When Black Hole Perturbation Theory Meets Amplitudes and CFT

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Talk @ IBS Workshop on QNM and BHPT

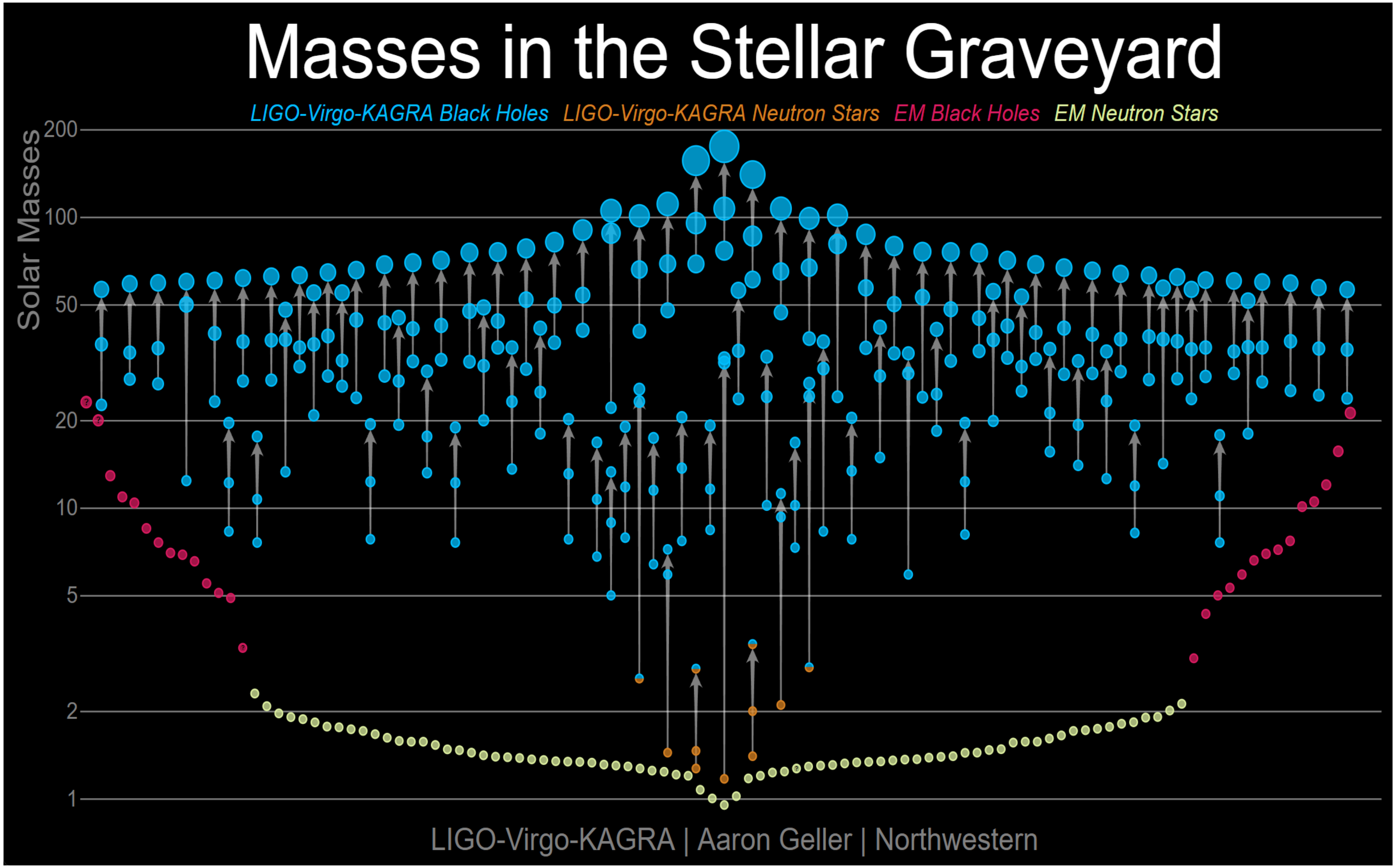
May 28th 2025

Department of Physics, Princeton University

Based on

- **“Black Hole Perturbation Theory Meets CFT_2: Kerr Compton Amplitudes from Nekrasov-Shatashvili Functions”**, with Y. F. Bautista, G. Bonelli, C. Iossa, A. Tanzini, arXiv:2312.05965
- **“Gravitational Raman Scattering in Effective Field Theory: A Scalar Tidal Matching at $\mathcal{O}(G^3)$ ”** with M. Ivanov, Y-Z. Li, J. Parra-Martinez, arXiv: 2401.08752
- **“Resummation of Universal Tails in Gravitational Waveforms”** with M. Ivanov, Y-Z. Li, J. Parra-Martinez, arXiv: 2504.07862
- “**5-Dimensional Gravitational Raman Scattering: Scalar Wave Perturbations in Schwarzschild-Tangherlini Spacetime**” with S. Akhtar, Y. F. Bautista, C. Iossa, see today’s arXiv.

Golden Time for Black Holes Physics



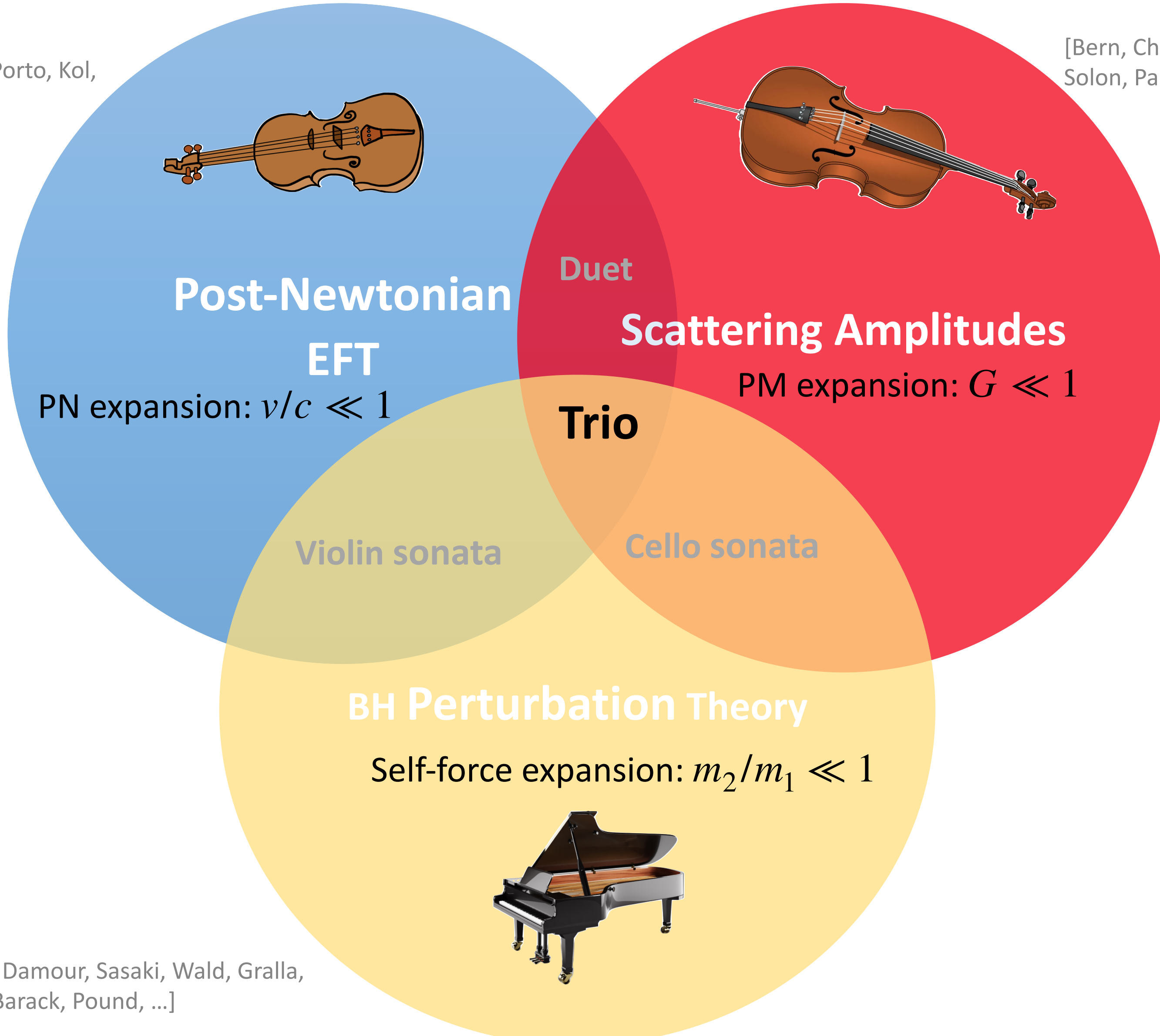
LIGO Virgo Collaboration

Over 200 events in total

Trio of Black Hole Dynamics

[Goldberger, Rothstein, Porto, Kol, Smolkin, …]

