



Stable dark matter from Pauli blocking in the degenerate fermion background with Quantum Field Theory

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Junghoon Joh (SungKyunKwan Univ.)

With Wonsub Cho (SKKU), Ki-Young Choi (SKKU) and Osamu Seto (Hokkaido U.)

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Motivation & Idea

- Dark matter's lifetime should be longer than the age of the Universe.
- If dark matter has interactions, some conditions (e.g., Z_2 parity, $U(1)$ symmetry) or constraints (e.g., minimal coupling, small mass) are needed.
- Main idea: If dark matter couples only to fermions, Pauli blocking can make dark matter live longer without other conditions.
- We calculate the effect of Pauli blocking using:
 - Boltzmann equations
 - Quantum Field Theory (QFT)

Motivation & Idea

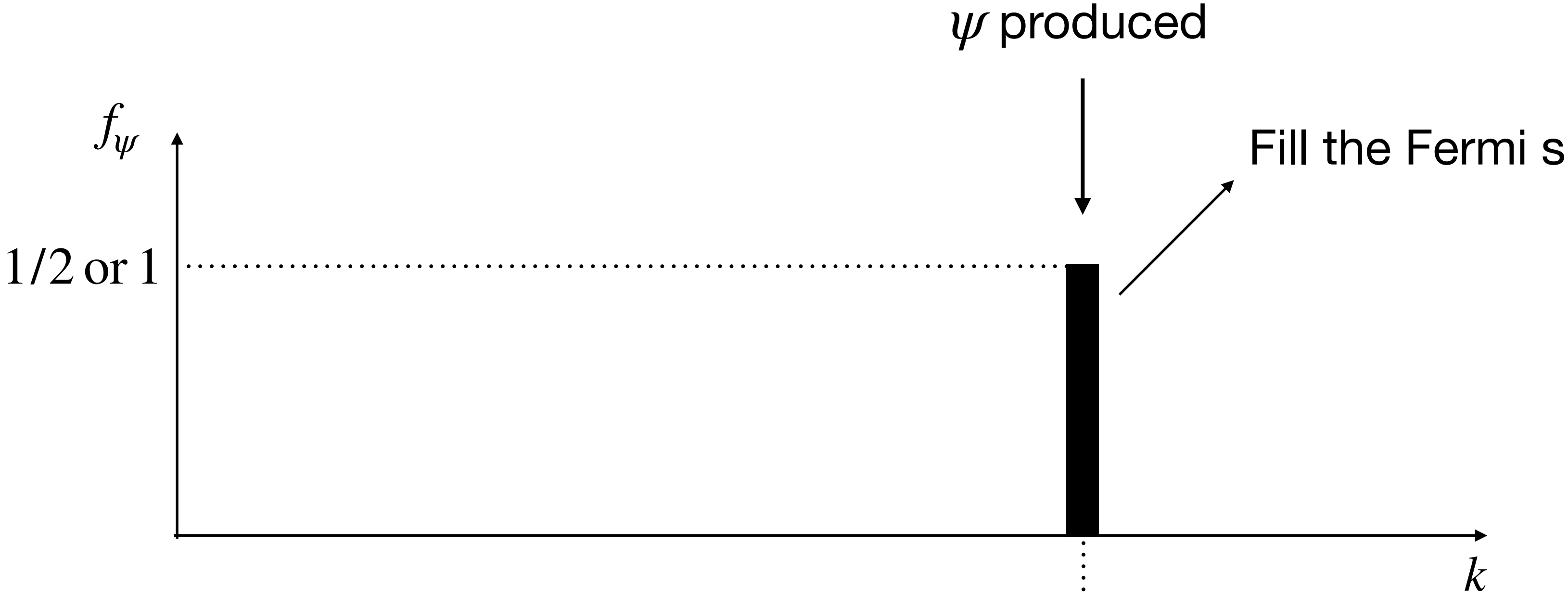
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- Main idea: If dark matter couples only to fermions, Pauli blocking can make dark matter live longer without other conditions.
➡ O.E.Bjaelde, S.Das, Phys.Rev.D 82 (2010), 043504
- We calculate the effect of Pauli blocking using:
 - Boltzmann equations ➡ B.Batell, W.Yin, Phys.Rev.D 110 (2024) 7, 075038
 - Quantum Field Theory (QFT) ➡ T.Moroi, W.Yin, JHEP 03 (2021), 296
- + Fermion production in preheating
➡ P.B.Greene, L.Kofman, Phys.Lett.B 448 (1999)
➡ P.B.Greene, L.Kofman, Phys.Rev.D 62 (2000)

