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Peccei–Quinn Quality of Warped Extra-Dimensional Axion

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Outline

Introduction to Axion

Strong CP problem

- In the Standard Model, the gauge symmetry allows the θ-term

$$\mathcal{L} = -i\left[\bar{Q}_L\bar{\sigma}^\mu D_\mu Q_L + \bar{u}_R\sigma^\mu D_\mu u_R + \bar{d}_R\sigma^\mu D_\mu d_R\right] - \left(y_u\bar{Q}_L\tilde{H}u_R + y_d\bar{Q}_L H d_R + h.c.\right)$$

$$-\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{g_s^2\theta_{\rm QCD}}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \cdots \quad \text{with} \quad \tilde{G}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G^a_{\rho\sigma}.$$

- However, the chiral anomaly of the quark sector gives an additional θ-term. One can define the observable (basis-independent) θ-term:

$$\bar{\theta}_{\rm QCD} = \theta_{\rm QCD} + {\rm Arg}\ {\rm det}(y_u y_d)\,.$$

Strong CP problem

- The θ-term violates P and CP ($\Leftrightarrow$ T).

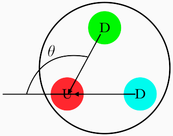
$$\mathcal{L} \supset \frac{g_s^2\,\theta}{32\pi^2} G^a_{\mu\nu}\,\tilde{G}^{a\mu\nu} \;\sim\; \theta\,\mathbf{E}_g \cdot \mathbf{B}_g$$

	P	C	T
$\mathbf{E}$	$-\mathbf{E}$	$-\mathbf{E}$	$\mathbf{E}$
$\mathbf{B}$	$\mathbf{B}$	$-\mathbf{B}$	$-\mathbf{B}$
q	q	$-q$	q

- It gives Nucleon Electric Dipole Moment at low energies consequence.

$$\mathcal{L}_{\text{eff}} = \bar{N}(i\gamma^\mu D_\mu - m_N)N + \bar{N}\sigma^{\mu\nu}(\mu_N F_{\mu\nu} + d_N \tilde{F}_{\mu\nu})N + \cdots$$

$$\Rightarrow \mathcal{H}_{\text{eff}} \supset -\mathbf{S}_N \cdot \left(\frac{g_N e}{2m_N}\mathbf{B} + d_N \mathbf{E}\right)$$



(A classical picture of the neutron)

Strong CP problem

- Experiment gives an upper bound of $\bar{\theta}_{\rm QCD}$

$$d_N \sim \frac{\bar{\theta}_{\rm QCD}}{8\pi^2}\frac{e}{m_N} \simeq 2.4\times 10^{-16}\,\theta\, e\,{\rm cm} \ < \ 1.8\times 10^{-26}\ e\,{\rm cm},$$

$$\Rightarrow \bar{\theta}_{\rm QCD} = \theta_{\rm QCD} + {\rm Arg}\ {\rm det}(y_u y_d) \ < \ 10^{-10}\,.$$

It implies a fine tuning between an unrelated quantities (QCD vacuum structure and phase of Yukawa matrix) in the SM!

Strong CP Problem

QCD axion solution

- Introduce a pseudo-scalar field $a(x)$ endowed with a global Peccei–Quinn (PQ) symmetry.

$$\mathcal{L} = \mathcal{L}_{\rm SM} - \frac{1}{2}\,\partial_\mu a\,\partial^\mu a + \frac{g_s^2}{32\pi^2}\,\frac{a}{f_a}\,G^a_{\mu\nu}\,\tilde{G}^{a\mu\nu}\,.$$

Using chiral perturbation theory, nonperturbative QCD generates a periodic potential for $a(x)$:

$$V_{\rm QCD}(a) \propto -\cos\left(\bar{\theta}_{\rm QCD} + \frac{a}{f_a}\right) \;\Rightarrow\; \langle a\rangle = -\,f_a\,\bar{\theta}_{\rm QCD}\,.$$

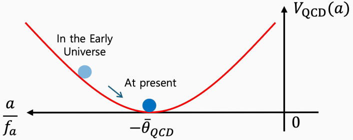
At the minimum of $V_{\rm QCD}(a)$, the CP–violating θ-term cancels, leading to a sufficiently small neutron EDM:

$$\left(\frac{\langle a\rangle}{f_a} + \bar{\theta}_{\rm QCD}\right)\frac{g_s^2}{32\pi^2}\,G^a_{\mu\nu}\tilde{G}^{a\mu\nu} = 0 \;\Rightarrow\; d_N \sim \frac{\theta_{\rm eff}}{8\pi^2}\frac{e}{m_N} \approx 0, \quad \theta_{\rm eff} = \bar{\theta}_{\rm QCD} + \frac{\langle a\rangle}{f_a}.$$

PQ symmetry:

$U(1)_{\rm PQ}:\; a \to a + \alpha f_a,\; \alpha \in \mathbb{R}$

This continuous symmetry forbids a potential for a at the perturbative level.