[Bern, Cheung, Roiban, Shen, Solon, Parra-Martinez, …]

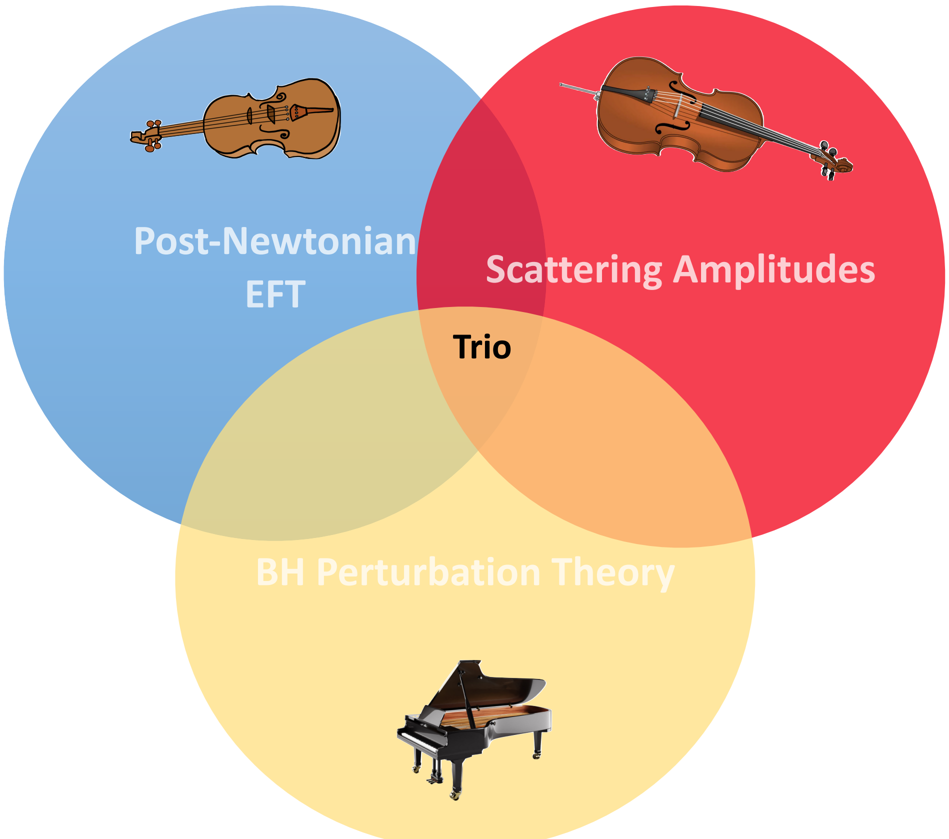


[Teukolsky, Damour, Sasaki, Wald, Gralla, Pretorius, Barack, Pound, …]

What Is Still Missing?

- **Detailed understanding of the internal structure of compact objects**
  - Most of the current waveforms only include static tides
- **Non-perturbative structure of GR**
  - The current waveforms only include Padé resummation

Plan of The Talk



- Recap of tidal deformation and dissipations
- Three ways of solving Teukolsky equation
- An exact anomalous dimension of multipole moments in GR
- Applications in GW waveform tail resummation

Recap on Tidal Deformation and Dissipation

$$S_{\rm tidal} = \frac{1}{2}\int d\tau Q^E_{ij}\cdot E^{ij} + \frac{1}{2}\int d\tau Q^B_{ij}\cdot B^{ij}$$

Microscopic description: $Q_{ij}(\vec{\xi})\ ,\quad \vec{\xi}:$ Stellar fluid Lagrangian displacement

Causal tidal response: $Q^E_{ab}(\tau) \propto M(GM)^4\Lambda E_{ab}(t-\tau_{\rm tidal})$

Love numbers: $\Lambda \sim \left(\frac{R}{GM}\right)^5$ —General ℓ→ $\Lambda_\ell \sim \left(\frac{R}{GM}\right)^{2\ell+1}$

Dissipation (Heating) numbers:

$H \sim \Lambda \times \frac{\tau_{\rm tidal}}{Gm}$ → $H \sim \left(\frac{R}{Gm}\right)^6 \times \left(\frac{\nu}{\nu_{\rm BH}}\right)$ $\qquad \frac{\nu}{\nu_{\rm BH}} \sim 0.1\ ,\ \ \nu_{\rm BH} = cr_s$

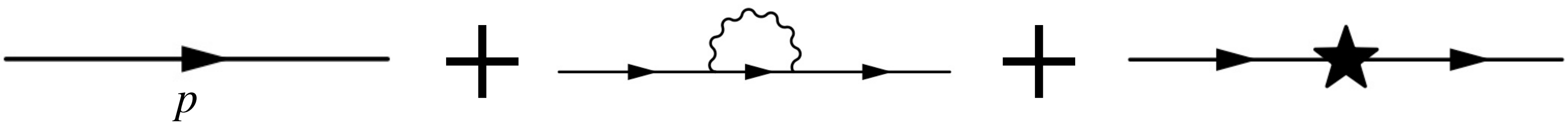
[Chia, **Zhou**, Ivanov (2024)]

[Saketh, **Zhou,** Ghosh, Steinhoff, Chatterjee (2024)]

Interesting Questions in the EFT

- Point particle couples to Einstein gravity is non-renormalizable. We expect UV divergences and renormalizations to happen

Recap of renormalization: electron self-energy



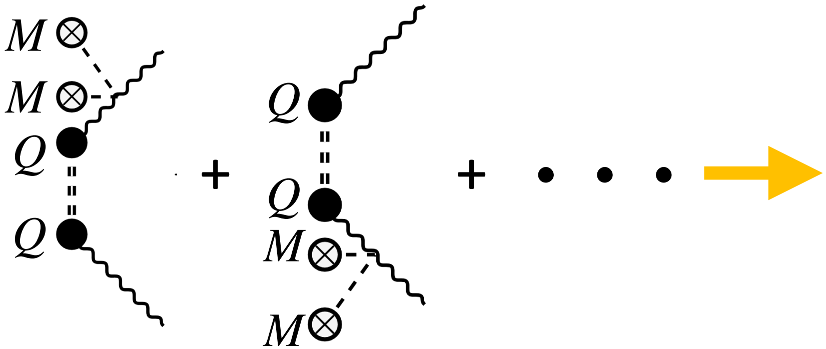
$$\sim \alpha\left(\frac{1}{\epsilon_{\text{UV}}} + \log\left(\frac{\mu^2}{m(p)^2}\right)\right) \qquad \delta = \sim -\alpha\frac{1}{\epsilon_{\text{UV}}}$$

→ $$m(\mu) \sim m(p)\left[1 - \alpha\log\left(\frac{\mu^2}{m(p)^2}\right)\right]$$

[M. Schwartz QFT book]

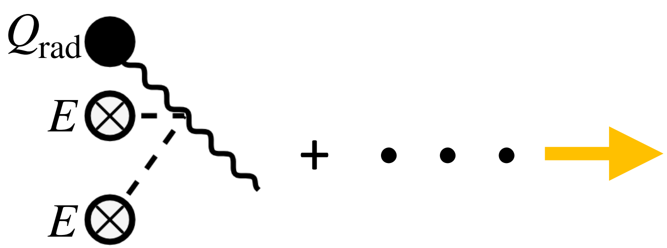
Interesting Questions in the EFT

- Tail effects of tidal correlators:



$$\frac{d\langle QQ\rangle_\ell(\omega;\mu)}{d\log\mu} = (\ ?\ )\langle QQ\rangle_\ell$$

