h_{\rm T}(t)$$

	GW spectrum	universality BH S-matrix	Initial data source
frequency domain			

$$\tilde{h}(\omega) = \tilde{h}_{\rm G}(\omega) \times \tilde{h}_{\rm T}(\omega)$$

~constant in frequency?



instantaneous source at the onset of ringdown?

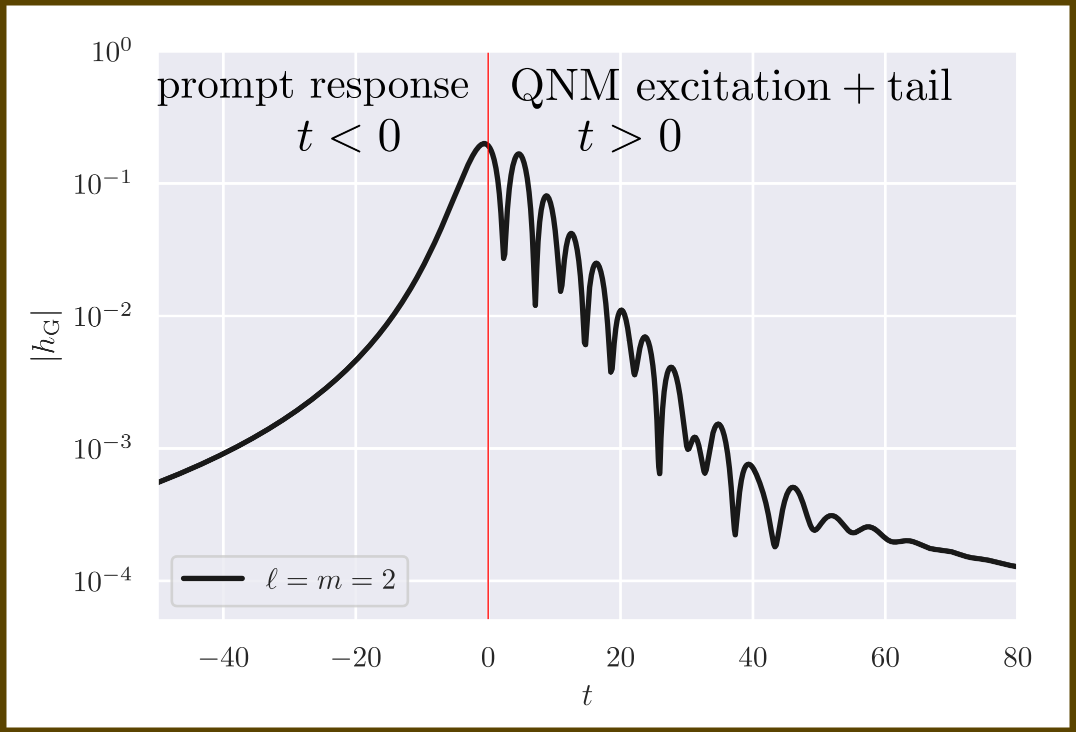
time domain

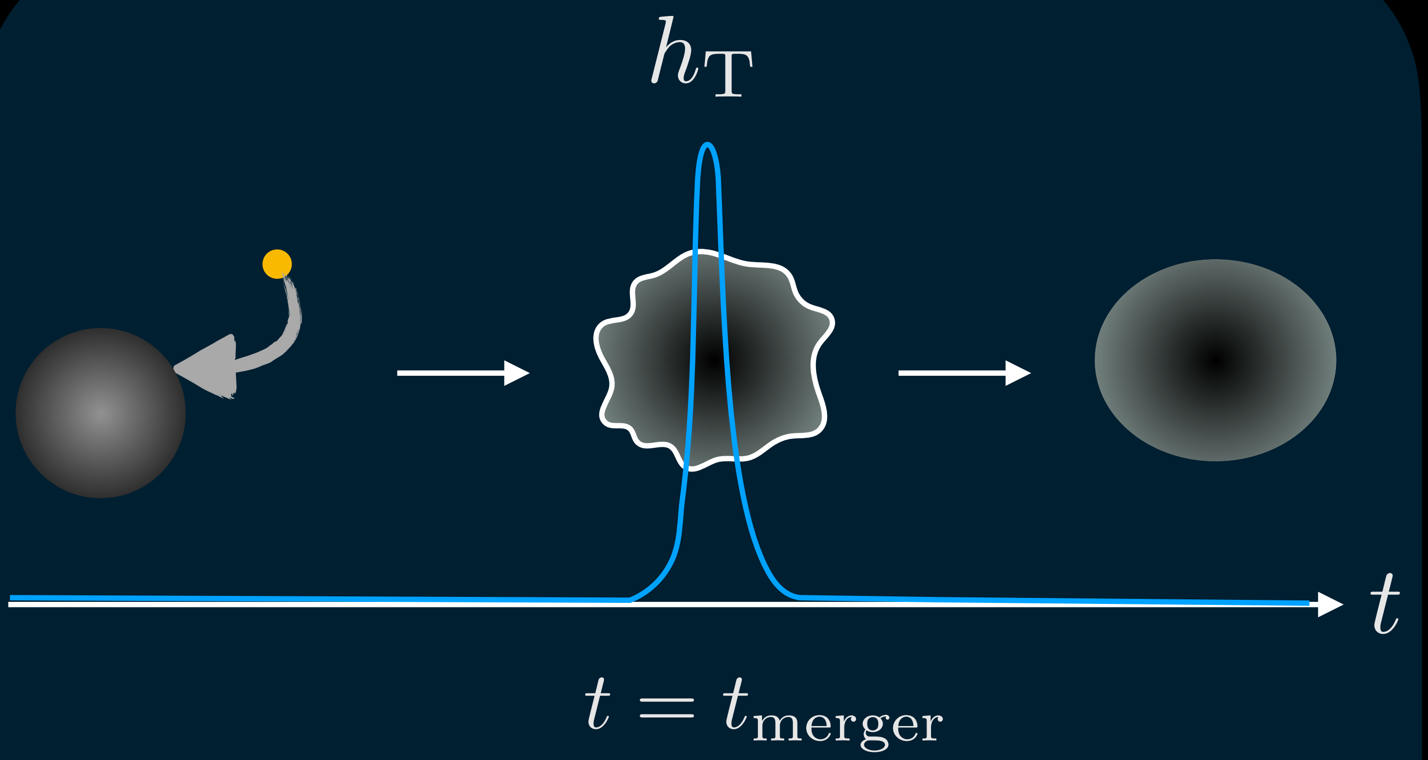
universality | Initial data

GW waveform | Green's function | source

$$h(t) = h_{\mathrm{G}}(t) * h_{\mathrm{T}}(t)$$

$$h_{\mathrm{G}} = \frac{1}{2\pi}\int d\omega \frac{A_{\mathrm{out}}(\omega)}{A_{\mathrm{in}}(\omega)} e^{-i\omega t}$$

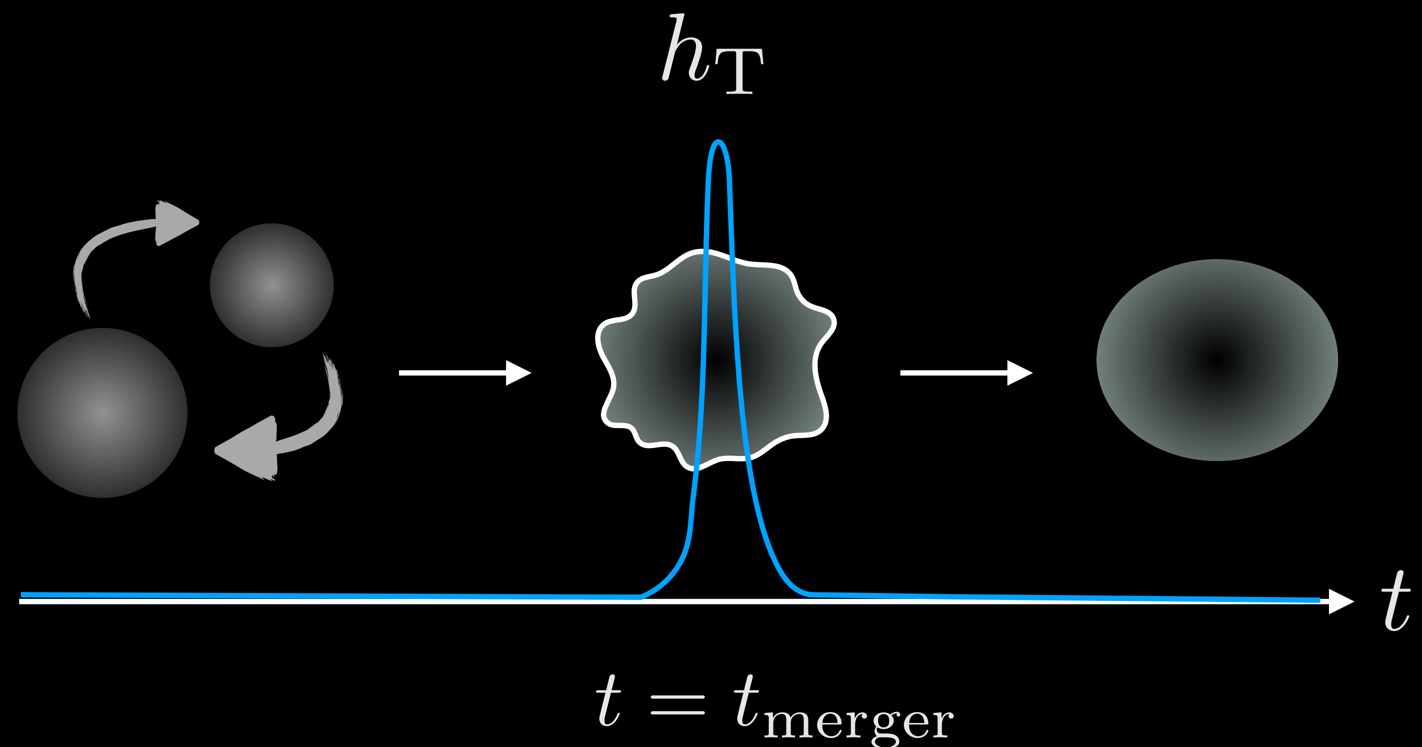




Systematically define h_{T} for non-linear systems.