UV completion of axion

- Due to the presence of a dimension-5 operator, the following effective Lagrangian cannot be a fundamental theory:

$$\mathcal{L} = \mathcal{L}_{\rm SM} - \frac{1}{2}\,\partial_\mu a\,\partial^\mu a + \frac{g_s^2}{32\pi^2}\,\frac{a}{f_a}\,G^a_{\mu\nu}\,\tilde{G}^{a\mu\nu}\,.$$

Then, what is the UV completion of the above effective Lagrangian?

Phase of a complex scalar (KSVZ, DFSZ)

Higher-dimensional gauge field

Axion quality problem

- Depending on the UV origin of the axion, one can have PQ-violating effects that are *not* tied to the QCD sector (e.g. quantum-gravitational contributions)

$$\mathcal{L} = -\frac{1}{2}\,\partial_\mu a\,\partial^\mu a + \frac{g_s^2}{32\pi^2}\,\frac{a}{f_a}\,G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \delta V_{\rm UV}(a) + \cdots\,.$$

For simplicity, shift the axion to zero VEV: $a \to a - \langle a \rangle$. The additional PQ-breaking source is parameterized as:

$$V(a) \;=\; V_{\rm QCD}(a) + \delta V_{\rm UV}(a) \;\simeq\; m_u\,\Lambda^3_{\rm QCD}\left[1 - \cos\left(\frac{a}{f_a}\right)\right] + V_{\rm UV}\cos\left(\frac{a}{f_a} + \delta_*\right),$$

where $\delta_* = \mathcal{O}(1)$, without fine-tuning.

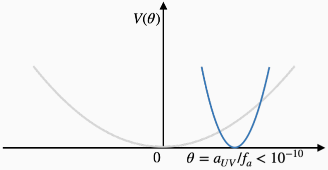
Axion quality problem

- At the low energy, the axion potential near the origin is

$$V(a) = V_{\rm QCD}(a) + \delta V_{\rm UV}(a) \simeq \frac{1}{2} m_a^2 (a - a_{\rm UV})^2 + \cdots \,, \quad m_a \simeq \mu{\rm eV} \left(\frac{10^{12}\,{\rm GeV}}{f_a}\right) + \delta_{\rm UV} m_a .$$

- To resolve the strong-CP problem one requires,

$$\theta \;=\; \frac{\langle a \rangle}{f_a} \;\simeq\; \frac{V_{\rm UV} \sin \delta_*}{m_u \, \Lambda_{\rm QCD}^3} \;<\; 10^{-10}.$$



Therefore the $V_{\rm UV}$ should be smaller than

$$V_{\rm UV} \;\lesssim\; \mathcal{O}(10^{-88}) \, M_{\rm P}^4 .$$

Axion Quality Problem

Extra-Dimensional Axion

Axion from higher-dimensional gauge field

- $U(1)$ gauge field and gluon field with Chern–Simons coupling in 5D,

$$S = \int_{M \times S^1} d^5x \left[-\frac{1}{4g_C^2} C_{MN} C^{MN} + \kappa \epsilon^{MNPQR} C_M G^a_{NP} G^a_{QR} \right], \quad M, N = 0, 1, 2, 3, 5$$

- One can define the 4D field $\theta(x) = a(x)/f_a$

$$\theta(x) \equiv \int_0^{2\pi R} dy \, C_5(x, y)$$

- Integrating the fifth dimension, the 4D action is

$$S = \int_M d^4x \left[-\frac{1}{4g^2} C_{\mu\nu} C^{\mu\nu} - \frac{1}{2} \partial^\mu a \partial_\mu a + \underbrace{2\pi R g \kappa \epsilon^{\mu\nu\rho\sigma} a G^a_{\mu\nu} G^a_{\rho\sigma}}_{\frac{1}{32\pi^2} \frac{a}{f_a} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}} + \cdots \right]$$

$$\boxed{\frac{2\pi R}{g_C^2} = \frac{1}{g^2}, \qquad f_a^2 = \frac{1}{2\pi R g_C^2}}$$

Axion from higher-dimensional gauge field

- Is $\theta(x)$ an axion candidate?

