Advances in the Parametrized QNM Framework and New Perspectives from Bound States

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Quasi Normal Mode and Black Hole Perturbation

Institute for Basic Science (IBS)
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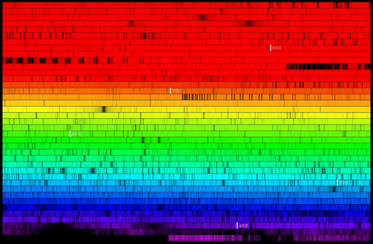
“Spectroscopy” of astrophysical objects



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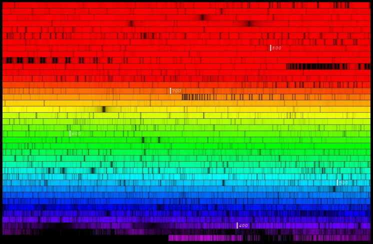

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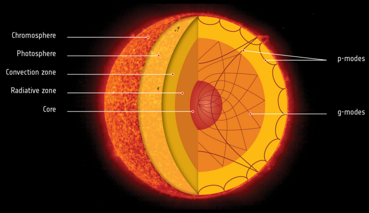

Interior and seismology of the Sun. ©ESA; (Suns chromosphere based on SOHO image; credit: SOHO (ESA & NASA))

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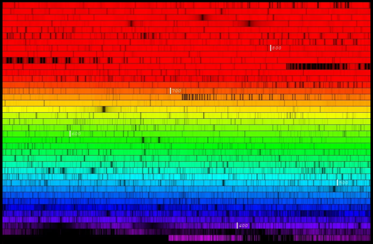

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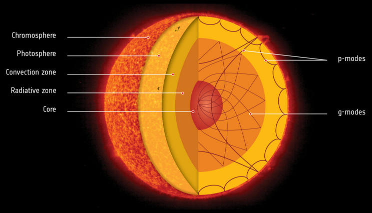

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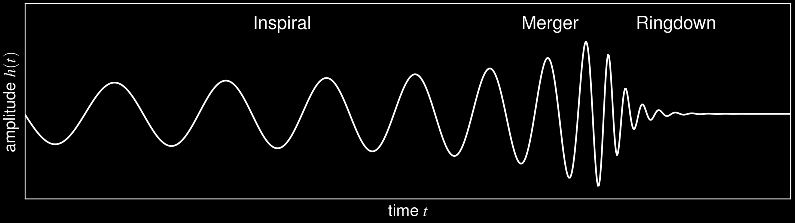

Waveform of a binary BH merger.

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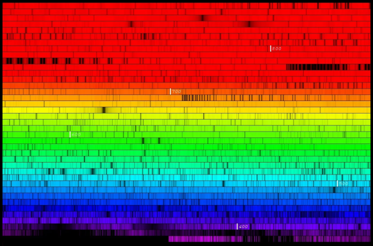

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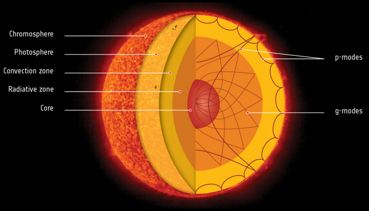

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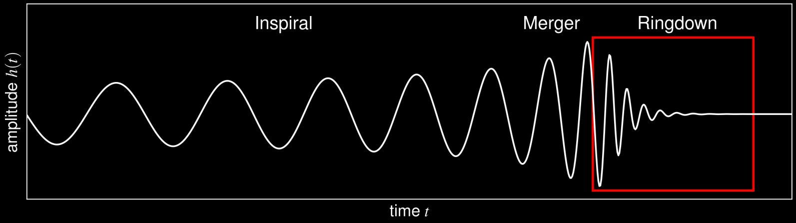

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QNMs test Kerr hypothesis' assumptions and thus general relativity (GR).

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$$\psi^{\mathrm{RD}}(t) = \psi^{\mathrm{direct}}(t) + \psi^{\mathrm{QNM}}(t) + \psi^{\mathrm{tail}}(t) \tag{1}$$

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$$\psi^{\mathrm{QNM}}(t, r_{\mathrm{obs}}) = \sum_{lmn} A_{lmn} e^{-\mathrm{i}\omega_{lmn} t} \tag{2}$$

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- spectral (in)stability: small "bumps" can be important
- how about rotating black holes beyond-GR?!
- ...