Non-linear and linear QNMs overlap.

ringdown starting time?



$$h_{\mathrm{T}}(t) = \int d\omega \left[\tilde{h}(\omega)/\tilde{h}_{\mathrm{G}}(\omega)\right] e^{-i\omega t}$$

source

SXS spectrum

BH S-matrix

Instantaneous source at the merger of BBHs

Oshita in prep.

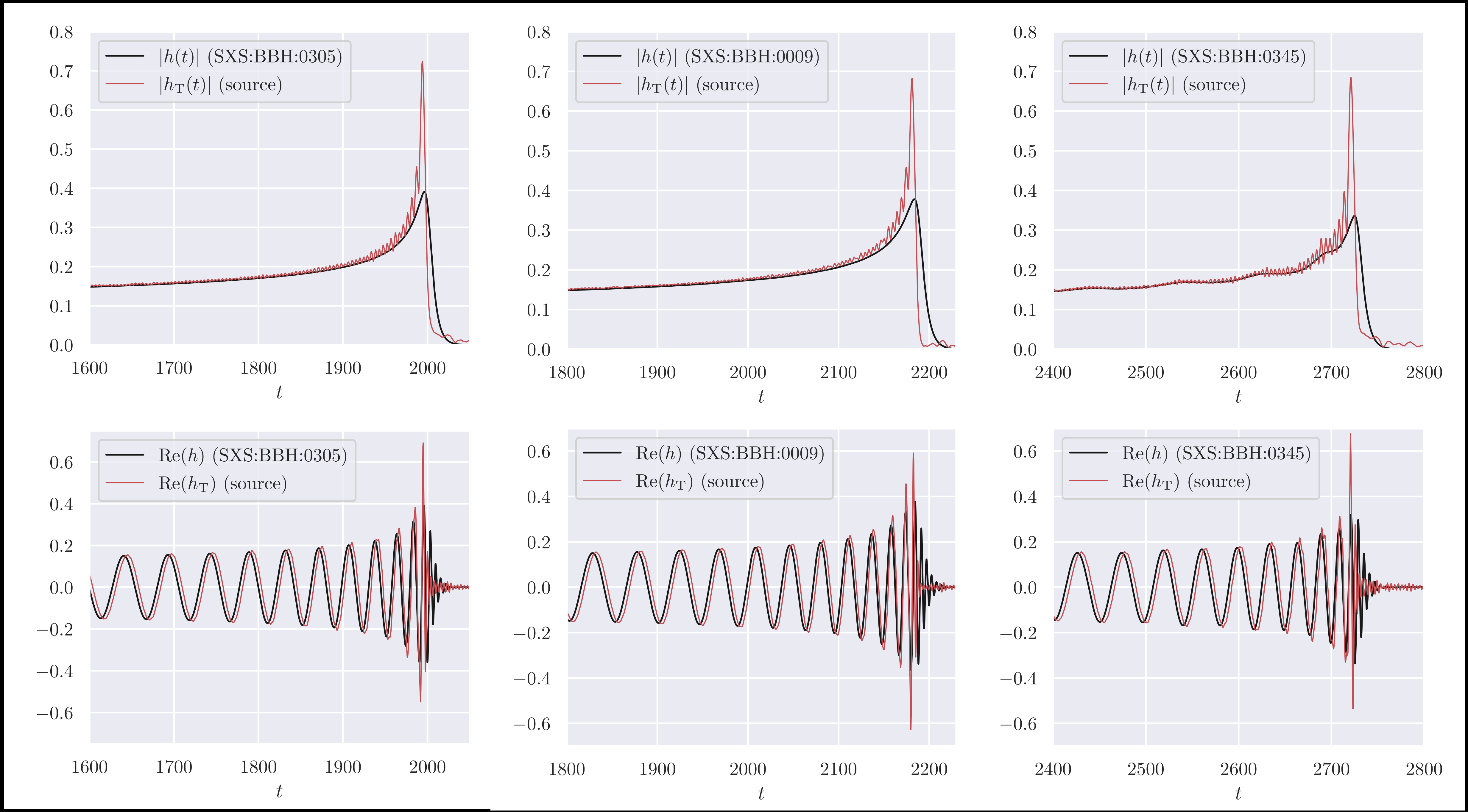
$$h_{\mathrm{T}}(t) = \int d\omega \left[\tilde{h}(\omega)/\tilde{h}_{\mathrm{G}}(\omega)\right] e^{-i\omega t}$$

source

SXS spectrum

BH S-matrix

$2M = 1$



Relation between QNM and greybody factor

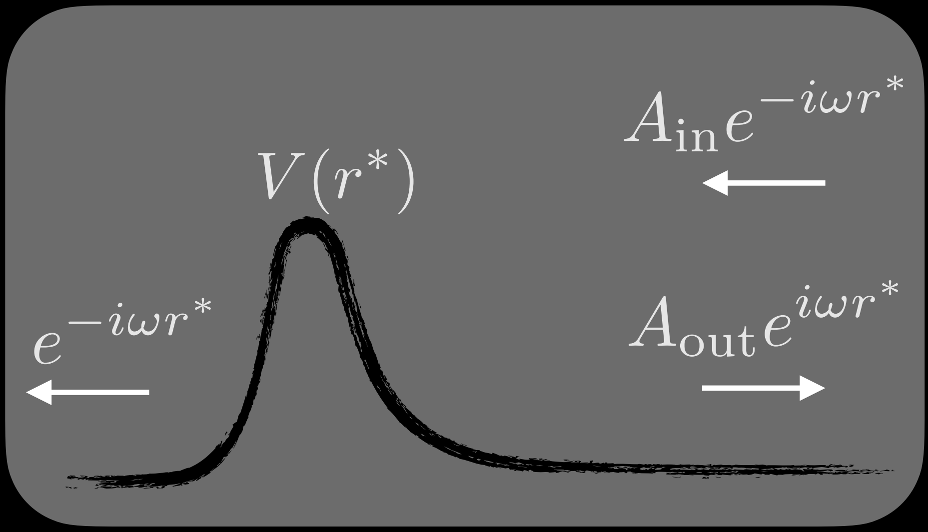
Ringdown reconstruction

NO, Cardoso (2024)

Full waveform

$$h_{\mathrm{G}}(t) = \frac{1}{2\pi}\int d\omega \frac{A_{\mathrm{out}}(\omega)}{A_{\mathrm{in}}(\omega)} e^{-i\omega t}$$

S-matrix (greybody factor)