  $U(1)_{\mathrm{PQ}}$ **transformation** : $\quad C \;\to\; C + d\left(\frac{\alpha\, y}{2\pi R}\right), \quad \alpha \in \mathbb{R}$

$$\theta(x) \;=\; \oint_{S^1} C \;\longrightarrow\; \oint_{S^1} C + \alpha \;=\; \theta(x) + \alpha \qquad \text{(shift symmetry)}$$

  $U(1)$ **large gauge transformation** : $\quad C \;\to\; C + d\left(\frac{y}{R}\right)$

$$\theta(x) \;=\; \oint_{S^1} C \;\longrightarrow\; \oint_{S^1} C + 2\pi \;=\; \theta(x) + 2\pi \qquad \text{(}2\pi\text{ periodicity)}$$

Axion from higher-dimensional gauge field

- Can $\theta(x)$ be a **good-quality** axion candidate?

$$U(1)_{PQ} \text{ transformation:} \qquad C \;\to\; C + d\Big(\frac{\alpha\, y}{2\pi R}\Big)\,, \quad \alpha \in \mathbb{R}$$

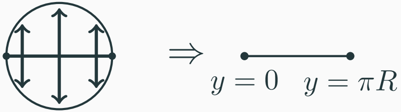
Locally this can be interpreted as a remnant of the $U(1)$ gauge transformation, but it is not globally well-defined for charged matter:

$$\Phi \;\to\; U\Phi = e^{iq_\Phi \alpha\, y/(2\pi R)}\,\Phi, \quad U(y) \neq U(y+2\pi R) \quad \text{unless} \quad \alpha = 2\pi n,\; n \in \mathbb{N}.$$

This implies that $U(1)_{PQ}$ can be broken only by **non-local effects** along the extra-dimensional direction.

Orbifold

- An **orbifold** is obtained by modding out a manifold by a discrete symmetry that leaves some points fixed. For example, the $S^1/\mathbb{Z}_2$ orbifold get from a circle S^1 and identify $y \sim -y$. The quotient gives an interval with fixed points $y = 0$ and $y = \pi R$.



- Fields on $[-\pi R, \pi R]$ are assigned the $\mathbb{Z}_2$ parities such as **even** or **odd** under $\mathbb{Z}_2$:

$$\text{even:} \quad \Phi_+(-y) = \Phi_+(y), \qquad \text{odd:} \quad \Phi_-(-y) = -\Phi_-(y).$$

Main goal

- We analyze how PQ–symmetry breaking on the orbifold $S^1/\mathbb{Z}_2$ affects axion *quality*, focusing on:
  1. $\mathbb{Z}_2$-parity assignments of gauge couplings,
  2. fixed–point (brane–localized) PQ breaking,
  3. warping.
- Why work on the **orbifold**?
  - A plain S^1 compactification is insufficient to obtain chiral fermions in 4D.
  - The $S^1/\mathbb{Z}_2$ orbifold projects out unwanted degrees of freedom and generates chiral zero modes.

$S^1/\mathbb{Z}_2$ orbifold parity of matter fields

- A general form of the orbifold parity is

$$\phi(x,-y) = \pm\phi^c(x,y), \qquad \tilde{\phi}(x,-y) = \pm\tilde{\phi}(x,y),$$

where ϕ^c is the charge–conjugated field of ϕ. Consistent charge assignments:

$$\phi : q_\phi, \qquad \tilde{\phi} : \epsilon(y)\,\tilde{q}_\phi.$$

- As discussed earlier, the PQ transformation corresponds to a gauge-field shift

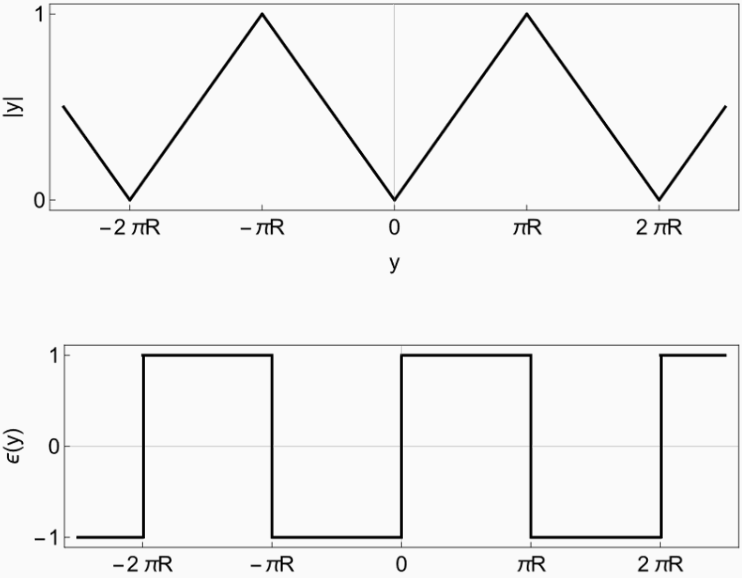
$$C_5 \;\to\; C_5 + d\Lambda, \qquad \Lambda = \frac{\alpha}{2\pi R}\,y,$$

under which the Lagrangian is invariant if the charged fields transform as

$$\phi \;\to\; \exp\left[i\,\frac{q_\phi\alpha}{2\pi R}\,y\right]\phi, \qquad \tilde{\phi} \;\to\; \exp\left[i\,\frac{\tilde{q}_\phi\alpha}{2\pi R}\,|y|\right]\tilde{\phi}\,.$$

Implication. Locally, the PQ symmetry is preserved on $S^1/\mathbb{Z}_2$ and does not induce explicit symmetry breaking.