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- could work completely within linear theory,
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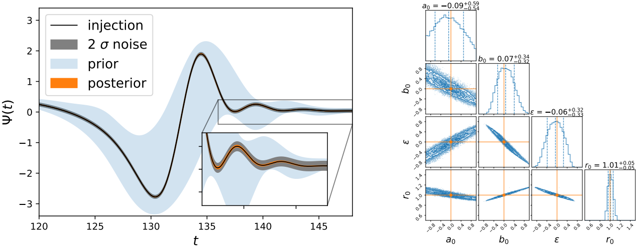

FIGURE 1: *MCMC inference on parametrized black hole metric from time-domain signal of known initial data in an analog gravity context. One might call this a ringdown analysis including prompt response, all QNMs, and all tails. Albuquerque, Völkel arxiv:2501.09000*

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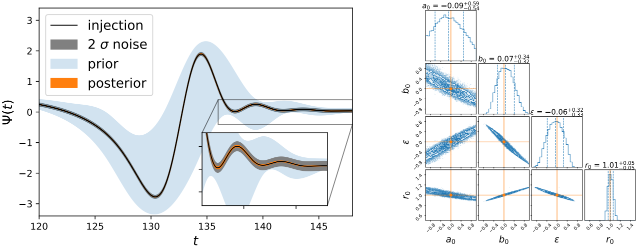

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Not possible for GW context, instead full inspiral, merger, ringdown analysis.

THEORY AGNOSTIC/SPECIFIC BH SPECTROSCOPY

Extraction of QNMs from simulations/data is "theory-agnostic or -specific":

[2] Talks by Bhagwat and Mitman

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Theory-specific workflow:

1. choose theory, e.g., GR
2. find black hole solution, e.g., Schwarzschild, Kerr
3. derive perturbation equations, e.g., Regge-Wheeler, Teukolsky
4. compute QNM spectrum, e.g, $\omega_{\ell mn}(M, a)$
5. connect excitation coefficients with NR simulations, e.g., $C_{\ell mn}(q, a_1, a_2, ...)$[2]
6. does this "model $=$ model (M, a)" describe data/simulations?

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6. does this "model = model (M, a)" describe data/simulations?

Final steps very hard, even in GR; vastly unexplored beyond-GR.

Typically $\omega_{\ell mn}(M, a)$ is the only theory-specific part, $C_{\ell mn}(M_{\text{final}}, a_{\text{final}})$ are free parameters with reasonable priors.

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Example for **theory-agnostic** workflow:

1. assume sum of QNMs describes signal reasonably well
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Many variations of agnostic/parametrized tests exist (examples next slide).

No strict borders, one can have a varying degree of theoretical/agnostic parametrizations.

EXAMPLES FOR AGNOSTIC FRAMEWORKS

Theory-agnostic frameworks for black hole spectroscopy (ringdown only):

- Parametrized Ringdown Spin Expansion Coefficients (ParSpec):
  measure coefficients from data/predict them from theories [3]
- Effective field theory of QNMs:
  start from an action [4]
- Post-Kerr black hole spectroscopy:
  relate Kerr metric deviations to QNMs via eikonal approximation/connection [5]
- parametrized QNM framework:
  modify/parametrize effective potentials and provide QNM shifts [6]

[3] Maselli+ 1910.12893 Maselli+ 2311.14803

[4] Tattersall+ 1711.01992, Franciolini+ 1810.07706, Hui+ 2111.02072

[5] Glampedakis+ 1706.07658, Glampedakis+ 1906.05455, Silva+ 1912.09286

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Each approach has advantages/disadvantages.

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OVERVIEW

First developed for non-rotating black holes[7] [8].

- Metric/scalar perturbations in GR and for Schwarzschild can be written as

$$f \frac{\mathrm{d}}{\mathrm{d}r}\left(f \frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{d}r}\right) + \left[\omega^2 - f\mathbf{V}\right]\mathbf{\Phi} = 0, \tag{3}$$

- with $f(r) = 1 - r_0/r$, r_0 being the location of the event horizon and

$$\mathbf{\Phi} = \left[\Phi^{\mathrm{scalar}}, \Phi^{\mathrm{polar}}, \Phi^{\mathrm{axial}}, \dots\right]. \tag{4}$$