- Tail effects of radiative multipoles:



$$\frac{d\langle Q_{\rm rad}\rangle_\ell(\omega;\mu)}{d\log\mu} = (\ ?\ )\langle Q_{\rm rad}\rangle_\ell$$

- Important for waveform tail resummation

To Answer these questions,
we consider the gravitational Raman scattering

Black Hole Raman (Compton) Scatterings

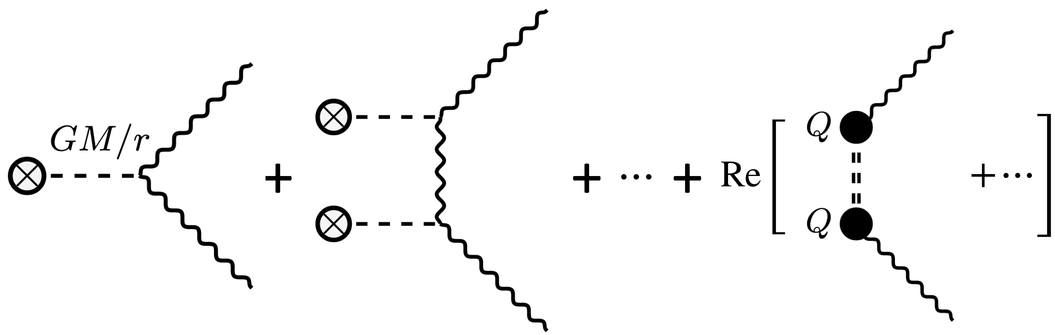
IR: EFT

$$S_{\mathrm{pp+EH}} = -m\int d\tau + \frac{1}{16\pi G}\int d^4x\sqrt{-g}R$$

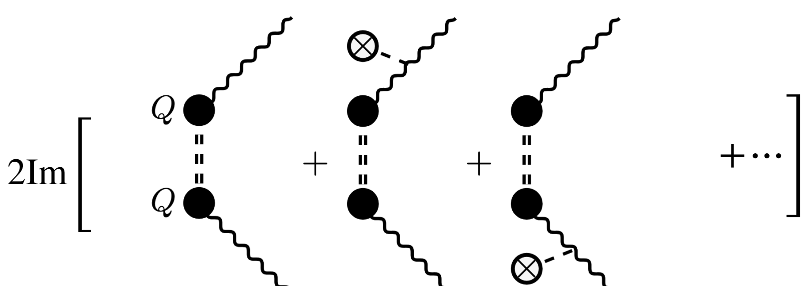
+

$$S_{\mathrm{tidal}} = \int d\tau Q_L^E \cdot E^L + \int d\tau Q_L^B \cdot B^L$$

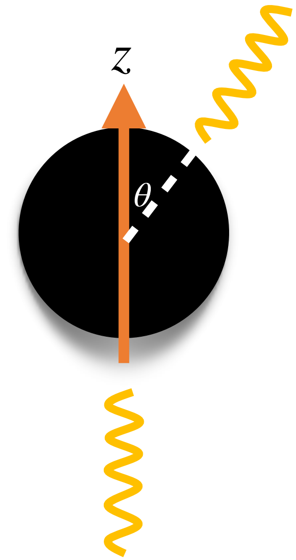
elastic:

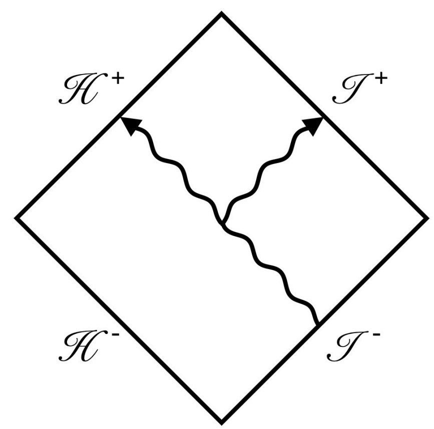


absorption:



UV: BHPT





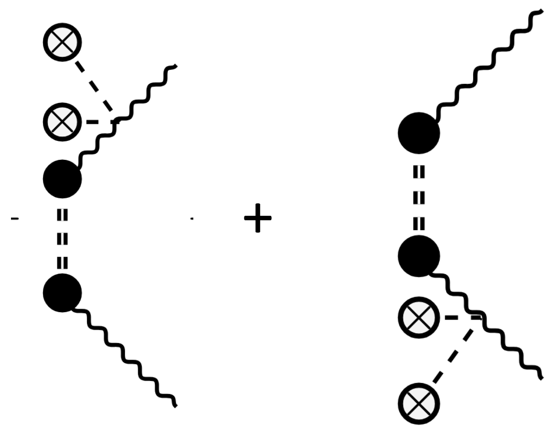
On The Way to Precision Physics

Scalar toy model: $$S_{\text{tidal}} = \int Q^L(\partial_L \phi), \quad \ell = 0,1,\cdots$$

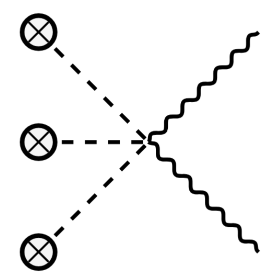
[Ivanov, Li, Parra-Martinez, **Zhou** (2024)]

[Caron-Huot, Correia, Isabella, Solon (2025)]

* Self-induced RG running: $$\frac{d\langle QQ\rangle_\ell(\omega;\mu)}{d\log\mu} = -(2Gm\omega)^2\left[\frac{-11+15\ell(1+\ell)}{(-1+2\ell)(1+2\ell)(3+2\ell)}\langle QQ\rangle_\ell\right]$$



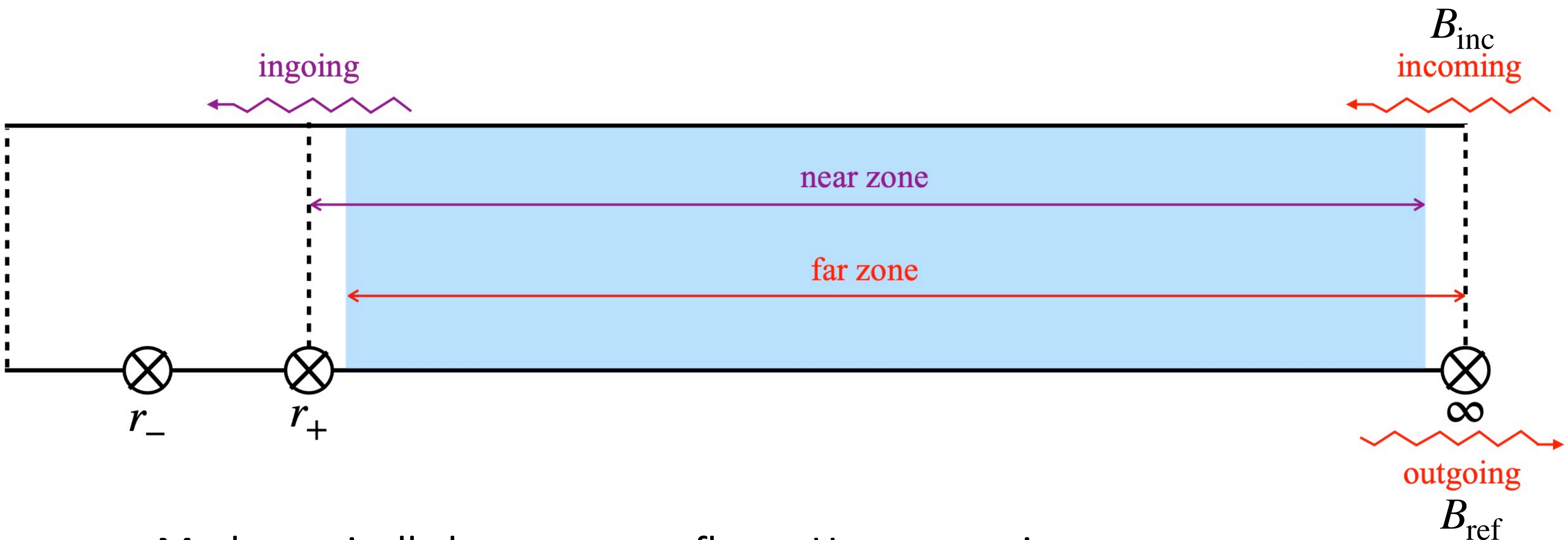
* Universal RG running: $$\frac{d\langle QQ\rangle_{\ell=0}(\omega;\mu)}{d\log\mu} = \text{self-induced} - 4\pi(2GM)^3$$