$|y|$ and $\epsilon(y)$



Sources of PQ breaking

- **Global consistency.** The PQ transformation is locally a gauge transformation, but it is globally allowed only for

$$\alpha = 2\pi n, \qquad n \in \mathbb{Z},$$

  otherwise the charged matter violates the 2π periodicity of ϕ.
- **Origin of breaking.** The failure of compact identification in the y direction implies a *global* PQ breaking that arises from *non-local* effects, which loops of charged matter propagating through the bulk.
- **Role of orbifold fixed points.** If $\tilde{\phi}$ preserves 2π periodicity, bulk loops do not generate an axion potential; explicit PQ breaking arises only from fixed-point operators, requiring communication between the two branes.

PQ breaking effects in the extra dimension model

Spectral function approach

- Define a spectral function $N(z; q\theta)$ whose zeros coincide with the KK masses $m_n(\theta)$:

$$N(z; q\theta) = 0, \qquad \text{for } z = m_n(\theta)\,.$$

- The one–loop effective potential for the axion is obtained via the Casimir energy:

$$V(\theta) \;=\; \frac{(-1)^F}{2}\int\frac{d^4k_E}{(2\pi)^4}\sum_n \ln\left[k_E^2+m_n^2(\theta)\right] \;=\; \frac{(-1)^{F+1}}{8\pi^2}\int_0^\infty dz\, z^3 \ln N(iz; q\theta)\,.$$

$F = 0$ (boson), $\quad F = 1$ (fermion)

Axion Potential without Fixed-Point Localized Potentials

- **Charged scalar action in flat geometry**

$$S = -\int d^5x \left[\eta^{MN}(D_M\Phi)^*(D_N\Phi) + M^2\Phi^*\Phi\right], \quad \Phi : \phi, \tilde{\phi},$$

where $D_M = \partial_M - i\, q_\Phi(y)\, C_M$. The orbifold condition is

$$\phi(x,-y) = \phi^c(x,y), \qquad \tilde{\phi}(x,-y) = \tilde{\phi}(x,y).$$

Field	**KK spectrum**: $m_n(\theta)$	**Axion potential**: $V(\theta)$
ϕ	$\sqrt{M^2 + \frac{1}{R^2}\left(n + \frac{q_\phi\theta}{2\pi}\right)^2}$	$-\frac{(MR)^2}{4\pi^4 R^4}\, e^{-2M\pi R}\left[1 - \cos\left(q_\phi\theta\right)\right] + \mathcal{O}(e^{-4M\pi R})$
$\tilde{\phi}$	$\sqrt{M^2 + \frac{n^2}{R^2}}$	0

Worldline approach

- Worldline formalism expresses the one-loop effective action as a path integral over particle trajectories (**worldlines**).
- One–loop effective action as a worldline path integral (Schwinger proper time):

$$\Gamma \;=\; \int d^5x \int_0^\infty \frac{dT}{T}\,\langle x|e^{-T\mathcal{O}_\Phi}|x\rangle \;=\; \int d^5x \int_0^\infty \frac{dT}{T}\int_{x(0)=x(T)} \mathcal{D}x(\tau)\; e^{-S_E[x,\dot{x}]}.$$

where KG operator (flat)

$$\mathcal{O}_\Phi = -\eta^{MN} D_M D_N + M^2,$$

and Euclidean worldline action

$$S_E = \int_0^1 d\tau \left(\frac{\eta_{MN}\dot{x}^M\dot{x}^N}{4T} + TM^2\right) - i\int_0^1 d\tau\, \dot{x}^N q_\Phi(y)\, C_N(x).$$

Worldline approach

- The real part of the action has a minimum:

$$\mathrm{Re}\, S_E \;\geq\; M \int_{x(0)}^{x(1)} \sqrt{\eta_{MN}\, dx^M dx^N}$$

$\rightarrow$ corresponds to the relativistic particle action with mass M.

- The imaginary part is given by

$$\mathrm{Im}\, S_E \;=\; -\int_{x(0)}^{x(1)} dx^N\, q_\Phi(y)\, C_N(x)$$

$\rightarrow$ describes a charged particle interacting with a gauge field.