- in GR, V is diagonal with Regge-Wheer/Zerilli/scalar potential
- the parametrized QNM framework writes the potentials ("natural extension")

$$V_{ij} = V_{ij}^{\mathrm{GR}} + \delta V_{ij}, \qquad \delta V_{ij} = \frac{1}{r_H^2} \sum_{k=0}^{\infty} \alpha_{ij}^{(k)} \left(\frac{r_H}{r}\right)^k \tag{5}$$

[7] Cardoso+, PRD 99 104077 (2019)
[8] McManus+, PRD 100 044061 (2019)

- non-zero off-diagonal δV_{ij} couple (new) fields
- for small deviations, approximate expression up to quadratic order in $\alpha^{(k)}$

$$\omega = \omega^0 + \alpha_{ij}^{(k)} d_{(k)}^{ij} + \alpha_{ij}^{(k)} \partial_\omega \alpha_{pq}^{(s)} d_{(k)}^{ij} d_{(s)}^{pq} + \frac{1}{2} \alpha_{ij}^{(k)} \alpha_{pq}^{(s)} e_{(ks)}^{ijpq} \tag{6}$$

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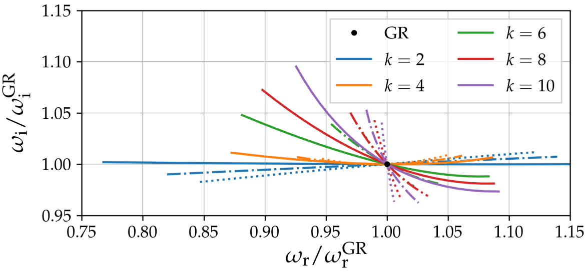

Relative change of the QNM spectrum along small $\alpha^{(k)}$ range. Völkel, Franchini and Barausse, PRD 105 084046 (2022).

PROGRESS IN THE PQNM FRAMEWORK

Main papers providing "foundations/coefficients" of the framework [9] [10] [11] [12] [13] [14] [15] [16] (more on "applications" later.)

	QNM-shift order	involved fields	time evolution
Schwarzsch.	lin.$^{(a)}$ & quadr.$^{(b)}$	single$^{(a,b)}$ & coupled$^{(b)}$	axial$^{(h)}$
non-Schwarzsch.*	lin. & quadr.$^{(e)}$	single$^{(e)}$	–
Kerr (Teukolsky)	lin.$^{(g)}$	single$^{(g)}$	–

- important observation for field redefinitions (c)
- computations of overtones in (d,f,g)

[9] (a) Cardoso+, 1901.01265
[10] (b) McManus+, 1906.05155
[11] (c) Kimura+, 2001.09613
[12] (d) Völkel+, 2202.08655
[13] (e) Franchini+, 2210.14020
[14] (f) Hirano+, 2404.09672
[15] (g) Cano+, 2407.15947
[16] (h) Thomopoulos+, 2504.17848

APPLICATIONS: INVERSE PROBLEM/INFERENCE

Assume we measure/provide some QNMs with errors:

- Can one reconstruct potentials and coupling functions from QNMs? [17]
- Can one reconstruct the lightring region? [18]

[17] Völkel, Franchini, Barausse 2202.08655

[18] Völkel, Franchini, Barausse, Berti 2209.10564

[19] Völkel, Kokkotas arxiv:1704.07517 Völkel, Kokkotas arxiv:1802.08525 Völkel, Konoplya, Kokkotas arxiv:1901.11262 Albuquerque, Völkel, Kokkotas, Bezerra 2309.11168 Albuquerque, Völkel, Kokkotas 2406.16670

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WKB-based strategy for inverse QNM problems for different type of compact objects with direct inversion studied in other works. [19]

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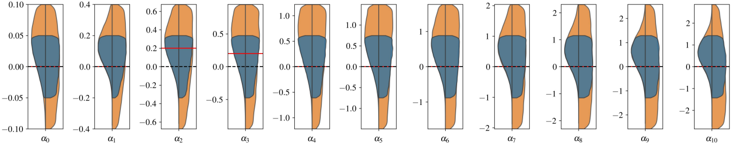
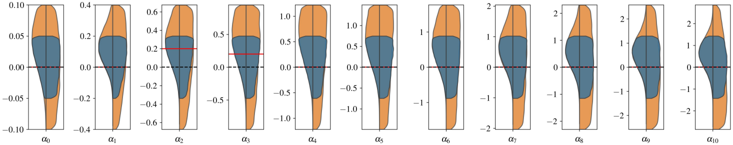


Non-GR: $n = 0,1$ QNMs $\vec{\omega}$ with non-zero $\alpha^{(k)}$ for $k = 2$ and $k = 3$ (shown in red), but other assumptions the same as in the GR case. Völkel, Franchini, Barausse, Berti PRD 106 (2022) 12, 124036

[20] Schutz and Will 1985, Iyer Will 1987, Konoplya 2003, ...