* Matching to BHs: $\lambda_{\ell=0} = \lambda_{\ell=1} = 0$ $\lambda^{\overline{MS}}_{\ell=0(\omega^2)} = -4\pi r_s^3\left[\frac{1}{4\epsilon_{\text{UV}}} + \ln(\mu r_s) + \frac{19}{24} + \gamma_E\right]$

Quasi-Exactly Solvable BHPT

✱ Teukolsky equation: $\mathcal{T}_{\mathrm{Kerr}}\,\psi^{[s]} = 0 \qquad \psi^{[s]} = e^{im\phi}e^{-i\omega t}{}_sS_\ell^m(\theta; a\omega)\,{}_sR_{\ell m}(r)$

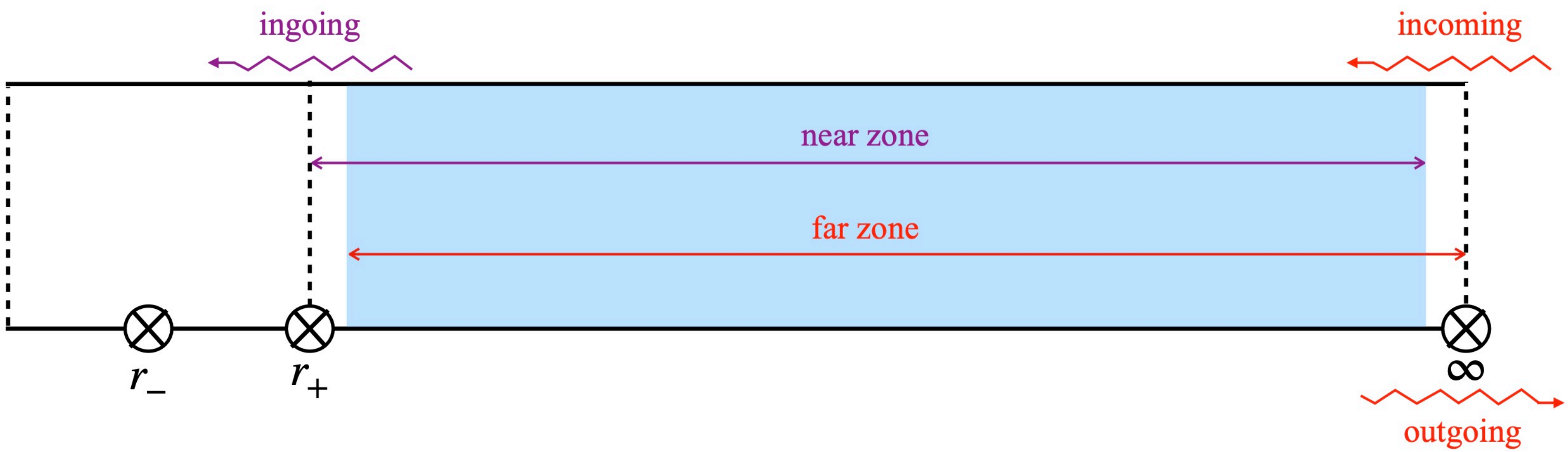


Mathematically known as confluent Heun equation.

✱ Scattering problem:

$\dfrac{B_{\mathrm{ref}}}{B_{\mathrm{inc}}}$ ➡ Connection problem between two singular points.

Matched Asymptotic Expansion (MST Method)



Near zone: $r_+ \leq r < \infty$

Far zone: $r_+ < r \leq \infty$

- Build near zone solutions in terms of series of hypergeometric functions
- Build far zone solutions in terms of series of Coulomb wave functions
- Match growing and decaying modes in the intermediate region

[Mano, Suzuki, Takasugi (1996)]

The Tale of Renormalized Angular Momentum: Stage 1

- First proposed in 1996 by S. Mano, H. Suzuki, E. Takasugi as the characteristic (Floquet) exponent of the Teukolsky equation

[Mano, Suzuki, Takasugi (1996)]

Progress of Theoretical Physics, Vol. 95, No. 6, June 1996

Analytic Solutions of the Teukolsky Equation and Their Low Frequency Expansions

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(Received March 6, 1996)

Analytic solutions of the Teukolsky equation for arbitrary spin weight in Kerr geometry are presented in the form of series of hypergeometric functions and Coulomb wave functions. Relations between these solutions are established. The solutions provide a very powerful method not only for examining the general properties of solutions and physical quantities both analytically and numerically. The solutions can be regarded as series expansions in terms of a small parameter $\epsilon \equiv 2M\omega$, M being the mass of black hole, which corresponds to the Post-Minkowski expansion by G and to post-Newtonian expansion when they are applied to the gravitational radiation from a particle in circular orbit around a black hole. It is expected that these solutions will become a powerful weapon to construct accurate theoretical templates for LIGO and VIRGO projects.

§ 1. Introduction

There are growing interests in analytic solutions of the Teukolsky equation[1] in the Schwarzshild and Kerr geometries in the connection with gravitational wave astrophysics. Since Teukolsky proposed the master equation for massless fields in the Kerr spacetime, many efforts have been made to obtain the analytic solutions. The analytic expressions valid for low frequencies were found by Page,[2] Starobinsky and Churilov[3] by matching the approximate solutions valid near horizon and far from it. Leaver[4] made a systematic study to obtain the analytic solutions of the Teukolsky equation in the form of series of various functions. He found the solution in the form of series of Coulomb wave functions which is valid in the region far from the horizon and established the relation between that solution and the one in the form of the Jaffe type series which is valid near the horizon.

Recently, Tagoshi and Nakamura[5] determined numerically the coefficients of the post-Newtonian expansion of the gravitational radiation by a particle traveling a circular orbit around a Schwarzshild black hole. Sasaki[6] proposed a method of post-Newtonian expansion to solve the homogeneous Regge-Wheeler equation by using Bessel functions. Subsequently, the extensive study on this line was made by Tagoshi and Sasaki[7] and the result was compared with the one by Tagoshi and Nakamura. The application of this method to the Kerr geometries was made by Shibata, Sasaki, Tagoshi and Tanaka.[8] Various other applications were discussed by Poisson and Sasaki.[9] Now the problem to obtain the analytic solutions and the examination of their behaviors in low frequencies became an important and urgent topic.

In this paper, we report that we obtained the analytic solutions of the Teukolsky equation in Kerr geometry in the form of series of hypergeometric functions and Coulomb wave functions. The series solution of hypergeometric type is shown to be

- The MST method has been widely used in the self-force calculation and waveform modeling

- It was unclear for almost 30 years what is the physical meaning of renormalized angular momentum.