Worldline approach

- Closed worldlines can wind along the extra dimension while 4D coordinates are fixed:

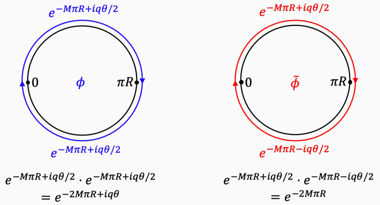
$$y(1) = y(0) + 2\pi n R, \qquad x^\mu(1) = x^\mu(0).$$

- Dominant (anti-)instanton correspond to $n = \pm 1$,

$$S_E \;=\; 2M\pi R \;\pm\; i\theta \oint \frac{dy}{2\pi R}\, q_\Phi(y).$$

- Depending on the profile of $q_\Phi(y)$, the worldline (anti-)instanton contribution generates axion potential.

Field	Im S_E	$V(\theta)$
ϕ	$\pm q_\phi \theta$	$e^{-2M\pi R} \cos(q_\phi \theta)$
$\tilde{\phi}$	0	0



(Worldline diagrams for ϕ and $\tilde{\phi}$.)

Axion Potential with Fixed-Point Localized Linear Potentials

- **Charged scalar action (flat geometry, brane linear terms)**

$$S = -\int d^5x \left[\eta^{MN}(D_M\Phi)^*(D_N\Phi) + M^2\Phi^*\Phi + \delta(y)\,V_0(\Phi) + \delta(y-\pi R)\,V_\pi(\Phi)\right],$$

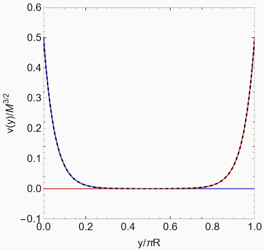
where $V_{0,\pi}(\Phi) = b_{L,\,0,\pi}\,\Phi + h.c.$

- **VEV in a constant axion background**

$$\langle\Phi(x,y)\rangle \equiv \Phi(y) = \exp\left(i\theta\int_0^y \frac{du}{2\pi R}\,q_\Phi(u)\right) v_\Phi(y;\theta).$$

$v_\Phi(y;\theta)$ depends linearly on localized sources at the fixed points,

$$v_\Phi(y;\theta=0) \simeq b^*_{L\,0}\cosh\left[M(\pi R - y)\right] + b^*_{L\,\pi}\cosh\left(My\right).$$

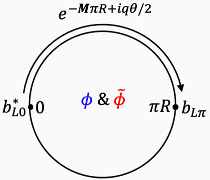


(An example of $v_\Phi(y;\theta=0)$ induced by the linear potential.)

Axion Potential with Fixed-Point Localized Linear Potentials

- Using the spectral-function, the axion potential induced by the localized linear terms is $(M\pi R \gg 1)$

$$V(\theta) = \frac{b^*_{L0}\, b_{L\pi}}{M}\, e^{-M\pi R + i\, q_\phi \theta/2} + h.c.$$



(Worldline diagrams from fixed -point localized linear terms)

PQ-breaking effects arise from bulk-field propagation between the orbifold fixed points.

Axion Potential with Fixed-Point Localized Quadratic Potentials

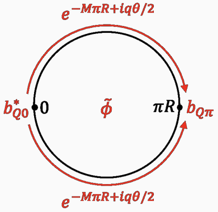
- **Charged scalar action (flat geometry, brane quadratic terms)**

$$S = -\int d^5x \left[\eta^{MN}(D_M\Phi)^*(D_N\Phi) + M^2\Phi^*\Phi + \delta(y)\,V_0(\Phi) + \delta(y-\pi R)\,V_\pi(\Phi)\right],$$

where $V_{0,\pi}(\Phi) = \frac{1}{2}\big(b_{Q\,0,\pi}\,\Phi^2 + h.c.\big)$.

- Using the spectral function, the axion potential induced by the localized quadratic terms is $(M\pi R \gg 1)$

$$V(\theta) = -\frac{b^*_{Q0}\,b_{Q\pi}}{M^2}\frac{(MR)^2}{8\pi^4 R^4}\,e^{-2M\pi R + i\,q_\phi\theta} + \text{h.c.}$$



$e^{-M\pi R+iq\theta/2}\cdot e^{-M\pi R+iq\theta/2} = e^{-2M\pi R+iq\theta}$

(Worldline diagrams from fixed -point localized quadratic terms)

Axion Potential (Warped case)

- The charged scalar action in the warped geometry is

$$S = -\int d^5x\sqrt{-g}\left[g^{MN}(D_M\Phi)^*(D_N\Phi) + M^2\Phi^*\Phi\right],$$

where

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2.$$

- Using the spectral function, the axion potential is ($M\pi R \gg 1$)

$$V(\theta) \simeq \left(\frac{M_{\text{eff}}^2 k^2}{\pi^2 e^{4k\pi R}}\right) e^{-2M_{\text{eff}}\pi R}\left[1-\cos(q_\phi\theta)\right], \quad M_{\text{eff}} = \sqrt{M^2+4k^2}.$$

Warped geometry amplifies the exponential suppression factor and also enlarges the worldline instanton action.