QNM decomposition

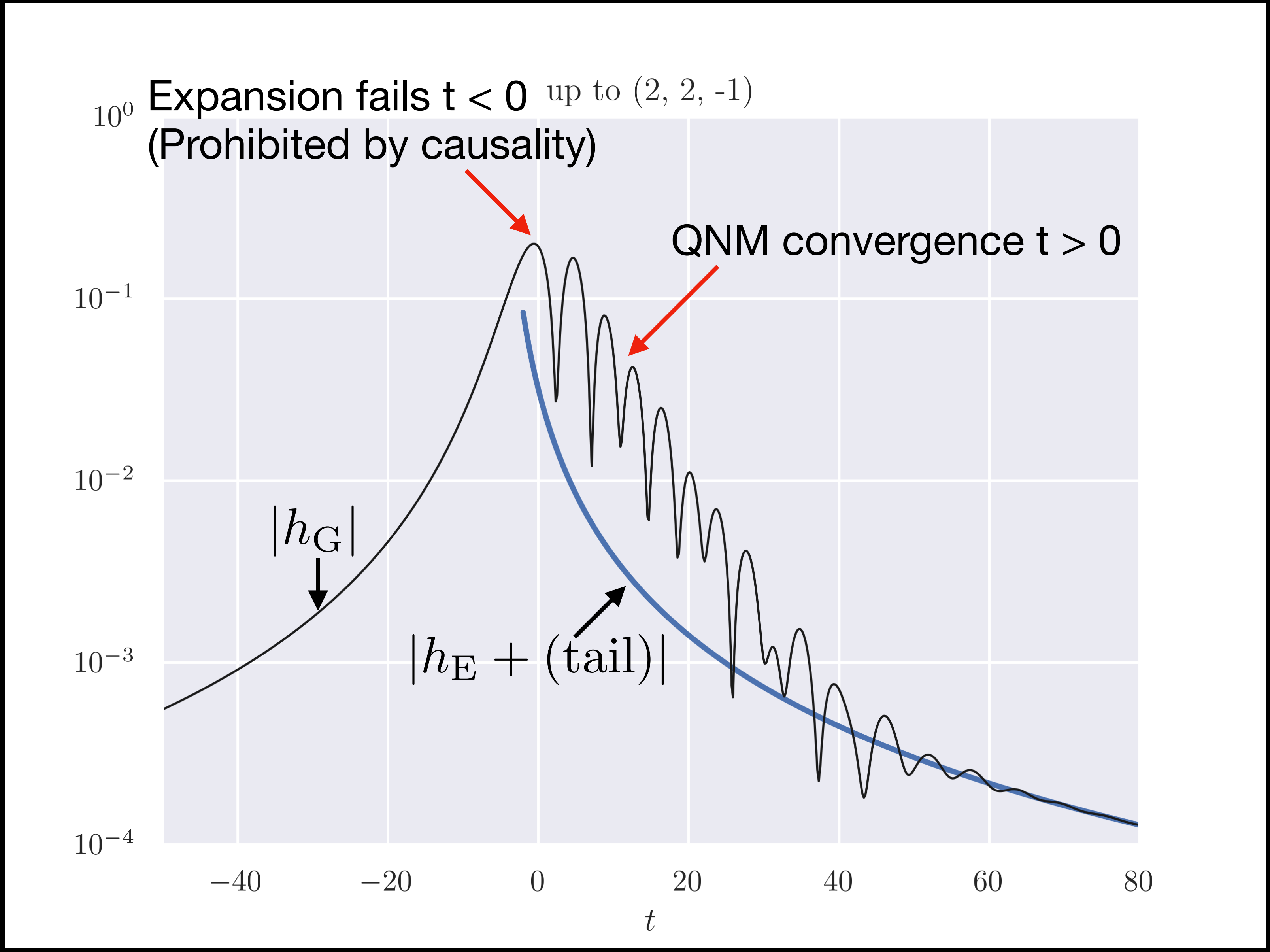
QNM excitation factor

$$h_{\mathrm{E}}(t) = \sum_n -i\frac{A_{\mathrm{out}}(\omega_n)}{A'_{\mathrm{in}}(\omega_n)} e^{-i\omega_n t} = \sum_n E_n e^{-i\omega_n t}$$

QNM convergence and causality

NO, Cardoso (2024)
NO, Berti, Cardoso (2025)

Caution: No QNM fit!



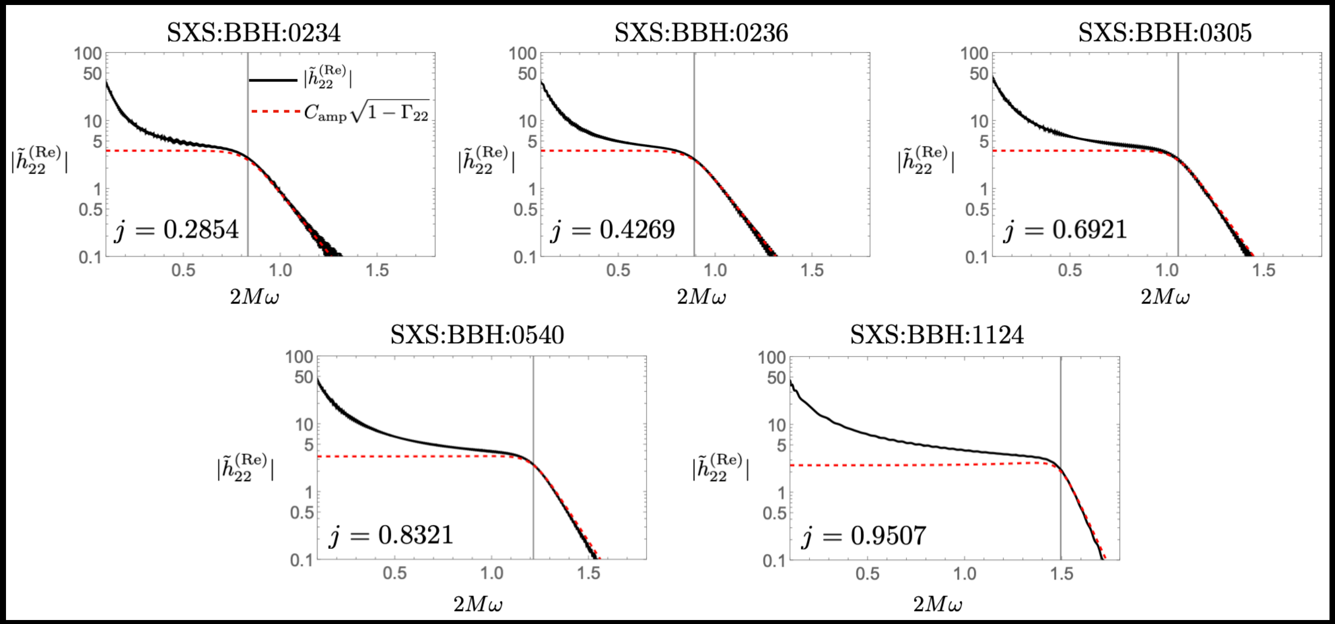
$$h_\mathrm{E}(t) = \sum_n -i \frac{A_\mathrm{out}(\omega_n)}{A'_\mathrm{in}(\omega_n)} e^{-i\omega_n t} = \sum_n E_n e^{-i\omega_n t}$$

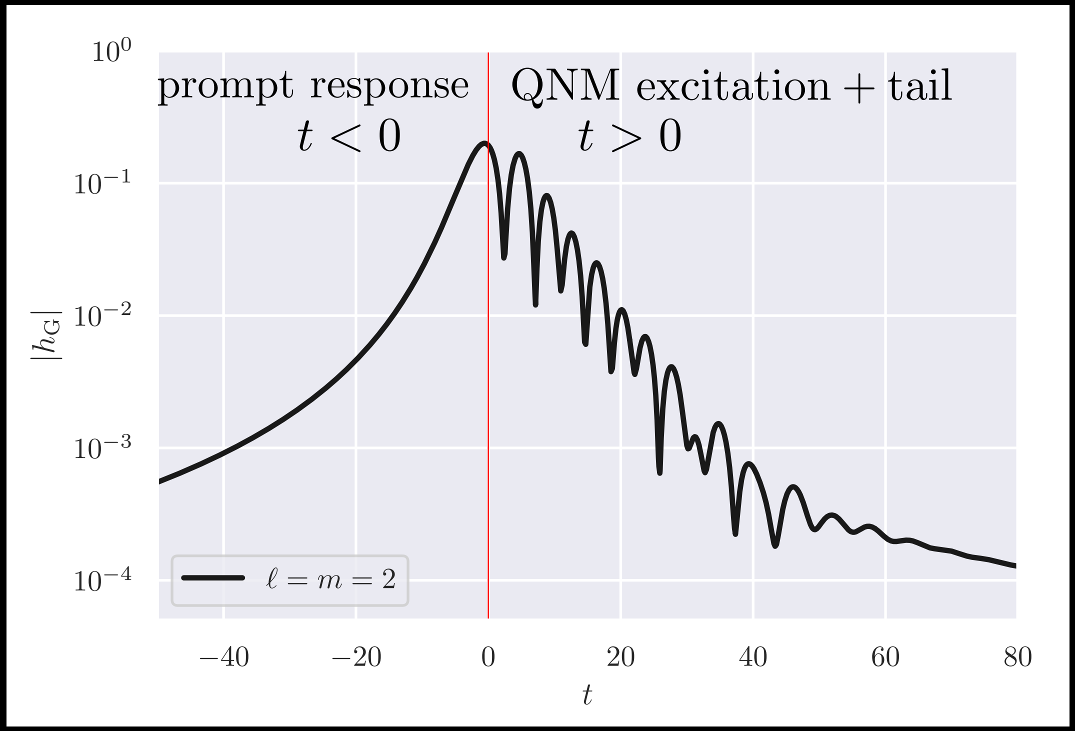
BH S-matrix / greybody factor

- universal interference among many overtones
- causality in QNM excitation (sudden transition from prompt to ringdown)
- tail

Rich information.

Compact characterization of QNM excitation.





Of course,

Time-domain analysis is important.