CFT Method + Trieste Formula

* Liouville correlator & BPZ:

$$\left(b^{-2}\partial_z^2 + U\left(z, z_i\right)\right)\left\langle V_{\alpha_\infty}(\infty)V_{\alpha_t}(t)V_{\alpha_1}(1)\Phi_{2,1}(z)V_{\alpha_0}(0)\right\rangle = 0$$

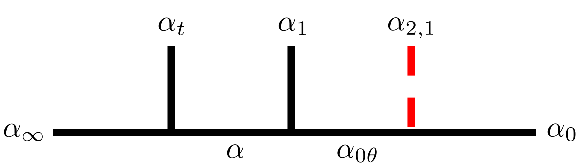
primary at $z_i \Rightarrow U \sim \frac{\Delta_i}{\left(z - z_i\right)^2} + \frac{c_i}{z - z_i}$, $\quad \Delta_i = \frac{1}{4}(b + b^{-1})^2 - \alpha_i^2, \quad c = 1 + 6(b + b^{-1})^2$

* Semi-classical limit

[Bonelli, Iossa, Lichtig, Tanzini (2022)]

$b \to 0, \quad \alpha_i \to \infty, \quad b\alpha_i = a_i$ finite

[Lisovyy, Naidiuk (2022)]



$$= \psi_\theta \exp\left(-\frac{1}{b^2}\mathscr{F}\right)$$

$\mathscr{F}$: quasi-classical conformal block

ψ satisfies the general Heun ODE.

* Trieste connection formula: $\psi_\theta^{[0]}(z) = \sum_{\theta'} C(\theta a_0, \theta' a_1, a)\psi_{\theta'}^{[1]}(z)$

$$C(a_0, a_1, a) = \underbrace{\frac{\Gamma(1 - 2a_0)\Gamma(2a_1)}{\Gamma\left(\frac{1}{2} + a_1 - a_0 + a_\infty\right)\Gamma\left(\frac{1}{2} + a_1 - a_0 - a_\infty\right)}}_{\text{Fusion Matrix}} \exp\left\{\frac{1}{2}\left(\frac{\partial \mathscr{F}}{\partial a_1} - \frac{\partial \mathscr{F}}{\partial a_0}\right)\right\}$$

BH Raman (Compton) Amplitudes: Near-Far Factorization

* Closed formula written in terms of Nekrasov-Shatashvili (NS) functions:

$$\eta e^{2i\delta} \sim \frac{{}_sB^{\text{ref}}_{\ell m}}{{}_sB^{\text{inc}}_{\ell m}} = \underbrace{\frac{e^{2i\epsilon(\log(|2\epsilon|)-1/2)}}{|2\omega|^{2s}} e^{\partial_{m_3}\mathcal{F}-\frac{L}{2}} e^{i\pi\left(a-\frac{1}{2}\right)} \frac{\Gamma\left(\frac{1}{2}-a-m_3\right)}{\Gamma\left(\frac{1}{2}-a+m_3\right)}}_{\text{far zone}} \times \underbrace{\frac{1+e^{-i\pi a}\mathcal{K}}{1+e^{i\pi a}\frac{\cos(\pi(m_3-a))}{\cos(\pi(m_3+a))}\mathcal{K}}}_{\text{near zone}};$$

Far zone: $r_+ < r \leq \infty$

Near zone: $r_+ \leq r < \infty$

$\mathscr{F}(m_1, m_2, m_3, a, L)$	$\epsilon = 2GM\omega$	$a = -\frac{1}{2} - \nu$	$\mathscr{K}$
NS function	PM counting	quantum A-period	tidal response

* Dictionary: $m_1 = i\frac{m\chi - \epsilon}{\kappa}$, $m_2 = -s - i\epsilon$, $m_3 = i\epsilon - s$, $L = -2i\epsilon\kappa$

$\chi = \frac{J}{GM^2}$: Dimensionless spin

$\kappa = \sqrt{1-\chi^2}$: Extremality

[Bautista, Bonelli, Iossa, Tanzini, **Zhou** (2023)]

[Ivanov, **Zhou** (2022)]

Far Zone Physics

Far zone: $r_+ < r \leq \infty$

* Far zone describes the BH in the spinning point-particle limit

* Far zone phase shift is purely elastic

* Far zone phase shift is polynomial in terms of BH spin

  - Aligns with spin multipole expansion in the EFT

* NS function works as PM resumption

  - Comparing to perturbative method

$$e^{\partial_{m_3}\mathcal{F}} = \frac{\sum_{n=-\infty}^{+\infty}(-1)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} a_n^\nu}{\sum_{n=-\infty}^{+\infty} a_n^\nu} \qquad a_n^\nu \sim (GM\omega)^n$$

[Mano, Suzuki, Takasugi (1996)] [Sasaki, Tagoshi (2003)]

Near Zone Physics

Near zone: $r_+ \leq r < \infty$

Tidal response function:

$$\mathcal{K} = |L|^{-2a} \frac{\Gamma(2a)\Gamma(2a+1)\Gamma\left(m_3 - a + \frac{1}{2}\right)\Gamma\left(m_2 - a + \frac{1}{2}\right)\Gamma\left(m_1 - a + \frac{1}{2}\right)}{\Gamma(-2a)\Gamma(1-2a)\Gamma\left(m_3 + a + \frac{1}{2}\right)\Gamma\left(m_2 + a + \frac{1}{2}\right)\Gamma\left(m_1 + a + \frac{1}{2}\right)} e^{\partial_a \mathcal{F}}$$

- Non-analytic in G
  - $L \sim GM\omega$
- Contain absorptions

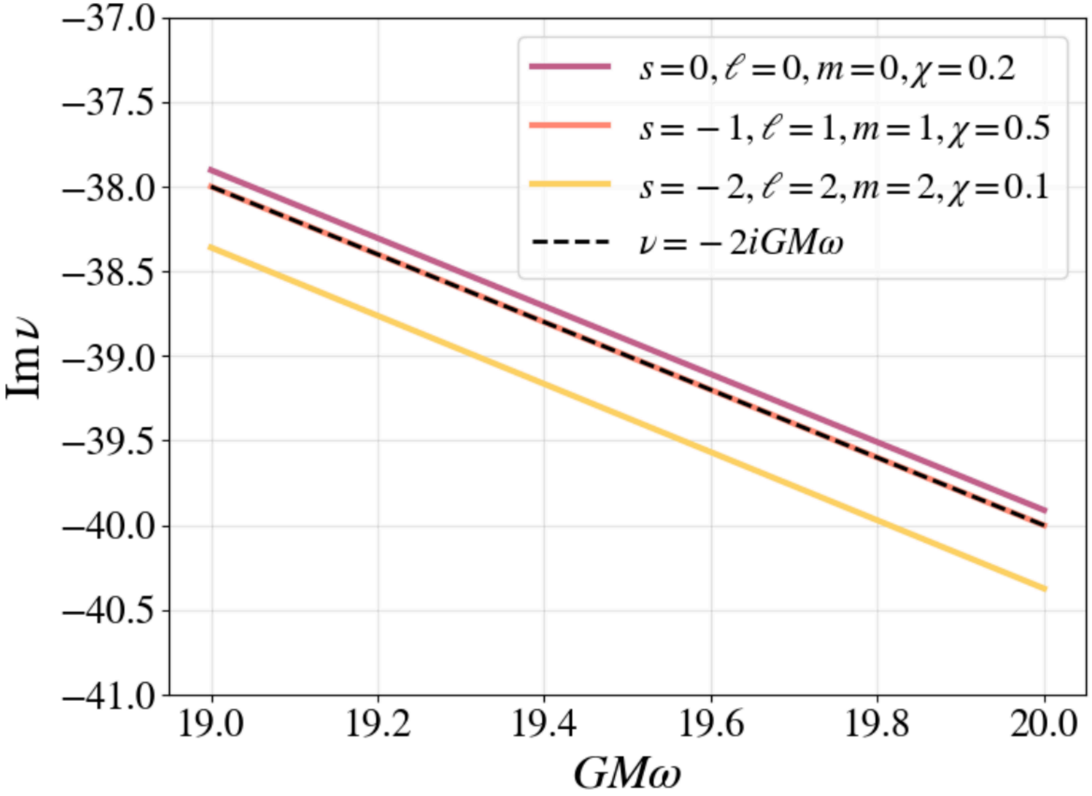
$${}_s\eta^P_{\ell m} = \left| \frac{1 + e^{-i\pi a}\mathcal{K}}{1 + e^{i\pi \alpha}\frac{\cos(\pi(m_3 - a))}{\cos(\pi(m_3 + a))}\mathcal{K}} \right|$$

- High Energy Limit: Exact QNM quantization condition

$$e^{i\pi a}\frac{\cos(\pi(m_3 - a))}{\cos(\pi(m_3 + a))}\mathcal{K} = 1$$

The Tale of Renormalized Angular Momentum: Stage 2

Quantum A-period: $a = -\frac{1}{2} - \nu$



$$\nu \simeq -2iGM\omega, \quad GM\omega \to \infty$$

[Bautista, Bonelli, Iossa, Tanzini, **Zhou** (2023)]

However, the physical meaning of ν still remains unclear.