The Toy Model: Yukawa Coupling

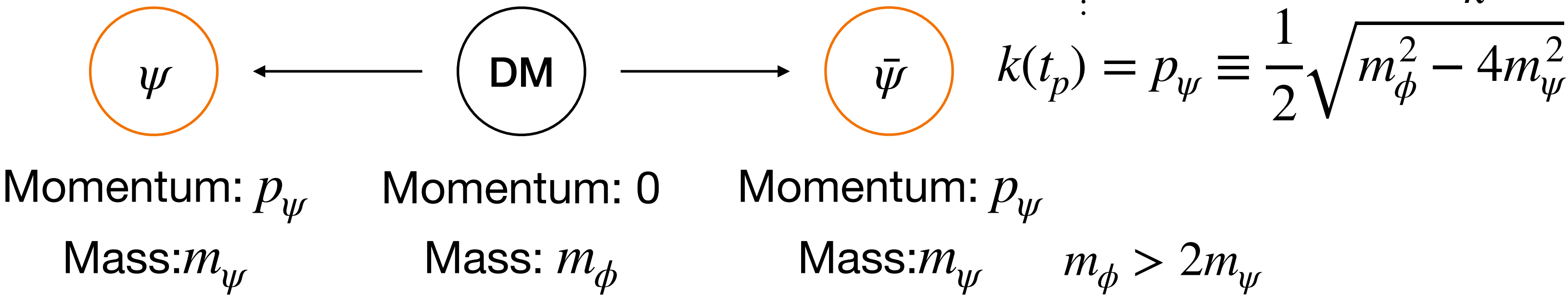
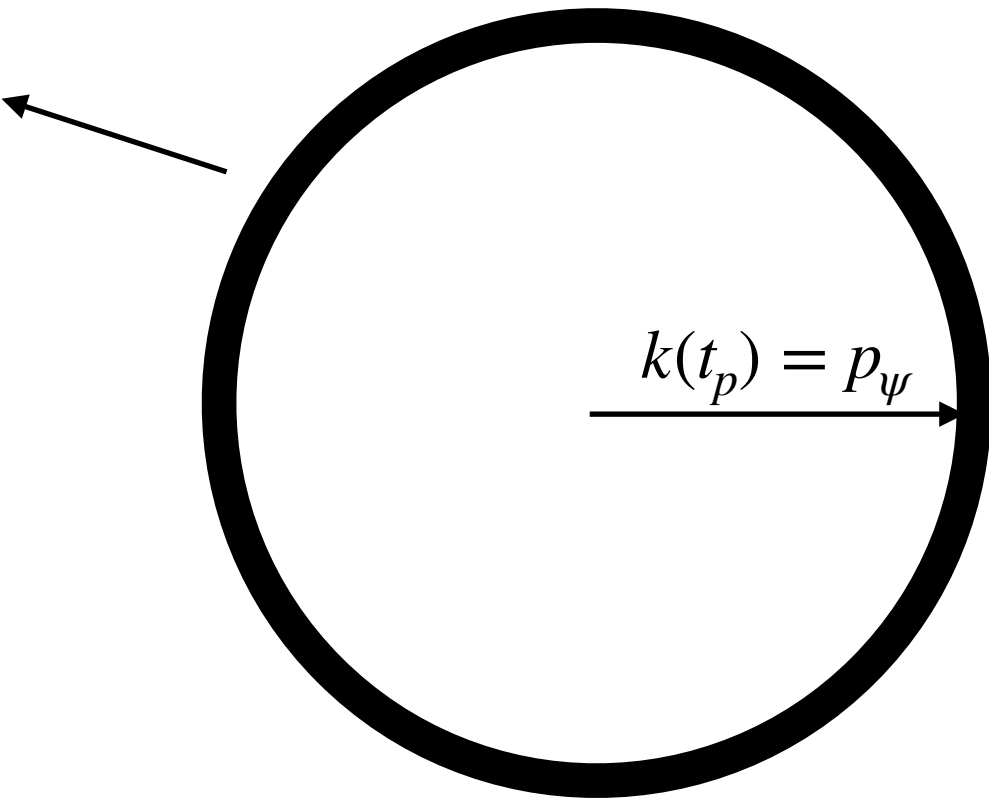
- $\mathcal{L} \ni \lambda \bar{\psi} \psi \phi$
- ϕ : scalar dark matter, ψ & $\bar{\psi}$: fermion & anti-fermion
- Assumption: Matter-dominated era
- Range of mass & coupling: $m_\phi \lesssim 0.01 \text{ eV}$, $\lambda \ll 2 \times 10^{-8} \left(\frac{m_\phi}{10^{-3} \text{ eV}} \right)^{3/4}$
- Lifetime w/o Pauli blocking:
$$\tau_{\phi,0} = \frac{1}{\Gamma_\phi} \simeq 1.6 \times 10^{13} \text{ sec} \left(\frac{10^{-12}}{\lambda} \right)^2 \left(\frac{10^{-3} \text{ eV}}{m_\phi} \right) \left(1 - \frac{4m_\psi^2}{m_\phi^2} \right)^{-3/2}$$
- Lifetime w/ Pauli blocking: $\tau_\phi \simeq t_i \left(\frac{\rho_\phi(t_i)}{m_\phi} \right)^{1/2} \left(\frac{p_\psi^3}{4\pi^2} \right)^{-1/2} \rightarrow \text{Result}$