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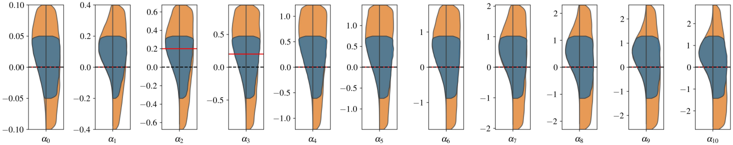

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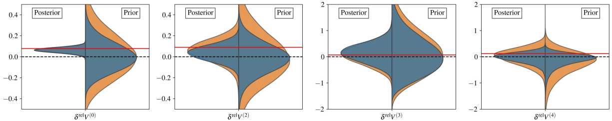

Non-GR: Sampling derivatives of the effective potential with respect to tortoise coordinate from the "all $\alpha^{(k)}$" posterior distributions (left sides) versus sampling from the priors of $\alpha^{(k)}$ (right sides). Völkel, Franchini, Barausse, Berti PRD 106 (2022) 12, 124036

[20] Schutz and Will 1985, Iyer Will 1987, Konoplya 2003, ...

PQNM FRAMEWORK FOR ROTATING BLACK HOLES

PQNM framework with QNM shifts "beyond-Teukolsky"[21]

$$\omega_{n\ell m} \simeq \omega^0_{n\ell m} + \sum_k d^{(k)}_{\omega,n\ell m} \alpha^{(k)}\,, \tag{8}$$

$$B_{\ell m}(a\omega) \simeq B^0_{\ell m}(a\omega) + \sum_k d^{(k)}_{B,\ell m} \alpha^{(k)}\,, \tag{9}$$

`github` with `Jupyter notebook` for non-rotating and beyond-Teukolsky data/tutorial [22]

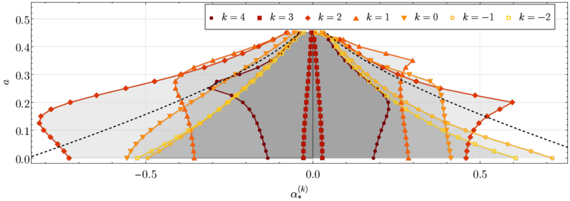

FIGURE 2: Here we show threshold values $\alpha_*^{(k)}$ of the beyond-Teukolsky framework against the spin for different values of k for the $n=0$, $\ell=m=2$ mode. The threshold describes the value at which the linear approximation and accurate result differ by 1%. An estimate for this error is shown as black dashed lines. Cano+, Phys.Rev.D 110 (2024) 10, 104007, DOI:10.1103/PhysRevD.110.104007.

[21] Cano, Capuano, Franchini, Maenaut, Völkel, arxiv:2407.15947

[22] `https://github.com/sebastianvoelkel/parametrized_qnm_framework/tree/v1.0`

PQNM FRAMEWORK IN THE TIME DOMAIN

With some caveats, let us explore $\omega^2 \to -\partial_t^2$ for single field perturbations

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\mathrm{d}^2}{\mathrm{d}x^2} - f(r)V(r)\right]\Phi(t,x) = 0\,, \tag{10}$$

with tortoise coordinate $x(r)$.

[23] Thomopoulos, Völkel, Pfeiffer 2504.17848

[24] de Medeiros, Corrêa, Macedo 2505.03892

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Scatter ingoing wave packets (Gaussian initial data) with various modifications:

- Are the perturbative QNM predictions correct and robust?
- What is the role of tails in QNM extraction and modified potentials?
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Studied in master thesis project of **Spyros Thomopoulos** (Natl. Tech. U. Athens) [23]

Complementary recent work with continued fraction comparison for single modifications.[24]