Near-Far Factorization in 5D Schwarzschild BH

$$\eta e^{2i\delta} \sim \quad e^{i\pi(2a+\ell+1)} \times \frac{1+e^{-2i\pi a}\mathcal{K}_5}{1+e^{2i\pi a}\mathcal{K}_5}$$

Far zone: $r_+ < r \leq \infty$

Near zone: $r_+ \leq r < \infty$

✱ Parameters:

$\mathscr{F}(a_0, a_1, a, L)$: NS function $L = i r_{s,5}\,\omega$: PM counting a : Floquet exponent $\mathscr{K}_5$:5D BH tidal response

✱ Tidal response function:

$$\mathcal{K}_5 = |L|^{-4a} 2^{4a} \frac{\Gamma(2a)\Gamma(1+2a)\Gamma(\frac{1}{2}+a_1+a_0-a)\Gamma(\frac{1}{2}+a_1-a_0-a)}{\Gamma(-2a)\Gamma(1-2a)\Gamma(\frac{1}{2}+a_1+a_0+a)\Gamma(\frac{1}{2}+a_1-a_0+a)} e^{\partial_a F}$$

[Akhtar, Bautista, Iossa, **Zhou,** appear today]

The Tale of Renormalized Angular Momentum: Stage 3

- **Resolve the mystery by a simple magic:**

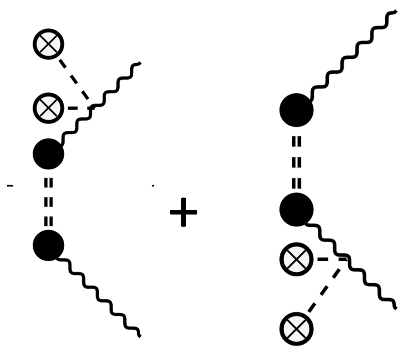
  - Dimensional analysis: $\Lambda \sim R^{2\ell+1}$
  - Newtonian tides: $\delta_\ell^{\mathrm{NZ}} \sim (R\omega)^{2\ell+1}$
  - Full GR tides: $\delta_\ell^{\mathrm{NZ}} \sim (R\omega)^{2\nu+1}$
  - Simple algebra: $(R\omega)^{2\nu+1} = (R\omega)^{2\ell+1}(R\mu)^{2(\nu-\ell)}\left(\frac{\omega}{\mu}\right)^{2(\nu-\ell)}$
  - RG equation: $\frac{d}{d\log\mu}\left[R^{2\ell+1}(R\mu)^{2(\nu-\ell)}\right] = 2(\nu-\ell)\left[R^{2\ell+1}(R\mu)^{2(\nu-\ell)}\right]$

Anomalous Dimension of Tidal Operators

- **EFT:** $S_{\text{tidal}} = \int d\tau Q_L^E \cdot E^L \quad + \quad \int d\tau Q_L^B \cdot B^L$

- **Self-induced RG running:**

$$\frac{d\langle QQ\rangle_\ell(\omega;\mu)}{d\log\mu} = 2(\nu(GM\omega) - \ell)\langle QQ\rangle_\ell$$



+ ⋯ + ⋯

2PM 4PM 6PM and more

- **Example:** $\nu(\omega) = \ell + \nu_2(\omega) + \nu_4(\omega) + \nu_6(\omega) + \cdots$

$s = 2, \ell = 2$ $\qquad \nu_2 = -\frac{214}{105}(GM\omega)^2 \qquad \nu_4 = -\frac{3390466}{1157625}(GM\omega)^4 \qquad \nu_6 = -\frac{1512394771238}{140390971875}(GM\omega)^6$

[Ross, Goldberger (2009)] [Saketh, **Zhou,** Ivanov (2023)] [Edison, Levi (2023)] [Ivanov, Li, Parra-Martinez, **Zhou** (2025)]

The Tale of Renormalized Angular Momentum: Answer to The 30-Year-Old Question

☆ Renormalized angular momentum (characteristic exponents / Floquet exponent) can be understood as the anomalous dimension for tidal operators, which accounts for the tail effects.

What are the implications for the radiative multipoles and the waveforms?

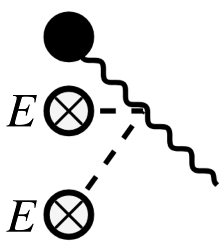
Anomalous Dimension of Radiative Multipoles

* **EFT:**

$$S_{\text{EFT}} = -\left[\int d\tau E(t) + \frac{1}{2}L^{\mu\nu}\Omega_{\mu\nu} - \frac{1}{2}Q^E_{ij}E_{ij} + \frac{2}{3}Q^B_{ij}B_{ij} + \cdots\right]$$

* **RG:**

$$\frac{d\langle Q\rangle_\ell(\omega;\mu)}{d\log\mu} = (\nu(\ell, E, \omega) - \ell)\langle Q\rangle_\ell$$



2PM + • • • + • • •

4PM

6PM and more

* **Solution:**

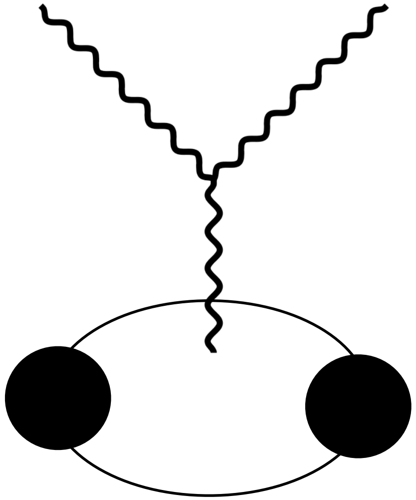
$$\langle Q_L\rangle(\omega) = (r_{\text{orbit}}\omega)^{\nu-\ell}\langle Q_L\rangle(\omega, r_{\text{scale}})$$

An Exact Formula For Anomalous Dimension

$$\gamma_{Q^{\rm rad}_{\ell m}} = -\frac{1}{\pi}\Big(\delta^{\rm BN}_{\ell m}(\omega) + \delta^{\rm BN}_{\ell m}(-\omega)\Big)$$

[Ivanov, Li, Parra-Martinez, **Zhou** (2025)]

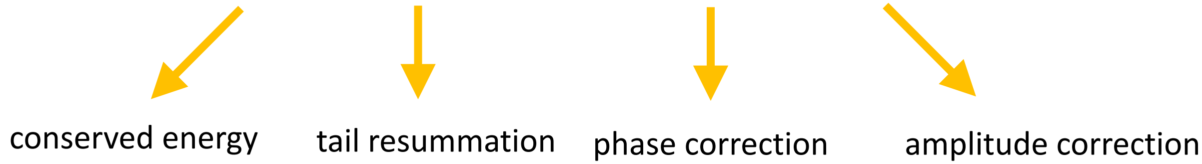
- **Compton from binary:**



[Initial study see Annulli, Bernard, Blas and Cardoso (2018)]

MPM Waveform Tail Resummation

$$h_{\ell m}(t) = H^{\mathrm{EOB}}_{\mathrm{eff}} \times T(\ell, \omega(t)) \times e^{i\delta_{\ell m}} \times f_{\ell m}(x) \;, \quad x = (GM\omega(t))^{1/3}$$



[Damour, Nagar (2007)]

- Leading tail resummation proposed by T. Damour and A. Nagar

$$T(\ell, \omega) = \mathcal{S} e^{i\delta^{\mathrm{tail}}_{\ell}} \qquad \mathcal{S} = e^{\pi GE\omega} \frac{|\Gamma(\ell + 1 - 2iGE\omega)|}{\Gamma(\ell + 1)}$$

$$\delta^{\mathrm{tail}}_{\ell} = \mathrm{Arg}\left[\Gamma(\ell + 1 - 2iGE\omega)\right] + (2GE\omega)\log(2\omega r_{\mathrm{orbit}})$$