Pauli-Blocking Idea

Value of distribution function

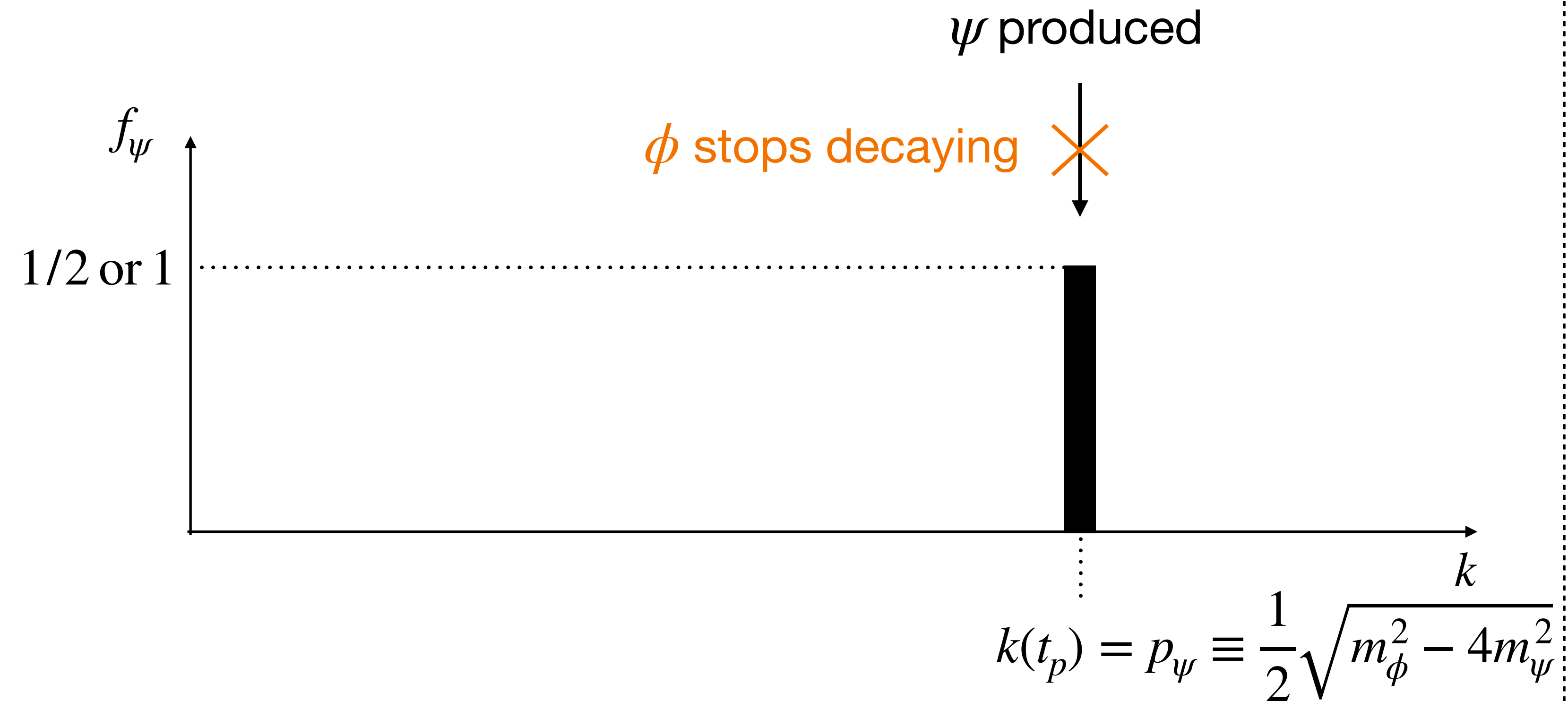


Fermi sphere
in momentum space