QNM model is precise as all source information comes into the fitting parameters.

e.g., QNM amplitude, phase

However,

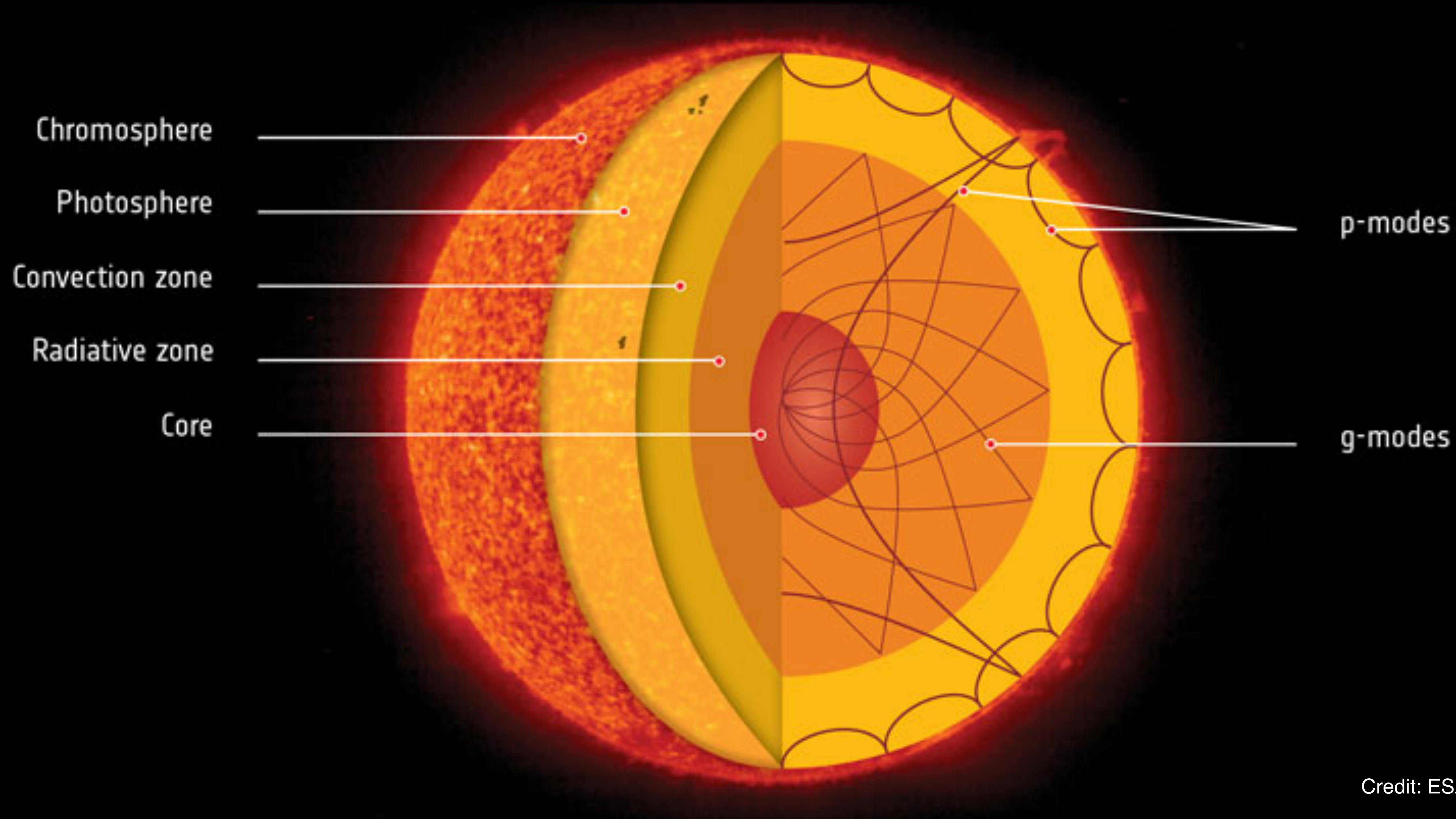
- QNM basis is NOT complete
- Q-value is smaller
- Many parameters

Sun

$$\text{Q value} = \frac{\text{frequency}}{\text{damping rate}}$$

$$\text{Q value} \sim 10^3$$

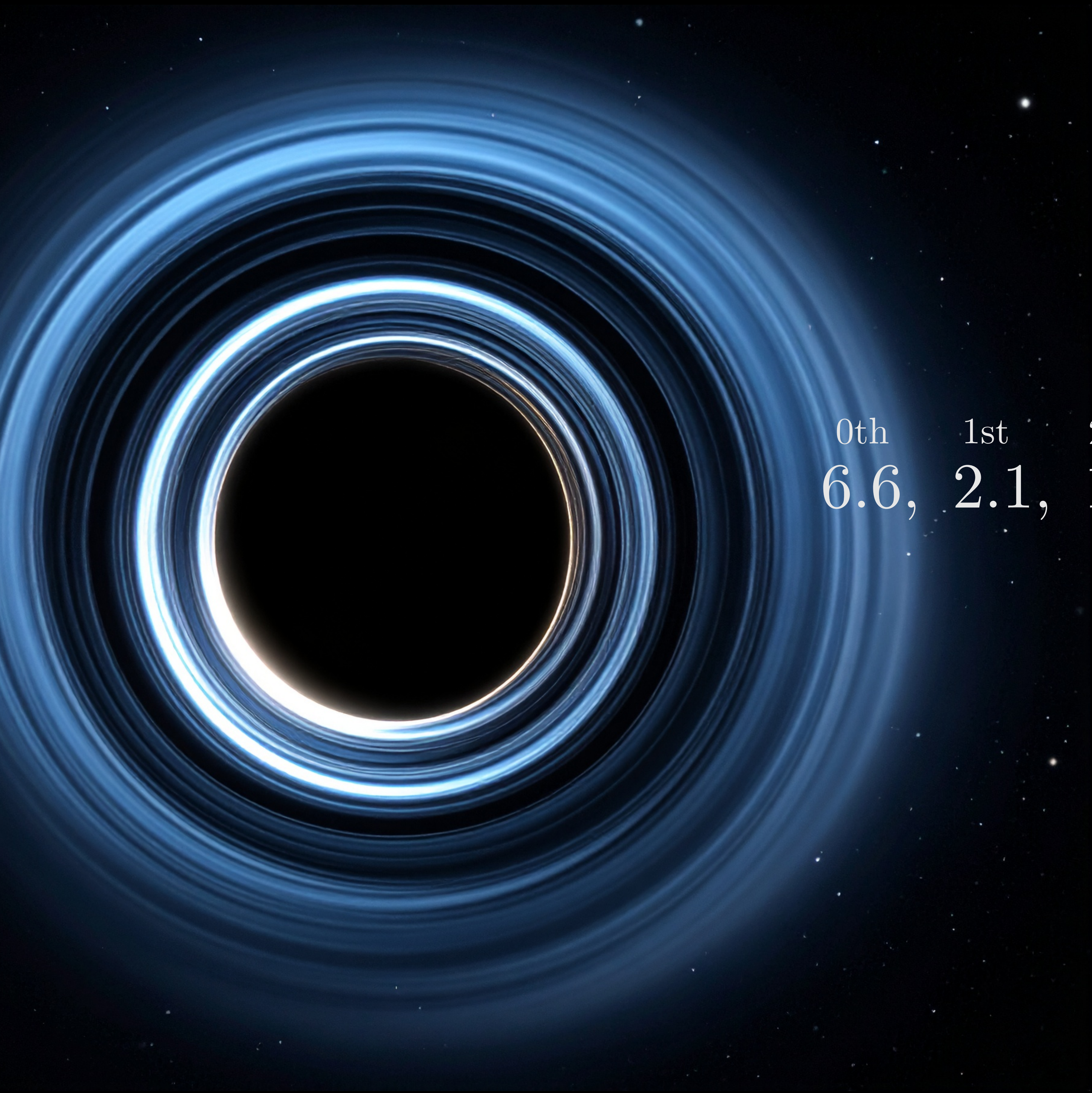
e.g. R. Kiefer et al. (2018)



Credit: ESA

Black holes

Q value $\lesssim \mathcal{O}(1)$



0th	1st	2nd	3rd	4th	
6.6,	2.1,	1.2,	1.1,	0.5,	...

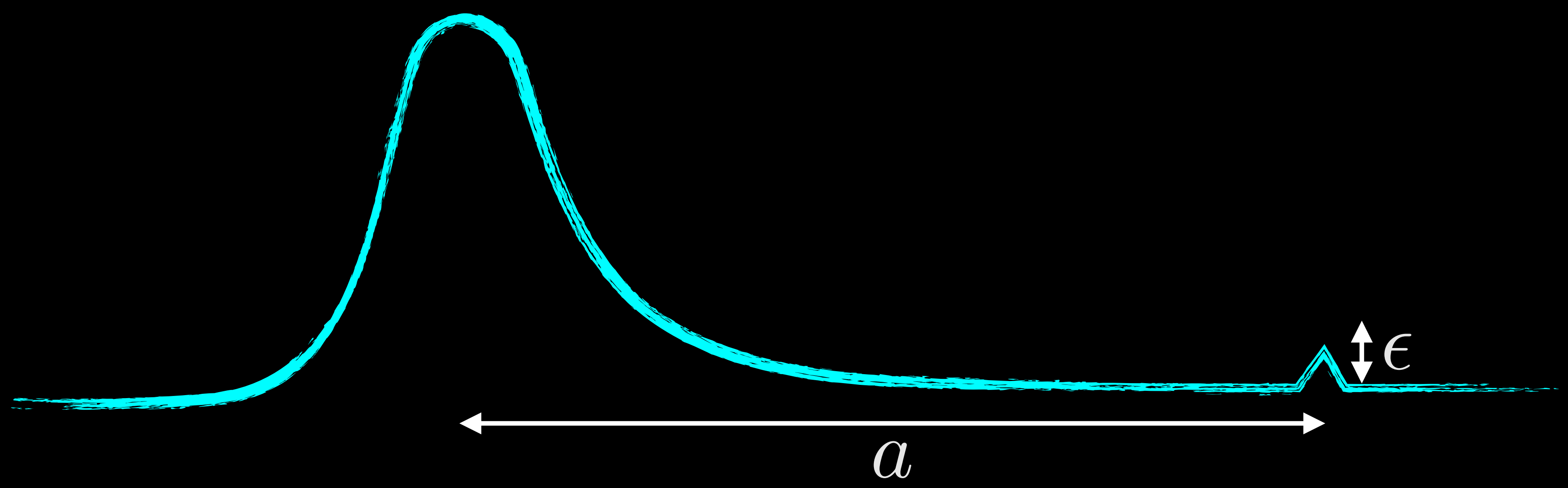
$(2, 2, n)$ and $j = 0.7$

- QNM basis is NOT complete
- Q-value is smaller
- Many parameters

Alternative basis?