[23] Thomopoulos, Völkel, Pfeiffer 2504.17848

[24] de Medeiros, Corrêa, Macedo 2505.03892

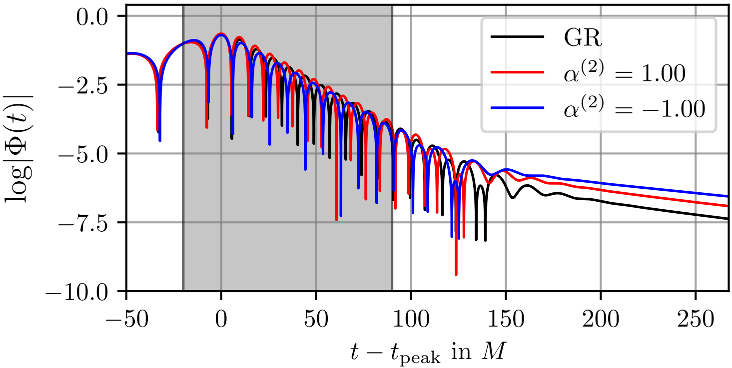

FIGURE 3: *Exemplary waveforms* Thomopoulos, Völkel, Pfeiffer 2504.17848

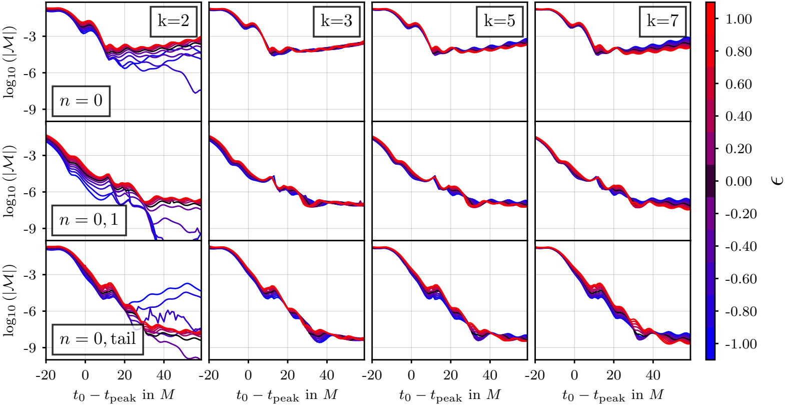

FIGURE 4: *Mismatches between gravitational waveforms and fit results.* In the top panels, we consider a model with one QNM; in the central panels, we use two QNMs; and in the bottom panels, we consider one QNM and the power law tail. From left to right, we consider different powers $k \in [2,3,5,7]$ of the modifications $\alpha^{(k)}$. In each panel, we show mismatches as a function of the starting time $t_0 - t_{\text{peak}}$. Different curves in each panel correspond to different amplitudes of $\alpha^{(k)}$ that are indicated by different colors in the colorbar. Thomopoulos, Völkel, Pfeiffer 2504.17848

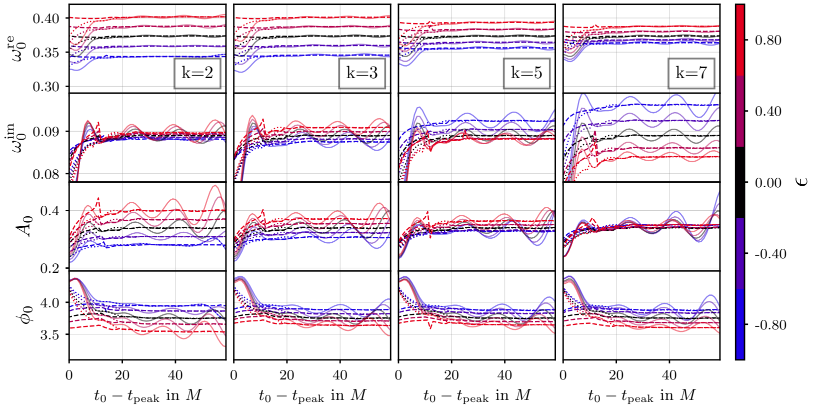

FIGURE 5: *Extracted parameters from gravitational waveforms.* We show from top to bottom the extracted real and imaginary parts of the fundamental QNM, its amplitude A_0 and phase ϕ_0, as function of the starting time $t_0 - t_{\text{peak}}$ when using different fitting models, $n = 0$ (solid lines), $n = 0,1$ (dashed lines) and $n = 0,\text{tail}$ (dotted lines). From left to right, we consider different powers $k \in [2,3,5,7]$ of the modifications $\alpha^{(k)}$. Thomopoulos, Völkel, Pfeiffer 2504.17848