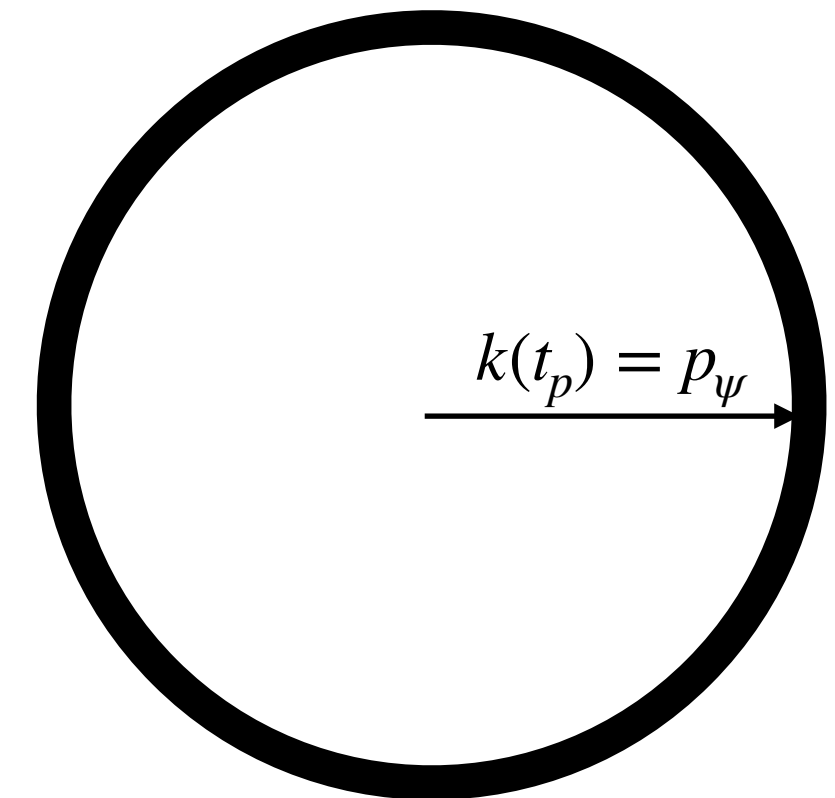


Pauli-Blocking Idea

Value of distribution function