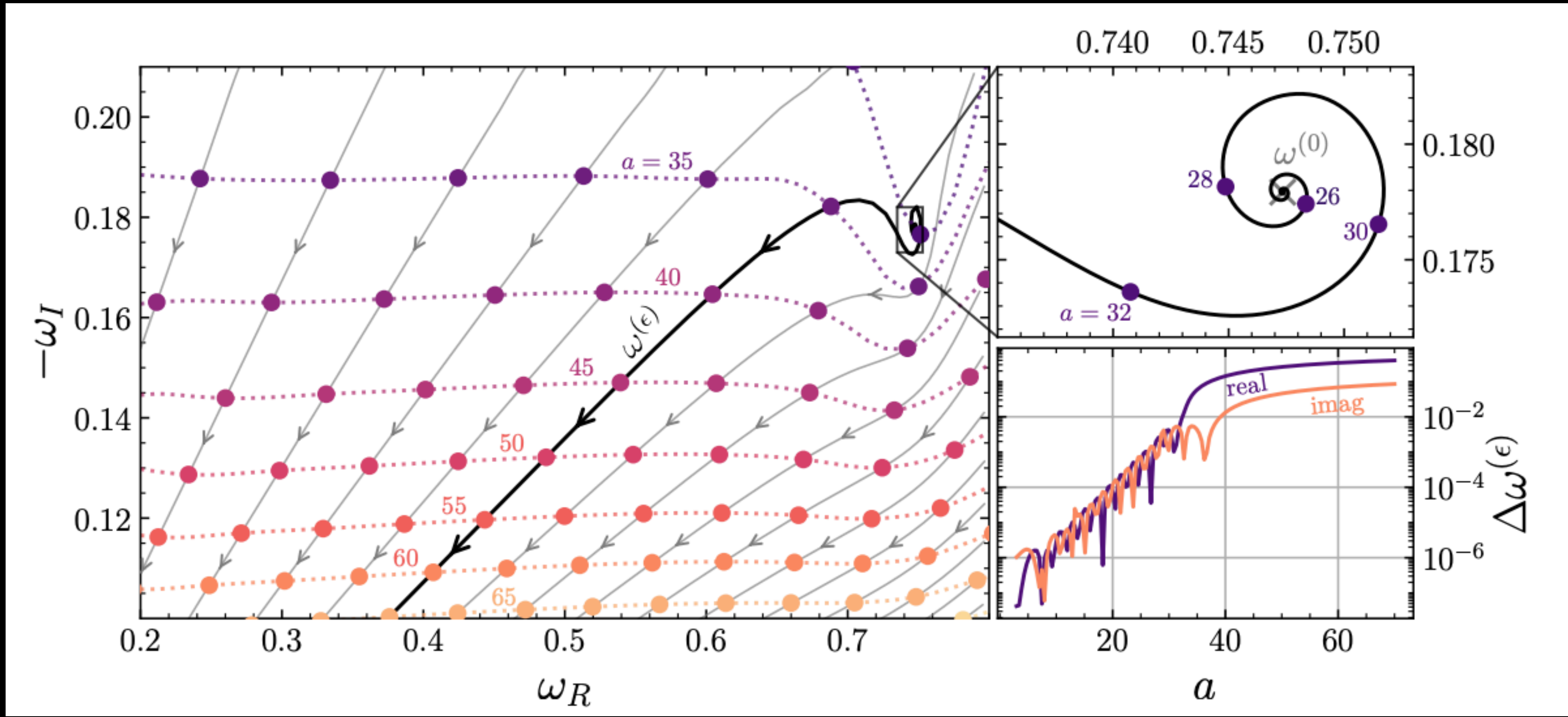
Destabilized QNMs

Nollert (1996); Barausse + (2014); Jaramillo (2021); Cheung+ (2022); Berti + (2022)



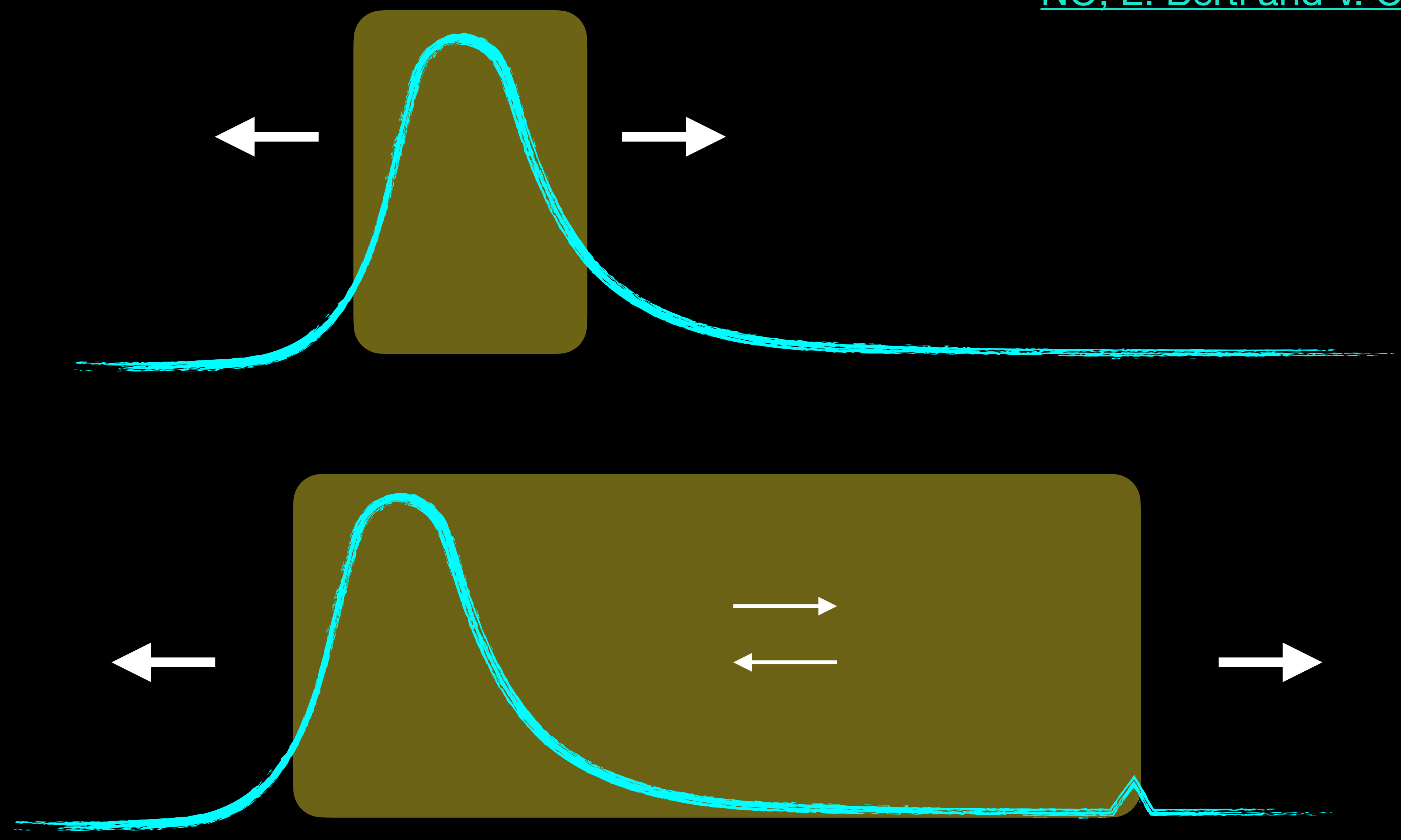
perturbed Regge-Wheeler equation $\epsilon = 10^{-6}$

Cheung et al. (2022)



A ringing black hole in an “echo room”