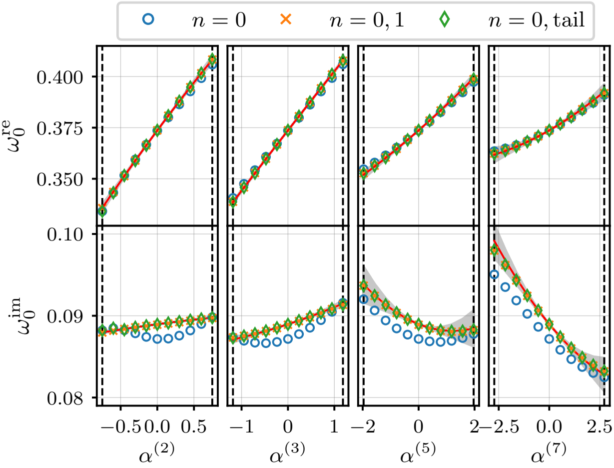

FIGURE 6: *Extraction of the $n=0$ QNM for $t_0 - t_{\text{peak}} = 50M$ and comparison with the perturbative prediction.* In the top panels, we show the real part, and in the bottom panels, the imaginary part. From left to right, we consider different powers $k \in [2,3,5,7]$ of the modifications $\alpha^{(k)} = \varepsilon \alpha^{(k)}_{\text{max}}$ and we also indicate each $\alpha^{(k)}_{\text{max}}$ value (vertical dashed lines). In each panel, we show the perturbative prediction (red line) obtained from Eq. (**??**) and an estimate for its accuracy (grey-shaded region) as a function of the coefficient $\alpha^{(k)}$. Different points correspond to the extracted QNM from the time-domain fits of different models. Thomopoulos, Völkel, Pfeiffer 2504.17848

Using Schutz-Will formula AND Taylor expansion of $V^{(0)}(\omega)$ and $V^{(2)}(\omega)$ around GR

$$V^{(0)} = \omega^{\mathrm{re}\,2} - \omega^{\mathrm{im}\,2}, \qquad V^{(2)} = -8\left(\omega^{\mathrm{re}}\omega^{\mathrm{im}}\right)^2. \tag{11}$$

$$V^{(0)}(\omega^{\mathrm{re}}) \approx V^{(0)}_{\mathrm{GR}} + 2\omega^{\mathrm{re}}_{\mathrm{GR}}\delta\omega^{\mathrm{re}}, \qquad V^{(2)}(\omega^{\mathrm{re}}\omega^{\mathrm{im}}) \approx V^{(2)}_{\mathrm{GR}} - 16\omega^{\mathrm{re}}_{\mathrm{GR}}\omega^{\mathrm{im}}_{\mathrm{GR}}\delta\omega^{\mathrm{re,im}}, \tag{12}$$

with $\delta\omega^{\mathrm{re}} = \omega^{\mathrm{re}} - \omega^{\mathrm{re}}_{\mathrm{GR}}$ and $\delta\omega^{\mathrm{re,im}} = \omega^{\mathrm{re}}\omega^{\mathrm{im}} - \omega^{\mathrm{re}}_{\mathrm{GR}}\omega^{\mathrm{im}}_{\mathrm{GR}}$.

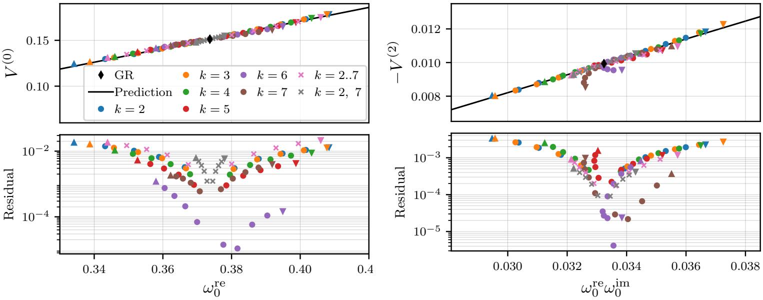

FIGURE 7: *Correlations of potential properties with fundamental mode parameters.* Thomopoulos, Völkel, Pfeiffer 2504.17848

CONCLUSIONS PART I

- parametrized QNM framework useful tool to explore QNM physics
- if one can map it to beyond-GR systems, small QNM shifts are trivial
- if interested in theory-agnostic studies, “general” parametrization
- works as expected in simple-time domain studies

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- works as expected in simple-time domain studies

Robust features of ringdown analyses are the **local properties of the potential**, inspired from WKB and eikonal approximation/lightring.