Axion Potential (Warped case)

- Since the axion is localized near the $y = \pi R$ brane, the axion decay constant f_a is red-shifted

$$f_a^2 = \frac{1}{g_C^2}\left(\frac{k}{e^{2k\pi R}-1}\right), \qquad M_P^2 = \frac{M_5^3}{k}(1-e^{-2k\pi R}).$$

- In terms of f_a, the parametric dependence of the axion potential is

$$V(\theta) \simeq 2\left(\frac{g_C^2 M_{\text{eff}}}{2\pi}\right)^2 f_a^4 e^{-2M_{\text{eff}}\pi R}\big[1-\cos(q_\phi\theta)\big].$$

With $k = 10^{17}\,\mathrm{GeV}$ and $M_5 \sim 1/g_C^2 = 5\times 10^{17}\,\mathrm{GeV}$, $kR = 5$ implies $f_a \simeq 3\times 10^{10}\,\mathrm{GeV}$. The PQ-quality condition is satisfied for $MR \geq 18$.

Summary

- Axion emerges as a 5D gauge field component.
  - PQ symmetry locally protected by gauge invariance.
  - Breaking only via nonlocal effects, strongly suppressed.
- Z_2-odd matter does not generate axion potential (orbifold symmetry).
- Worldline formalism results:
  - Bulk loops: $\propto e^{-2M_{\text{eff}}\pi R}$
  - Brane-localized potential: $\propto e^{-M_{\text{eff}}\pi R}$ (requires two branes)
- Warping improves PQ quality:
  - Red-shifted mass scale
  - Enlarged instanton action

Backup

Kaluza–Klein theory

- The action of a massive complex scalar field Φ on $M \times S^1$ is

$$S \;=\; -\int d^5x \left[\, \partial_M \Phi^*(x,y)\, \partial^M \Phi(x,y) + m^2\, \Phi^*(x,y)\Phi(x,y) \,\right].$$

Since the extra dimension is compactified, $y \sim y + 2\pi R$, the field value is periodic, $\Phi(x, y + 2\pi R) = \Phi(x,y)$. Hence one may perform the Fourier–decomposition along y:

$$\Phi(x,y) = \frac{1}{\sqrt{2\pi R}} \sum_{n\in\mathbb{Z}} \Phi^{(n)}(x)\, e^{iny/R}.$$

Plugging this series into the action gives the 4D KK-reduced form

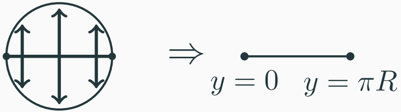
$$S = -\int d^4x\, \partial_\mu \Phi^{*(0)} \partial^\mu \Phi^{(0)} + m^2 \Phi^{*(0)} \Phi^{(0)} + \sum_{n\neq 0} \Big(\partial_\mu \Phi^{*(n)} \partial^\mu \Phi^{(n)} + \big(m^2 + \tfrac{n^2}{R^2}\big)\, \Phi^{*(n)} \Phi^{(n)} \Big).$$

From the 4D viewpoint the theory describes a KK tower with masses

$$m_n^2 = m^2 + \frac{n^2}{R^2}\,.$$

Kaluza–Klein theory on orbifold

- An **orbifold** is obtained by modding out a manifold by a discrete symmetry that leaves some points fixed. For example, the $S^1/\mathbb{Z}_2$ orbifold is obtained from a circle S^1 and identifying $y \sim -y$. The quotient gives an interval with fixed points $y = 0$ and $y = \pi R$.



- Fields on $[-\pi R, \pi R]$ decompose into **even** and **odd** under $\mathbb{Z}_2$:

$$\text{even:} \quad \Phi_+(-y) = \Phi_+(y), \qquad \text{odd:} \quad \Phi_-(-y) = -\Phi_-(y).$$

The $\mathbb{Z}_2$ parity controls whether a zero mode exists

$$\Phi_+(x,y) \propto \Phi_+^{(0)}(x) + \sum_{n=1}^{\infty} \Phi_+^{(n)}(x) \cos\left(\frac{n}{R} y\right), \quad \Phi_-(x,y) \propto \sum_{n=1}^{\infty} \Phi_-^{(n)}(x) \sin\left(\frac{n}{R} y\right).$$

Axion Potential without Fixed-Point Localized Potentials

- **Charged scalar action in flat geometry**

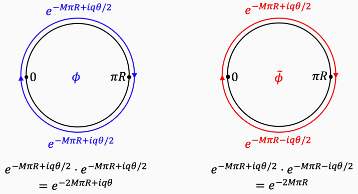
$$S \;=\; -\int d^5x \left[\eta^{MN}(D_M\Phi)^*(D_N\Phi) + M^2\Phi^*\Phi\right], \quad \Phi : \phi, \tilde{\phi},$$

where $D_M = \partial_M - i\, q_\Phi(y)\, C_M$.