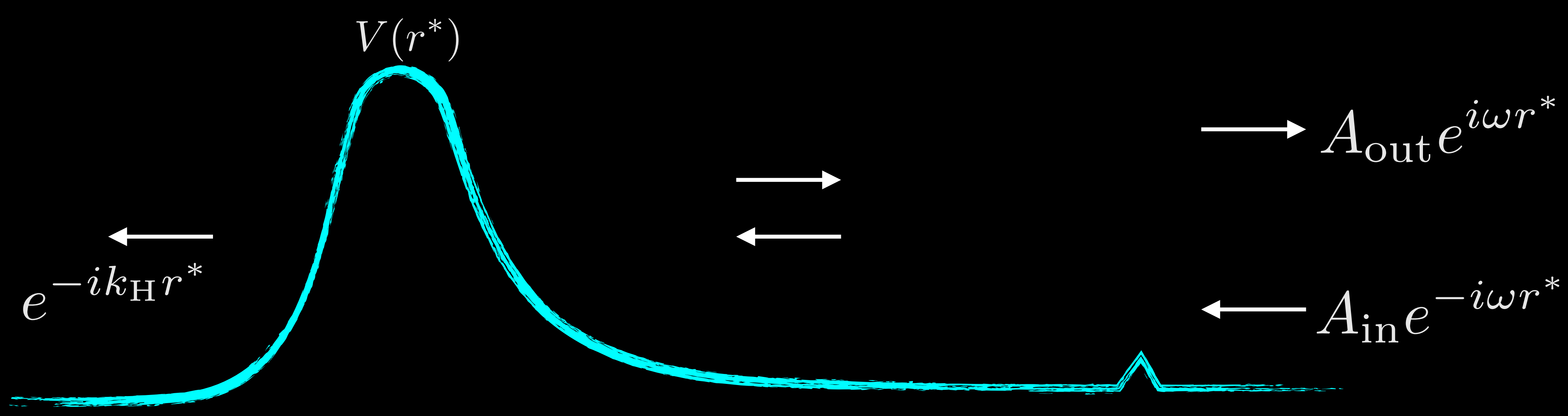
NO, E. Berti and V. Cardoso (2025)



Destabilized (trapped) QNMs:

- high Q value of overtones
- capture a broader structure of the potential barrier

Ringdown reconstruction



Full waveform (prompt + QNM + tail)

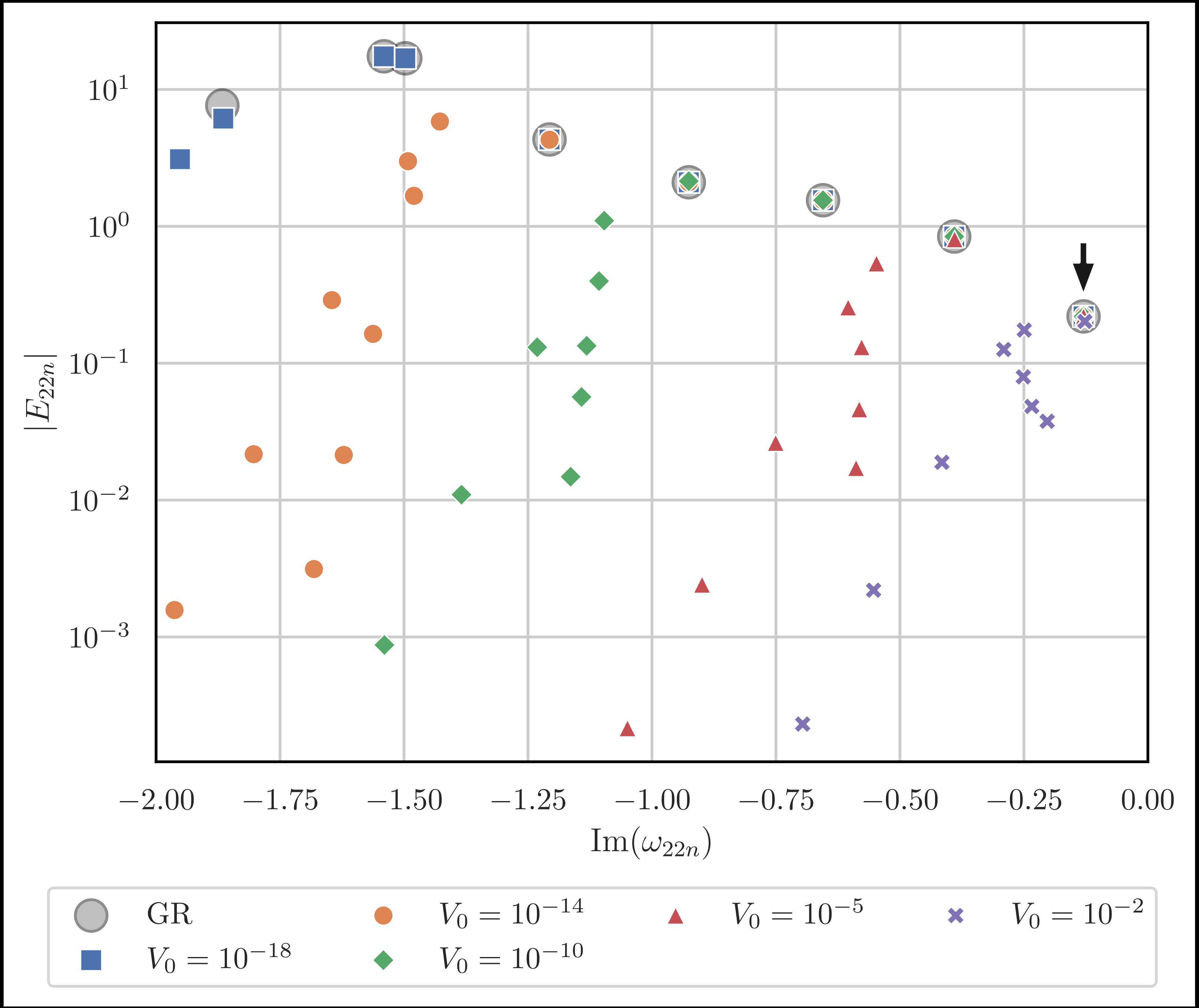
$$h_{\rm G} = \frac{1}{2\pi} \int d\omega \frac{A_{\rm out}(\omega)}{A_{\rm in}(\omega)} e^{-i\omega t}$$

QNM excitation

$$h_{\rm E} = \sum_n -i \frac{A_{\rm out}(\omega_n)}{A'_{\rm in}(\omega_n)} e^{-i\omega_n t} = \sum_n E_n e^{-i\omega_n t}$$

Destabilized QNM chords

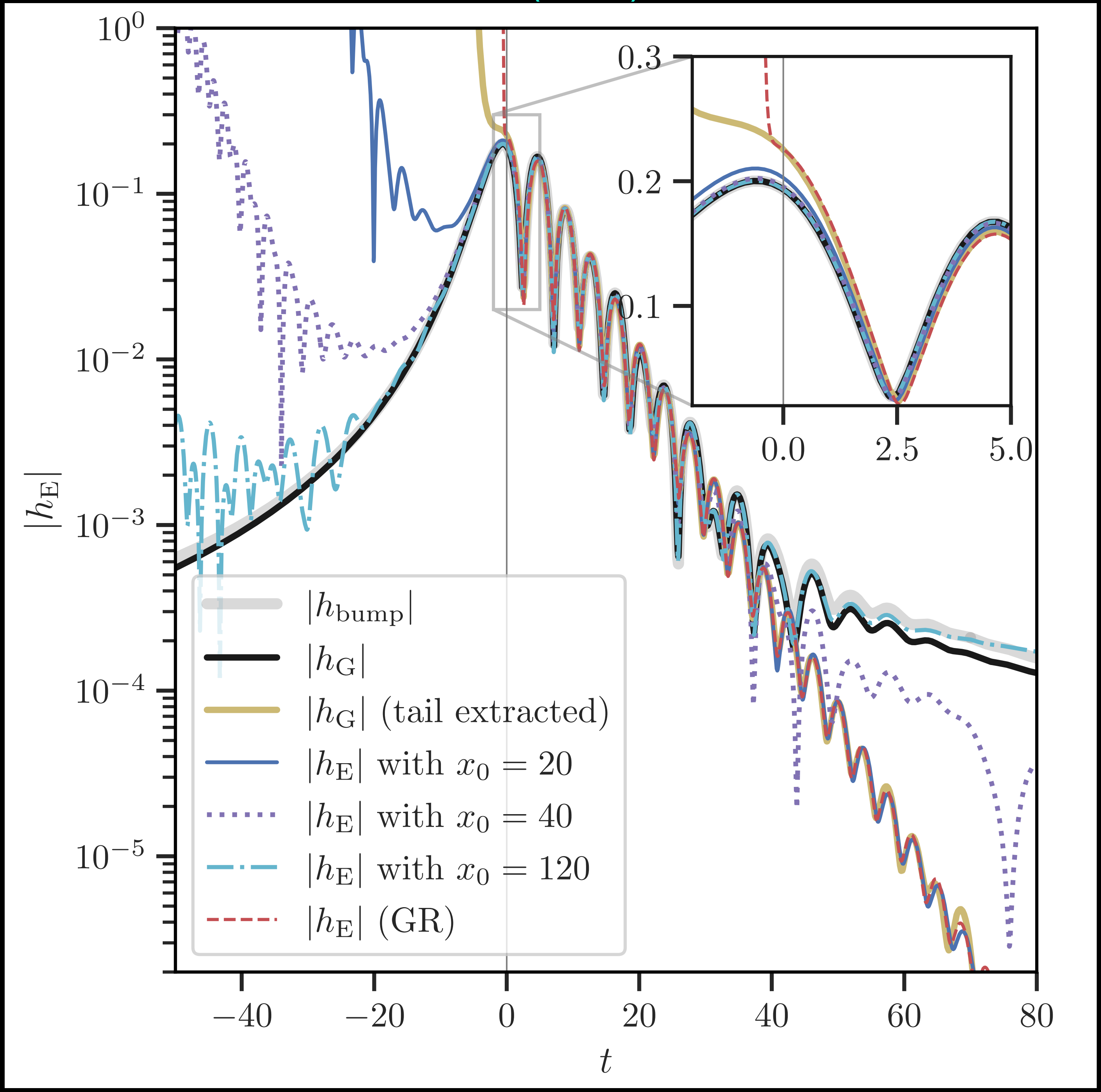
NO, E. Berti and V. Cardoso (2025)



Reconstruction of the tail and prompt part with QNMs

Caution: No QNM fit!