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2025 is celebrated as **100 years of quantum mechanics**, not in this talk?!

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- numerical computations difficult, shooting methods extremely unstable
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What can we learn from bound states?

MASHHOON'S MAPPING

Mashhoon mapped QNMs to bound states of an inverted potential[25]:

- apply $x \to -ix$ in wave equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \left[\omega_n^2 - V_\ell(r,P)\right]\psi(x) = 0\,, \tag{13}$$

- demand that $V(x,P) = V(-ix,P')$
- define $\Omega^2(P) \equiv -E_n(P)$
- one finds QNMs from bound states via

$$\omega_n(P) \equiv \Omega_n(\pi^{-1}(P))\,. \tag{14}$$

- $P' = \pi(P)$ depends on the specific potential.

In early works for parabolic, Pöschl-Teller and Eckart potentials.

WKB-based and numerical approach without analytic solution introduced recently[26]

[25] Mashhoon, 3rd Marcel Grossmann Meeting (1982), Blome and Mashhoon, Physics Letters A 100, 231 (1984), Ferrari and Mashhoon, Phys. Rev. Lett. 52, 1361 (1984), Ferrari and Mashhoon, Phys. Rev. D 30, 295 (1984)

[26] Hatsuda arxiv:1906.07232, Völkel, arxiv:2210.02069

NUMERICAL RESULTS

Here are some of the bound states of a Schwarzschild black hole (M=1).

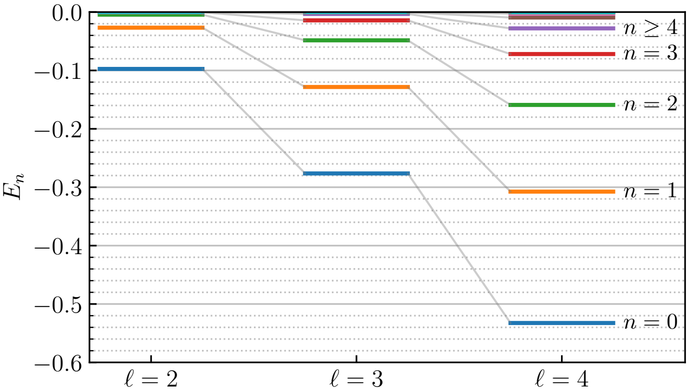

FIGURE 8: Spectrum of bound states E_n of the inverted Regge-Wheeler potential for $\ell \in [2,3,4]$ and $M = 1$. Gray lines connect states with the same n (same color) for different ℓ. Völkel arxiv:2505.17186

Eigenfunctions and potential ($\ell = 2$) with and without an artificial bump.

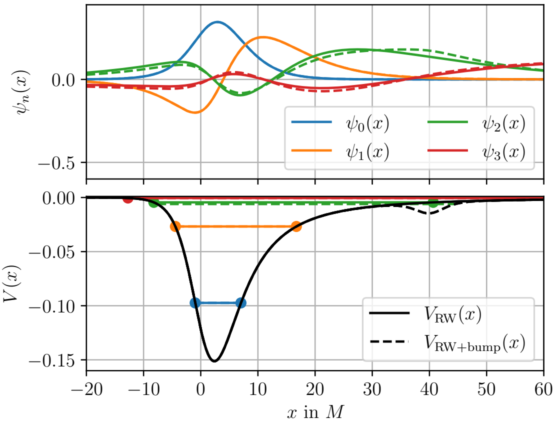

FIGURE 9: Inverted Regge-Wheeler potential without (black solid) and with an artificial bump (black dashed) together with the first four eigenfunctions (colored solid and dashed, respectively). The energy levels of both potentials are indicated as horizontal lines that start and end at the classical turning points of the unperturbed Regge-Wheeler potential. Völkel arxiv:2505.17186

Separation of classical turning points $V(x_{0,1}) = E_n$.