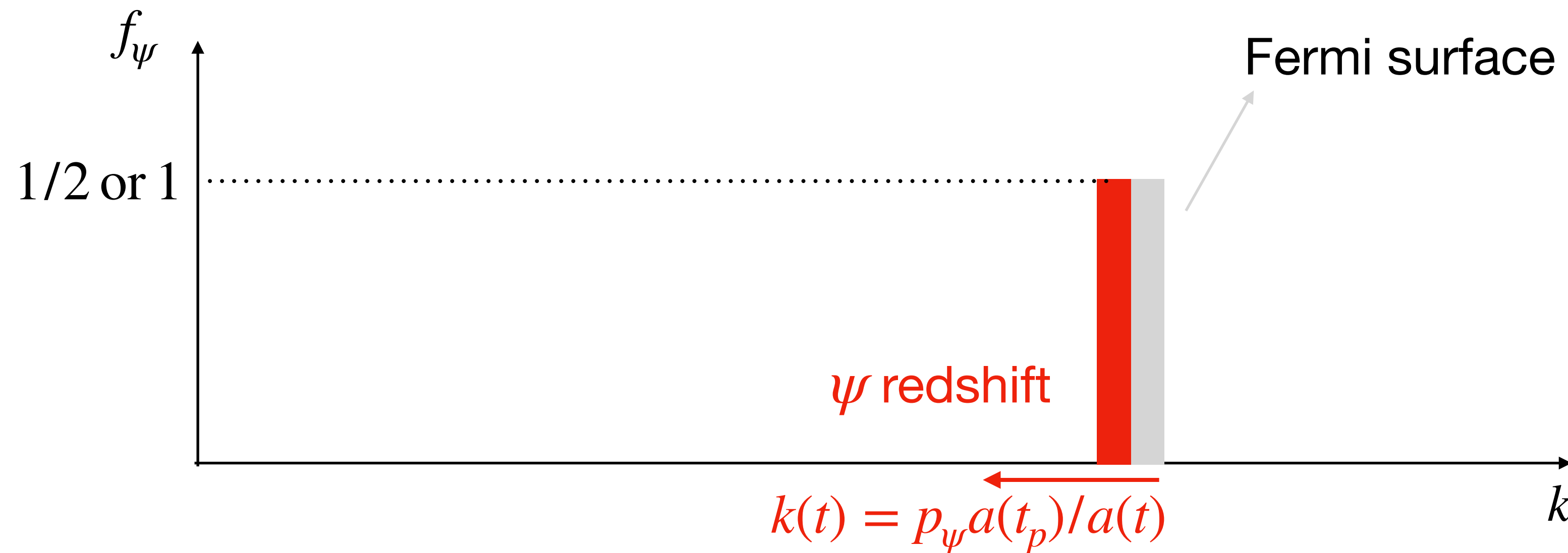


Fermi sphere
in momentum space



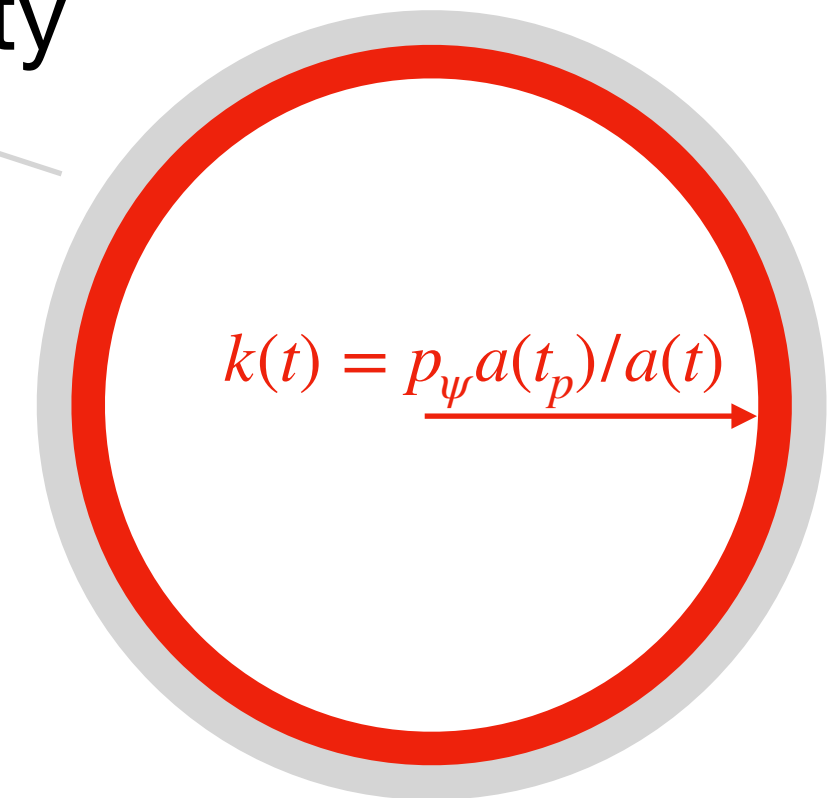
Pauli-Blocking Idea