- Improved tail resummation

$$T(\ell, \omega) = \mathcal{S} e^{i\delta^{\mathrm{tail}}_{\ell}} (r_{\mathrm{orbit}}\omega)^{\nu - \ell} \qquad \mathcal{S} = e^{\pi GE\omega} \frac{|\Gamma(\nu + 1 - 2iGE\omega)|}{\Gamma(\nu + 1)}$$

$$\delta^{\mathrm{tail}}_{\ell} = \mathrm{Arg}\left[\Gamma(\nu + 1 - 2iGE\omega)\right] + (2GE\omega)\log(2\omega r_{\mathrm{orbit}}) + \frac{\ell - \nu}{2}\pi$$

[Ivanov, Li, Parra-Martinez, **Zhou** (2025)]

MPM Waveform Tail Resummation: Some Checks

- Probe limit checks to all order in G

[Fucito, Morales, Russo (2024)]

- Beyond probe limit

$$
\begin{aligned}
H_{22} = {} & 1 + \left(-\frac{107}{42} + \frac{55}{42}\nu\right)x + 2\pi x^{3/2} + \left(-\frac{2173}{1512} - \frac{1069}{216}\nu + \frac{2047}{1512}\nu^2\right)x^2 + \left[-\frac{107\pi}{21} + \left(\frac{34\pi}{21} - 24\,\mathrm{i}\right)\nu\right]x^{5/2} \\
& + \left[\frac{27027409}{646800} - \frac{856}{105}\gamma_\mathrm{E} + \frac{428\,\mathrm{i}\,\pi}{105} + \frac{2\pi^2}{3} + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96}\right)\nu - \frac{20261}{2772}\nu^2 + \frac{114635}{99792}\nu^3 - \frac{428}{105}\ln(16x)\right]x^3 \\
& + \left[-\frac{2173\pi}{756} + \left(-\frac{2495\pi}{378} + \frac{14333\,\mathrm{i}}{162}\right)\nu + \left(\frac{40\pi}{27} - \frac{4066\,\mathrm{i}}{945}\right)\nu^2\right]x^{7/2} \\
& + \left[-\frac{846557506853}{12713500800} + \frac{45796}{2205}\gamma_\mathrm{E} - \frac{22898}{2205}\mathrm{i}\pi - \frac{107}{63}\pi^2 + \frac{22898}{2205}\ln(16x)\right. \\
& \quad + \left(-\frac{336005827477}{4237833600} + \frac{15284}{441}\gamma_\mathrm{E} - \frac{219314}{2205}\mathrm{i}\pi - \frac{9755}{32256}\pi^2 + \frac{7642}{441}\ln(16x)\right)\nu \\
& \quad \left. + \left(\frac{256450291}{7413120} - \frac{1025}{1008}\pi^2\right)\nu^2 - \frac{81579187}{15567552}\nu^3 + \frac{26251249}{31135104}\nu^4\right]x^4 + \mathcal{O}(x^{9/2})\,. \qquad (6.17)
\end{aligned}
$$

[Blanchet et al (2023)]

Summary

- Worldline EFT gives a model-independent way to study tidal effects of compact objects.

- Gravitational Raman scattering provides a gauge-invariant way to study the structure of compact objects.

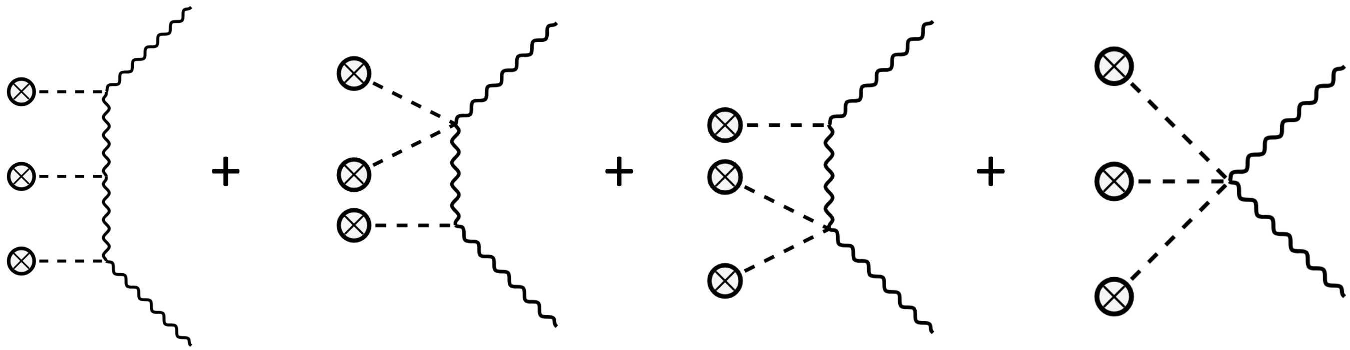
- The analytic continuation of gravitational Raman phase shift gives an exact anomalous dimension for GR multipoles

- Renormalized angular momentum provides an efficient way to perform tail resummation in the GW waveform

Thank You !

Backup Slides

EFT Calculation with On-shell Techniques



- Exp Rep of S-matrix with reverse unitarity:

$$S = e^{i\Delta}$$

- Gauge invariant basis:

$$i\Delta = i\Delta_V V^2 + i\Delta_H H^2 \qquad V^2 : \text{helicity reversing} \qquad H^2 : \text{helicity preserving}$$

- Two-loop amplitudes ($x = \sin(\theta/2)$):

$$i\Delta_{V^2,G^3} = i(GM)^3\omega^2\pi\left[\frac{\left(-15x^6+7x^4+7x^2-15\right)}{16x\left(x^2-1\right)^4}J_1(x) - \frac{\left(x^8+2x^6-3x^4+2x^2+1\right)}{3x^2\left(x^2-1\right)^4}J_2(x) + \frac{\left(-134x^6+153x^4+72x^2-36\pi^2\left(2x^4-3x^2+2\right)-91\right)}{108\left(x^2-1\right)^4} + \frac{\left(88x^6-81x^4+90x^2-81\right)\log(x)}{36\left(x^2-1\right)^4}\right],$$

$$i\Delta_{H^2,G^3} = i(GM)^3\omega^2\pi\left[\frac{(2-x^2)(10-10x^2+x^4)}{3x^8}J_2(x) + \frac{2(60-60x^2+11x^4)}{9x^6}\log(x) - \frac{360-450x^2+121x^4}{54x^6}\right]$$

$$J_1(x) \equiv 2\mathrm{Li}_2(-x) - 2\mathrm{Li}_2(x) + \log(x^2)\log\left(\frac{1+x}{1-x}\right), \quad J_2(x) \equiv \mathrm{Li}_2\left(x^2\right) + \log(x^2)\log(1-x^2)$$

Perturbative Near Zone

BHPT:

Near zone $\sim (GM\omega)^{2\nu+1}\left(1 \quad + \quad (iGM\omega) \quad + \quad (iGM\omega)^2 + \cdots \right)$

"renormalized" angular momentum $\nu(\omega) = -\frac{1}{2} - a = \ell + \mathcal{O}((M\omega)^2)$

[Mano, Suzuki, Takasugi (1996)]

[Sasaki, Tagoshi (2003)]

Relativistic Stars:

Near zone

$$\sim (R\omega)^{2\nu+1}\Bigg[1 + \frac{GM}{R} + \left(\frac{GM}{R}\right)^2 + \cdots$$
$$+(iR\omega)\left(1 + \frac{GM}{R} + \left(\frac{GM}{R}\right)^2 + \cdots\right)$$
$$+(iR\omega)^2\left(1 + \frac{GM}{R} + \left(\frac{GM}{R}\right)^2 + \cdots\right) + \cdots\Bigg]$$

[Saketh, **Zhou,** Ghosh, Steinhoff, Chatterjee (2024)]

Derivation of Connection Formula

✱ Liouville correlator & BPZ:

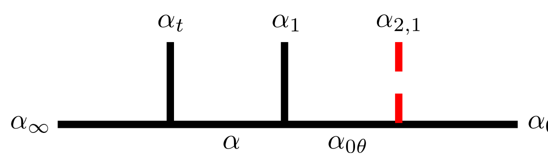
BPZ differential operator:

$$\mathscr{D}_{\mathrm{BPZ}} = \frac{1}{b^2}\frac{\partial^2}{\partial z^2} - \left(\frac{1}{z} + \frac{1}{z-1}\right)\frac{\partial}{\partial z} + \frac{t\,(t-1)}{z(z-1)(z-t)}\frac{\partial}{\partial t} + \sum_{k=0,1,t}\frac{\Delta_k}{\left(z-k\right)^2} + \frac{\Delta_\infty - \Delta_{2,1} - \sum_{k=0,1,t}\Delta_k}{z(z-1)}$$

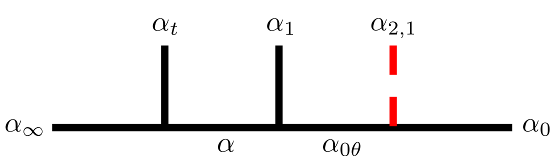
Correlation function with degenerate field:

$$\mathscr{G}(z) = \left\langle V_{\alpha_\infty}(\infty)V_{\alpha_t}(t)V_{\alpha_1}(1)\Phi_{2,1}(z)V_{\alpha_0}(0)\right\rangle$$

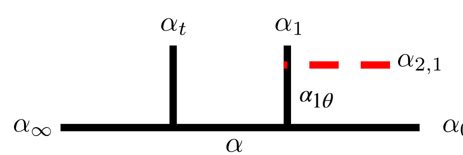
$$= \sum_{\theta=\pm}\int d\alpha\, C_{\alpha_{0\theta}\alpha_{2,1}\alpha_0}C_{\alpha\alpha_1\alpha_{0\theta}}C_{\alpha_\infty\alpha_t\alpha}$$



Locality of fusion transformation:



$$= \sum_{\theta'=\pm} F_{\theta\theta'}(\alpha_0, \alpha_1, \alpha)$$



Fusion matrix:

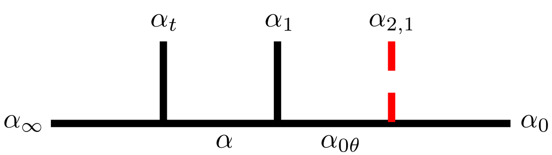
$$F_{\theta\theta'}(\alpha_0, \alpha_1, \alpha) = \frac{\Gamma(1-2b\alpha_0)\Gamma(2b\alpha_1)}{\Gamma\left(\frac{1}{2} + b(\alpha_1 - \alpha_0 + \alpha)\right)\Gamma\left(\frac{1}{2} + b(\alpha_1 - \alpha_0 - \alpha)\right)}$$

[Bonelli, Iossa, Lichtig, Tanzini (2022)]

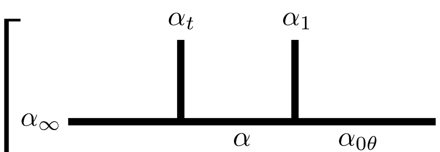
[Lisovyy, Naidiuk (2022)]

Derivation of Connection Formula

✱ OPE:



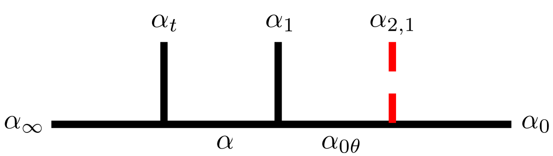
$$= z^{\frac{1+b^2}{2}-\theta b\alpha_0}\Big[$$



$$+O(z)\Big], \qquad z \to 0^+$$

✱ Quasi-classical limit:

$b \to 0, \alpha_i \to \infty, b\alpha_i = a_i$ finite.



$$= \psi_\theta \exp\left(-\frac{1}{b^2}\mathcal{F}\right)$$

$\mathcal{F}$: quasi-classical conformal block

Heun equation: $$\left[\frac{d^2}{dz^2} + \sum_{k=0,1,t} \frac{\delta_k}{(z-k)^2} + \frac{\delta_\infty - \sum_{k=0,1,t}\delta_k}{z(z-1)} + \frac{(t-1)\,\mathcal{E}}{z(z-1)(z-t)}\right]\psi_\pm(z) = 0, \qquad \delta_k = \frac{1}{4} - a_k^2, \qquad \mathcal{E} = t\,\frac{\partial \mathcal{F}}{\partial t}$$

[Bonelli, Iossa, Lichtig, Tanzini (2022)]

[Lisovyy, Naidiuk (2022)]

Derivation of Connection Formula

* OPE + Quasi-classical limit:

$$\psi_{\pm}^{[0]} = \mathcal{N}_{\pm}^{[0]} z^{\frac{1}{2}\mp a_0}\,, \quad z \to 0 \qquad\qquad \psi_{\pm}^{[0]} = \mathcal{N}_{\pm}^{[1]} (1-z)^{\frac{1}{2}\mp a_1}\,, \quad z \to 1$$

$$\frac{\mathcal{N}_{\theta'}^{[1]}}{\mathcal{N}_{\theta}^{[0]}} = \exp\left\{ \frac{\theta'}{2}\frac{\partial \mathcal{F}}{\partial a_1} - \frac{\theta}{2}\frac{\partial \mathcal{F}}{\partial a_0} \right\}$$

* OPE + Quasi-classical limit + Fusion:

$$\psi_{\theta}^{[0]}(z) = \sum_{\theta'} C(\theta a_0, \theta' a_1, a)\psi_{\theta'}^{[1]}(z)$$

$$C(a_0, a_1, a) = F_{\text{cl}}(a_0, a_1, a)\, \exp\left\{ \frac{1}{2}\left(\frac{\partial \mathcal{F}}{\partial a_1} - \frac{\partial \mathcal{F}}{\partial a_0} \right) \right\}$$

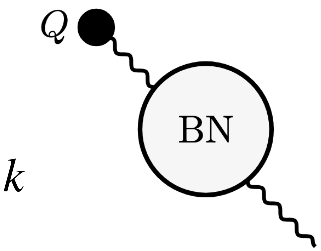
[Bonelli, Iossa, Lichtig, Tanzini (2022)]

[Lisovyy, Naidiuk (2022)]

An Exact Formula For Anomalous Dimension

$$S_{\text{rad}} \simeq \int d\tau \, \underbrace{Q_L^E \cdot E^L}_{\mathbb{O}(\tau)} + (E \leftrightarrow B) \qquad F(\omega) = \langle \vec{k}, h \,|\, \mathbb{O}(\omega) \,|\, M, \vec{S} \rangle$$

$$\omega = -u \cdot k$$



[Follow Caron-Huot and Wilhelm (2016)]

- Construct an in-in observable (symmetric (Kelydish) correlator)

$$G_S(\omega) = \frac{1}{2} \langle M, \vec{S} \,|\, \{ \mathbb{O}(-\omega), \mathbb{O}(\omega) \} \,|\, M, \vec{S} \rangle$$

- Dilatation operator

$$G_S(e^{i\pi}\omega) = e^{i\pi D} G_S(\omega) = G_S(\omega)^* = \frac{1}{2} \langle M, \vec{S} \,|\, \{ \mathbb{O}^\dagger(-\omega), \mathbb{O}^\dagger(\omega) \} \,|\, M, \vec{S} \rangle$$

- Making Connections with far-zone S-matrix from unitarity

$$\mathbb{O}^\dagger = S_{\text{BN}}^\dagger \, \mathbb{O} \, S_{\text{BN}}^\dagger$$

- Exact Anomalous Dimension

$$e^{i\pi D} G_S(\omega) = G_S(\omega) e^{-2i(\delta_{\ell m}^{\text{BN}}(\omega) + \delta_{\ell m}^{\text{BN}}(-\omega))}$$

$$\boxed{\gamma_{Q_{\ell m}} = -\frac{1}{\pi} \left(\delta_{\ell m}^{\text{BN}}(\omega) + \delta_{\ell m}^{\text{BN}}(-\omega) \right)}$$