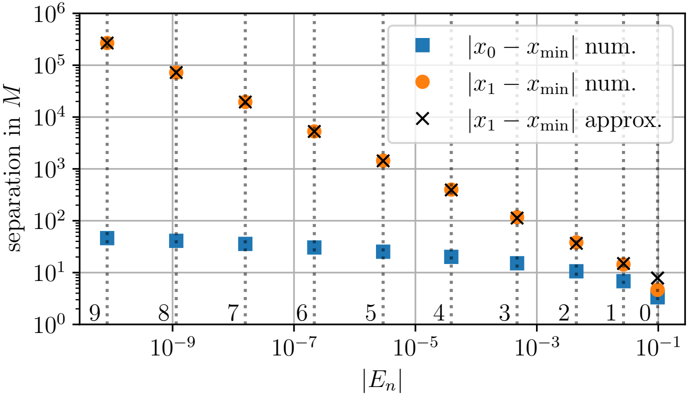

FIGURE 10: Here we relate the Regge-Wheeler bound states E_n with their associated left ($x_0(E_n)$, blue squares) and right turning points ($x_1(E_n)$, orange circles). For comparison, we also show the asymptotic approximation Eq. (**??**) for the right turning point (black crosses). We indicate the value of the bound states E_n (vertical black dotted) and its number. Völkel arxiv:2505.17186

EIGENFUNCTIONS AND BOUND STATES

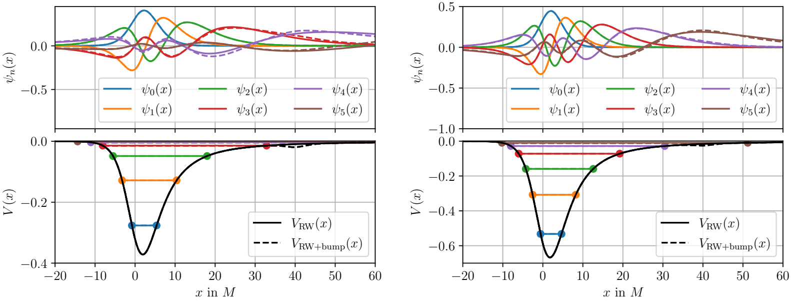

FIGURE 11: Inverted Regge-Wheeler potential for $\ell = 3$ (left panel) and $\ell = 4$ (right panel) without (black solid) and with an artificial bump at $x_0 = 40M$(black dashed) together with the first six eigenfunctions (colored solid and dashed, respectively). Völkel arxiv:2505.17186

ASYMPTOTIC SPACING FOR LARGE n

Exponential scaling for large n states, (Hydrogen atom only $1/n^2$)

$$E_{n+1} \approx E_n e^{-K}, \quad K = \frac{2\pi}{\sqrt{\ell(\ell+1)-1/4}}. \tag{15}$$

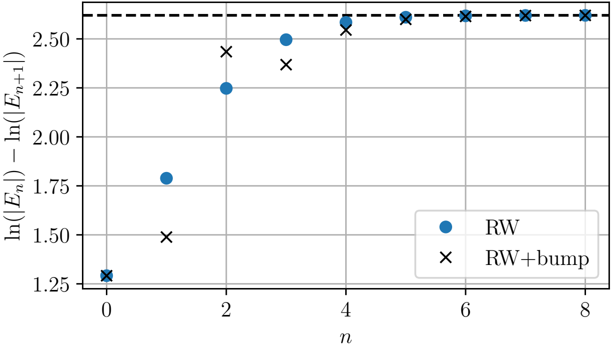

FIGURE 12: Here we show the separation of subsequent energy levels $\ln(|E_n|) - \ln(|E_{n+1}|)$ for the two potentials, as well as the asymptotic spacing expected from Eq. (15) (black dashed). Völkel arxiv:2505.17186

CONCLUSIONS PART II

Some (open) questions that one should think about:

- Implications of bound states being loosely (marginally?) bound, outer turning point "divergence".
- If large overtones are sensitive to far distances, how could we excite 7 (correct) overtones "close" to the black hole?
- Asymptotic spacing of ΔE_n depends on ℓ ($\Delta\omega_n$ does not), like tails!
- Small deviations in the potential only cause small shifts in E_n. However, one could think of tunneling effects between two potential wells for suitable levels.
- Deeper connections to Maggiore's interpretation of the QNM spectrum?[27]

Thank you very much for listening!

[27] Maggiore, Phys. Rev. Lett. 100, 141301 (2008), arxiv:0711.3145