Value of distribution function



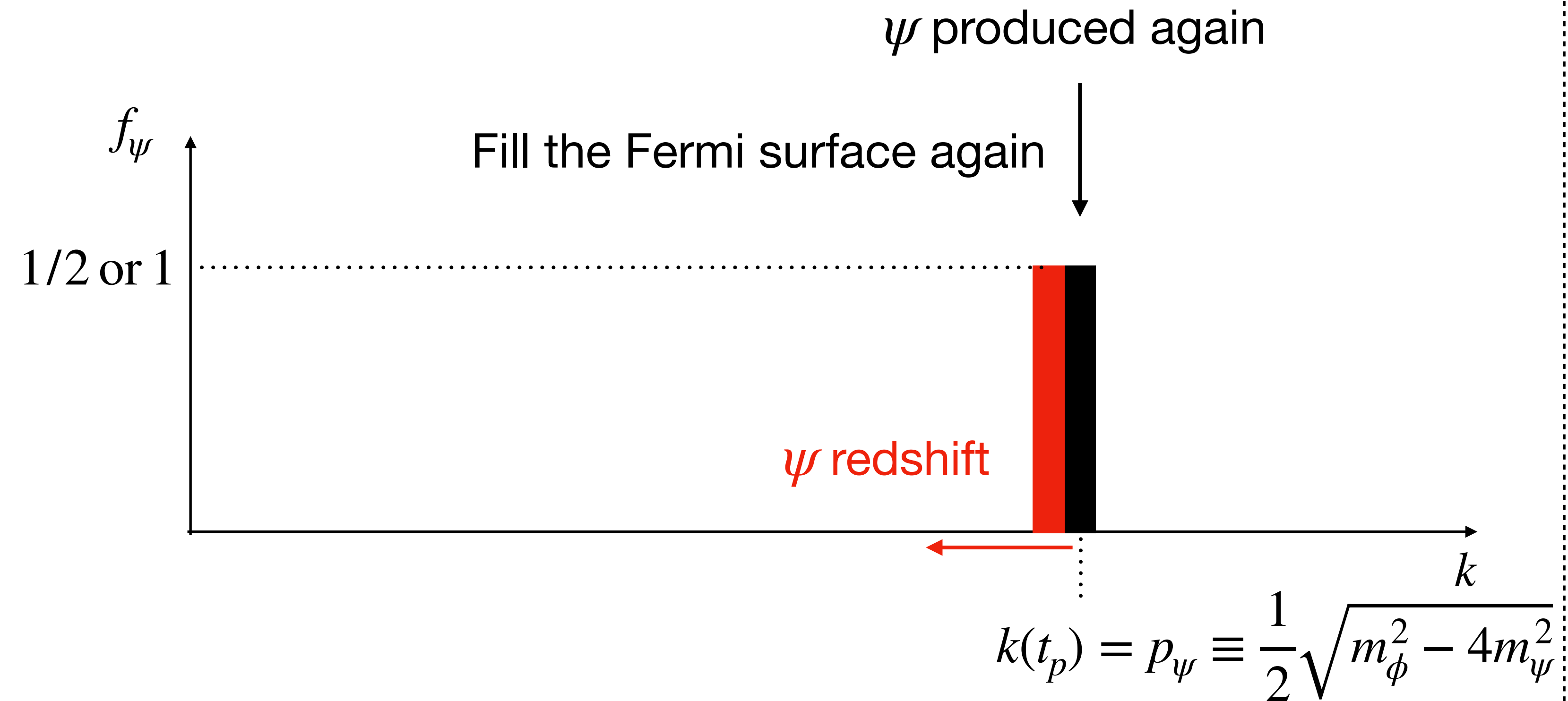
Fermi surface becomes empty

Fermi sphere