- **KK mass spectra**

$$\phi : \quad m_n(\theta) = \sqrt{M^2 + \frac{1}{R^2}\left(n + \frac{q_\phi\,\theta}{2\pi}\right)^2},$$

$$\tilde{\phi} : \quad m_n = \sqrt{M^2 + \frac{n^2}{R^2}}.$$ (no θ-dependence)



- **Axion potential via spectral-function ($M\pi R \gg 1$)**

$$V(\theta) \;=\; -\frac{(MR)^2}{4\pi^4 R^4}\, e^{-2M\pi R}\left[1 - \cos\!\left(q_\phi\theta\right)\right] \;+\; \mathcal{O}\!\left(e^{-4M\pi R}\right).$$

(2) Worldline approach

- Worldline formalism evaluates the one-loop effective action as a quantum mechanical path integral over the particle's trajectories (**worldlines**).
- Using Schwinger's proper time method, the one-loop effective action is

$$\Gamma = \int d^5x \int_0^\infty \frac{dT}{T} \langle x| e^{-T\mathcal{O}_\Phi} |x\rangle = \int d^5x \int_0^\infty \frac{dT}{T} \int_{x(0)=x(T)} \mathcal{D}x(\tau)\, e^{-S_E[x,\dot{x}]},$$

where the Euclidean worldline action is

$$S_E = \int_0^1 d\tau \left(\frac{1}{4T} \eta_{MN} \dot{x}^M \dot{x}^N + TM^2 \right) - i \int_0^1 d\tau\, \dot{x}^N q_\Phi(y(\tau)) C_N(x(\tau)),$$

and the Klein-Gordon operator in flat geometry

$$\mathcal{O}_\Phi = -\eta^{MN} D_M D_N + M^2.$$

(2) Worldline Approach

- The real part of the worldline action has a minimum with respect to T:

$$\frac{1}{4T}\,\eta_{MN}\,\dot{x}^M\dot{x}^N + T\,M^2 \;\geq\; M\,\sqrt{\eta_{MN}\,\dot{x}^M\dot{x}^N}\,,$$

  which implies

$$\mathrm{Re}\,S_E \;\geq\; M\int_{x(0)}^{x(1)}\sqrt{\eta_{MN}\,dx^M dx^N}\,.$$

  Therefore, the minimum corresponds to the relativistic particle action with mass M. The imaginary part evaluates to

$$\mathrm{Im}\,S_E \;=\; -\int_{x(0)}^{x(1)} dx^N\,q_\Phi(y)\,C_N(x)\,.$$

(2) Worldline Approach

- The path satisfying $x(0) = x(1)$ may wind non-trivially along the y-direction, while the 4D spacetime coordinates remain fixed,

$$y(1) = y(0) + 2n\pi R, \quad x^\mu(1) = x^\mu(0).$$

The dominant contribution to the axion potential arises from the winding number $n = \pm 1$. The worldline instanton action becomes

$$S_E = 2M\pi R \pm i\theta \oint \frac{dy}{2\pi R} q_\Phi(y).$$

- Depending on the profile of $q_\Phi(y)$, we find

$$\oint \frac{dy}{2\pi R} q_\Phi(y) = q_\phi, \quad \oint \frac{dy}{2\pi R} q_\Phi(y) = \tilde{q}_\phi \oint \frac{dy}{2\pi R} \epsilon(y) = 0.$$

Thus, the worldline (anti-)instanton contribution generates axion potential for ϕ, while it vanishes for $\tilde{\phi}$,

$$V(\theta) \sim e^{-2M\pi R \pm i q_\phi \theta}.$$

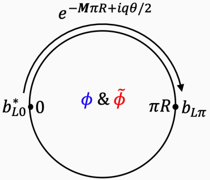
Axion Potential with Fixed-Point Localized Linear Potentials

- Using the spectral-function, the axion potential induced by the localized linear terms is $(M\pi R \gg 1)$

$$V(\theta) \; = \; \frac{b_{L0}^{*}\, b_{L\pi}}{M}\, e^{-M\pi R + i\, q_{\phi}\theta/2} \; + \; h.c.$$

- A perturbative expansion $(b_{0,\pi} \ll M)$ of the effective potential yields:

$$V(\theta) = \frac{1}{2}\int d^4x' \;\left[b_{L\,0}^{*} b_{L\,\pi}\langle \Phi(x,\pi R)\Phi^{*}(x',0)\rangle + h.c.\right]$$



(Worldline diagrams from fixed -point localized linear terms)

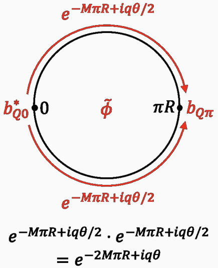
PQ-breaking effects arise from bulk-field propagation between the orbifold fixed points.