NO, E. Berti and V. Cardoso (2025)

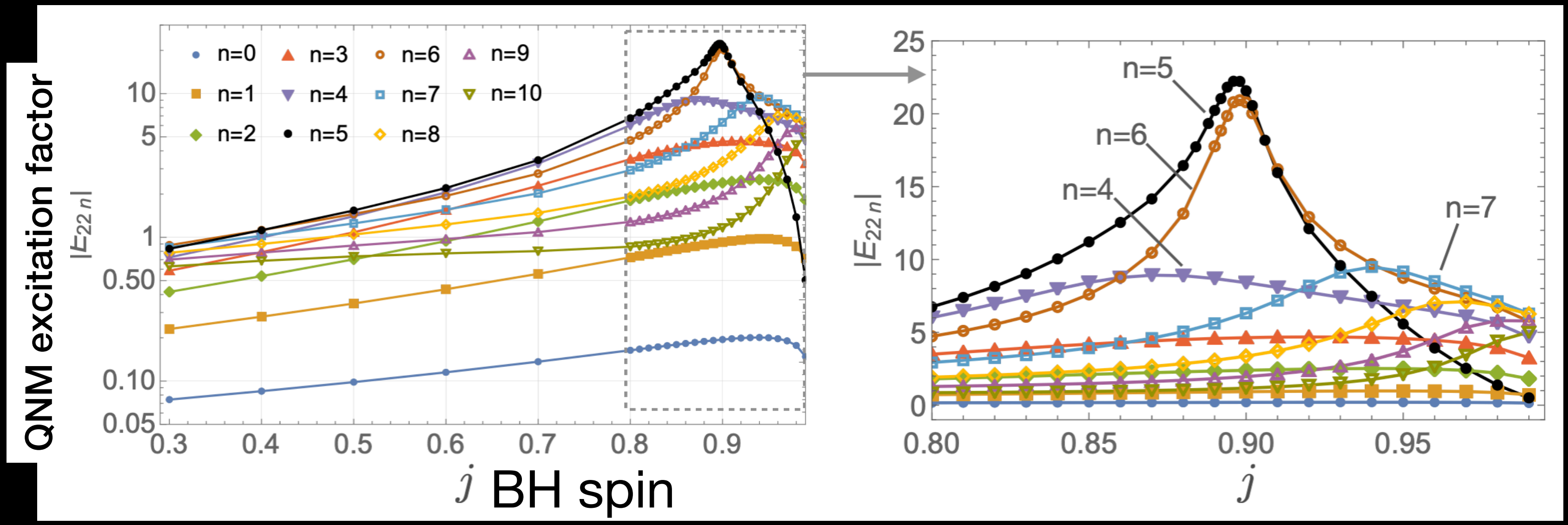


Destabilized QNMs improves the convergence of QNM expansion.

Reproduce the tail contribution and effectively complete basis.

Enhanced excitation factors

excitation factor for (2,2,5) and (2,2,6)



Oshita (2021)

Avoided crossing / Exceptional point

H. Motohashi (2024) NO, E. Berti and V. Cardoso (2025) R. K. L. Lo, L. Sabani, V. Cardoso (2025) Y. Yang, E. Berti and N. Franchini (2025)

Eigenstates of a non-Hermitian two-level system

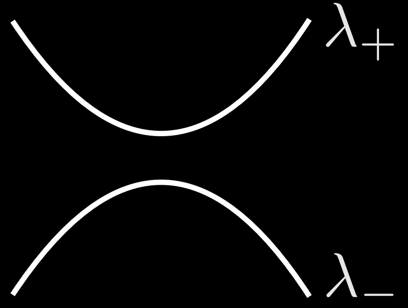
Hamiltonian

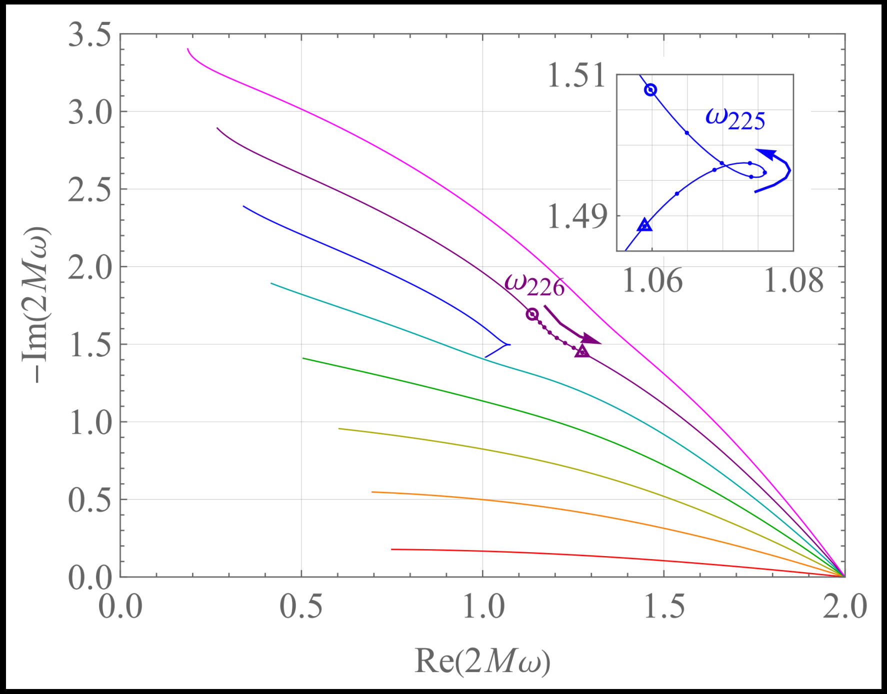
$$H_\epsilon = \begin{pmatrix} \epsilon_1 + i\epsilon_2 & 1 \\ 1 & -\epsilon_1 - i\epsilon_2 \end{pmatrix} \quad \epsilon_{1,2} \in \mathbb{R}$$

Eigenvalues

$$\lambda = \pm\sqrt{1 + (\epsilon_1 + i\epsilon_2)^2}$$

avoided crossing





H. Motohashi (2024)

Divergent excitation factors at exceptional point

Adding another parameter

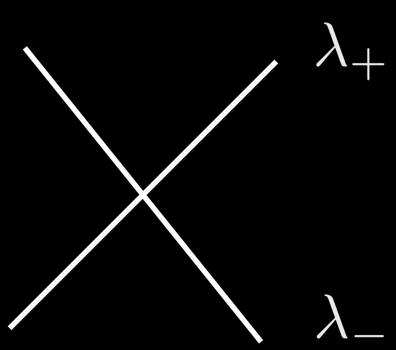
Hamiltonian

$$H_\epsilon = \begin{pmatrix} \epsilon_1 + i\epsilon_2 & 1 \\ 1 & -\epsilon_1 - i\epsilon_2 \end{pmatrix} \qquad \epsilon_{1,2} \in \mathbb{R}$$