Axion Potential with Fixed-Point Localized Quadratic Potentials

- **Charged scalar action (flat geometry, brane quadratic terms)**

$$S = -\int d^5x \left[\eta^{MN}(D_M\Phi)^*(D_N\Phi) + M^2\Phi^*\Phi + \delta(y)\,V_0(\Phi) + \delta(y-\pi R)\,V_\pi(\Phi)\right],$$

where $V_{0,\pi}(\Phi) = \frac{1}{2}\big(b_{Q\,0,\pi}\,\Phi^2 + h.c.\big)$.



- Using the spectral-function,

$$V(\theta) = -\,\frac{b^*_{Q0}\,b_{Q\pi}}{M^2}\,\frac{(MR)^2}{8\pi^4R^4}\,e^{-2M\pi R+i\,q_\phi\theta} \;+\; \text{h.c.}$$

- A perturbative expansion ($b_{0,\pi} \ll M$) of the effective potential yields:

$$V(\theta) = \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\left[b^*_{Q\,0}b_{Q\,\pi}G_p(\pi R,0)G_p(\pi R,0) + h.c.\right].$$

Axion Potential — Fermion case

- We consider two types of fermions, ψ and $\tilde{\psi}$, with the following *orbifold* parities:

$$\psi(x,-y) = \psi^c(x,y) = -\, i\gamma^2\, \psi^*(x,y), \qquad \tilde{\psi}(x,-y) = \pm\gamma^5\, \tilde{\psi}(x,y).$$

- These lead to different charge/mass profiles in the 5D action:

$$S \;=\; -\int d^5x\, \sqrt{|g|}\; \bar{\Psi}\big(\Gamma^M D_M + M_\psi\big)\Psi, \qquad \Psi = \psi, \tilde{\psi}.$$

- Analogous to the scalar case, $\tilde{\psi}$ has an axion–independent KK spectrum (symmetry protected). Using the spectral function, the axion potential for ψ is

$$V(\theta) \;\simeq\; \left(\frac{16\, M_L^2 M_R^3 k^2}{\pi^2\, e^{4k\pi R}\, (M_L + M_R)^4}\right) e^{-(M_L+M_R)\pi R}\, \cos\!\big(q_\psi\, \theta\big),$$

where

$$M_L = \left| M + \frac{k}{2} \right|, \qquad M_R = \left| M - \frac{k}{2} \right|.$$

Axion Potential — Fermion case

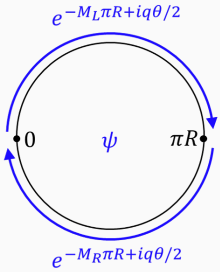
- The exponential suppression factor $e^{-(M_L+M_R)\pi R}$ can be understood in the **worldline** picture.
- Klein–Gordon operator for a 5D fermion (after squaring the Dirac operator):

$$\mathcal{O}_\psi = -\left(\partial_y - iq_\psi C_5\right)^2 + M_{\rm eff}^2(y) + e^{2k|y|}\,p^2.$$

- The bulk (position–dependent) mass takes

$$M_{\rm eff}(y) = \left| M + \frac{k}{2}\,\epsilon(y) \right|,$$

so that the effective mass felt by a winding worldline differs between $0 < y < \pi R$ and $-\pi R < y < 0$.



$e^{-M_L\pi R+iq\theta/2}\cdot e^{-M_R\pi R+iq\theta/2}$
$= e^{-(M_L+M_R)\pi R+iq\theta}$

Fermion action and parity in 5D

- For a 5D fermion doublet
$$\Psi = (\chi, \bar\psi)^T,$$
the kinetic action is
$$S = \int d^5x\, \bar\Psi\, i\Gamma^M \partial_M \Psi = \int d^4x\, dy \Big(i\,\bar\chi\,\bar\sigma^\mu \partial_\mu \chi + i\,\psi\,\sigma^\mu \partial_\mu \bar\psi + \psi\,\partial_5 \chi - \bar\chi\,\partial_5 \bar\psi \Big).$$
- Since ∂_5 is odd under $y \to -y$, ψ and χ must carry *opposite* orbifold parities. Hence only one has a zero mode $\Rightarrow$ the zero mode is chiral.
- Gauge interaction (from $\bar\Psi\Gamma^M A_M \Psi$):
$$\bar\Psi\Gamma^M A_M \Psi \supset \bar\chi\,\bar\sigma^\mu A_\mu \chi + \psi\,\sigma^\mu A_\mu \bar\psi + \psi\, A_5 \chi - \bar\chi\, A_5 \bar\psi.$$
- A bulk mass term is allowed if the parities of χ and ψ are opposite:
$$m\,\bar\Psi\Psi = m\,(\psi\chi + \bar\chi\,\bar\psi).$$
On an orbifold, $m = m(y)$ must be *odd* under $y \to -y$. Such a mass localizes the zero mode like a domain wall.