Eigenvalues

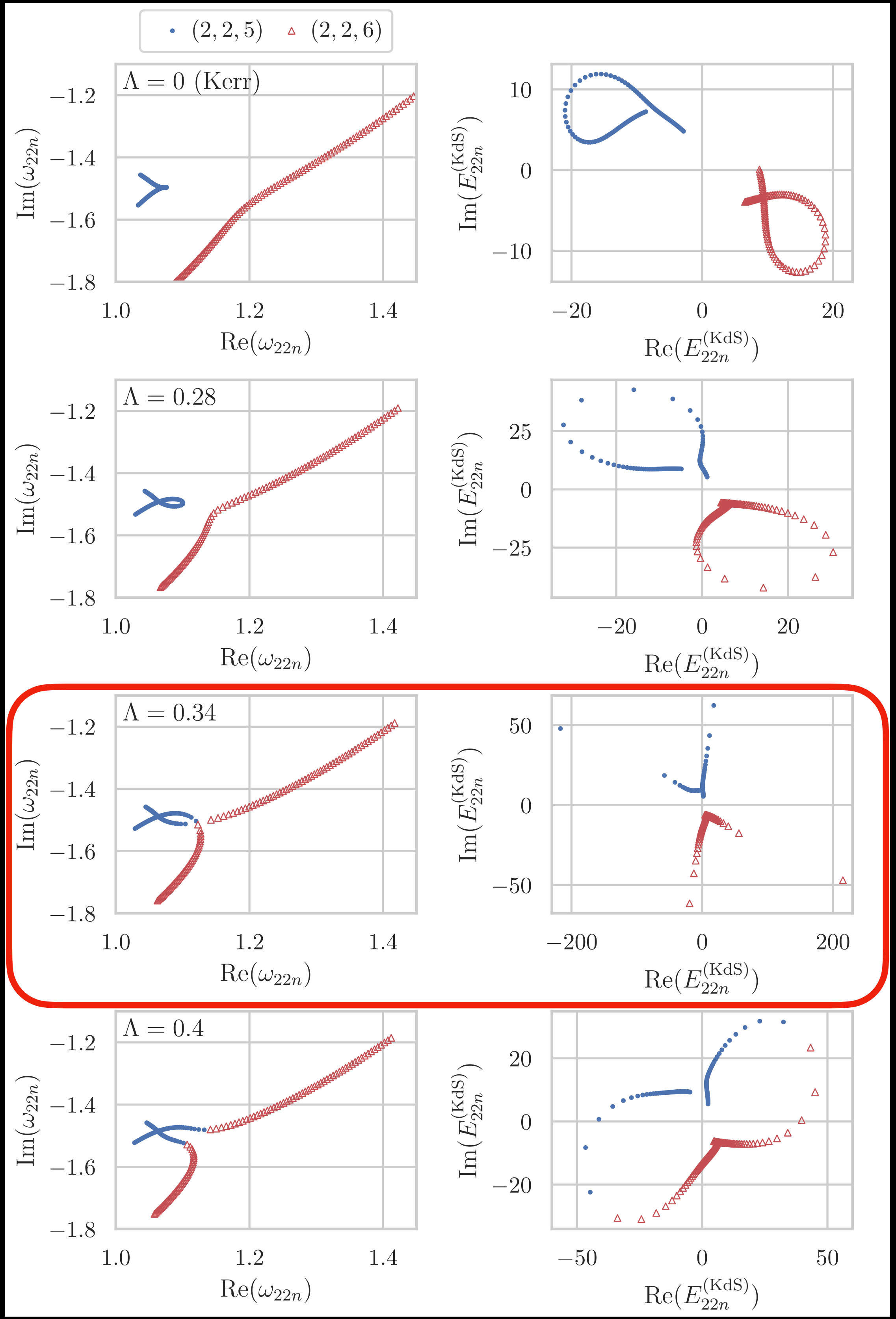
$$\lambda = \pm\sqrt{1 + (\epsilon_1 + i\epsilon_2)^2}$$

exceptional point



$$\epsilon_1 = 0 \qquad \epsilon_2 = \pm 1$$

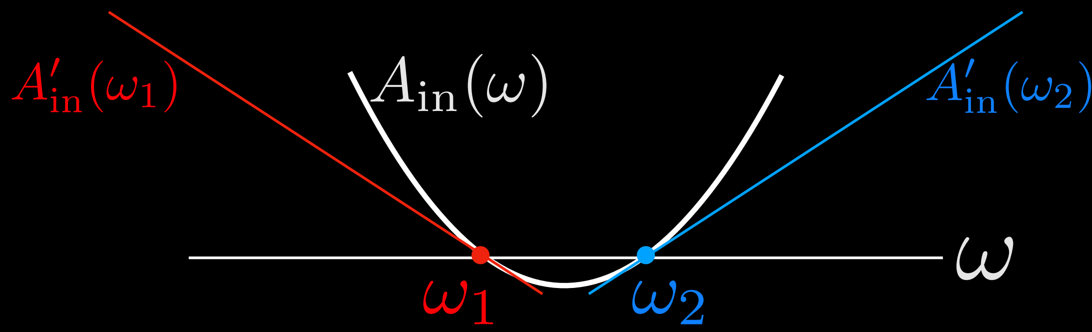
Kerr + cosmological constant



NO, E. Berti and V. Cardoso (2025)

Destructive resonant excitation of QNMs

Does the avoided crossing or exceptional point impact the ringdown waveform?



source, spheroidal harmonics, etc

ringdown spectrum $\dfrac{f(\omega)}{A_{\rm in}(\omega)} e^{-i\omega t}$

S-matrix with QNM poles

$$A_{\rm in}(\omega) = (\omega - \omega_1)(\omega - \omega_2)\underline{g(\omega)}$$

regular around $\omega \sim \omega_1 (\sim \omega_2)$

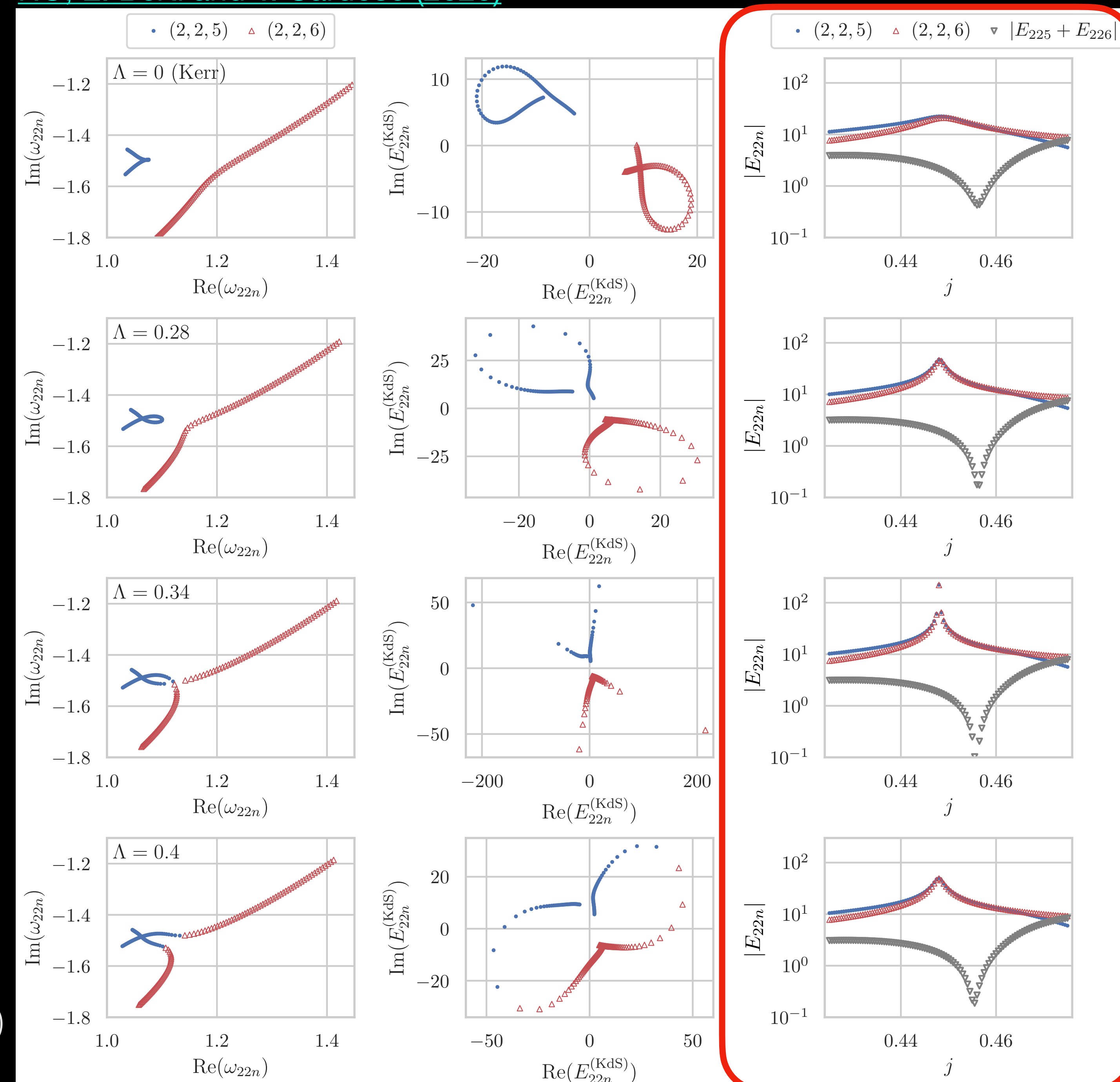
$\omega_2 - \omega_1 = \delta\omega \quad |\delta\omega| \ll |\omega_1|$

sum of QNMs

$$\frac{f(\omega_1)}{A'_{\rm in}(\omega_1)} e^{-i\omega_1 t} + \frac{f(\omega_2)}{A'_{\rm in}(\omega_2)} e^{-i\omega_2 t} = \left.\frac{d(f/g)}{d\omega}\right|_{\omega=\omega_1 \simeq \omega_2} + \mathcal{O}(\delta\omega)$$

$\propto 1/\delta\omega$ $\quad$ $\propto 1/\delta\omega$ $\quad$ $\propto (\delta\omega)^0$

NO, E. Berti and V. Cardoso (2025)



Destructive interference of the enhanced QNMs.

Is this really advantageous in GW observations?

Conclusions

"Instantaneous excitation" of QNMs at the merger of BBHs

- universal interference among QNM overtones
- instantaneous excitation of QNMs

→ ringdown efficiently modeled by the greybody factor

Set of destabilized QNMs exhibit its better convergence, high Q values, and captures the prompt part and late-time tail

→ A better QNM basis.

QNMs in the avoided crossing or near the exceptional point are "destructively" excited. Is this really advantageous in GW observations? Rather, another evidence of the time-domain ringdown stability.