in momentum space

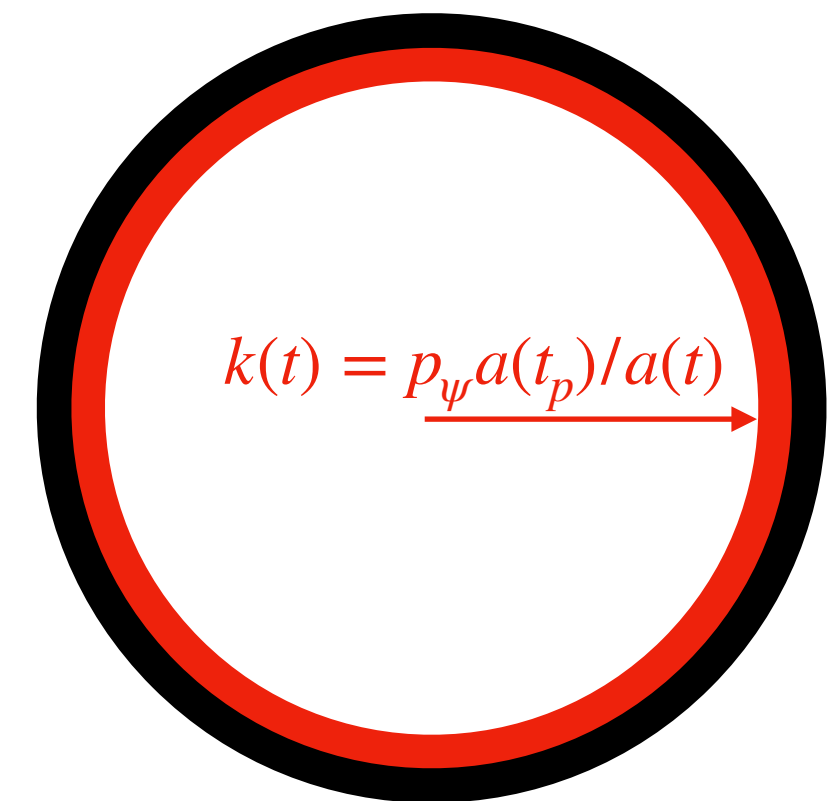


Pauli-Blocking Idea

Value of distribution function

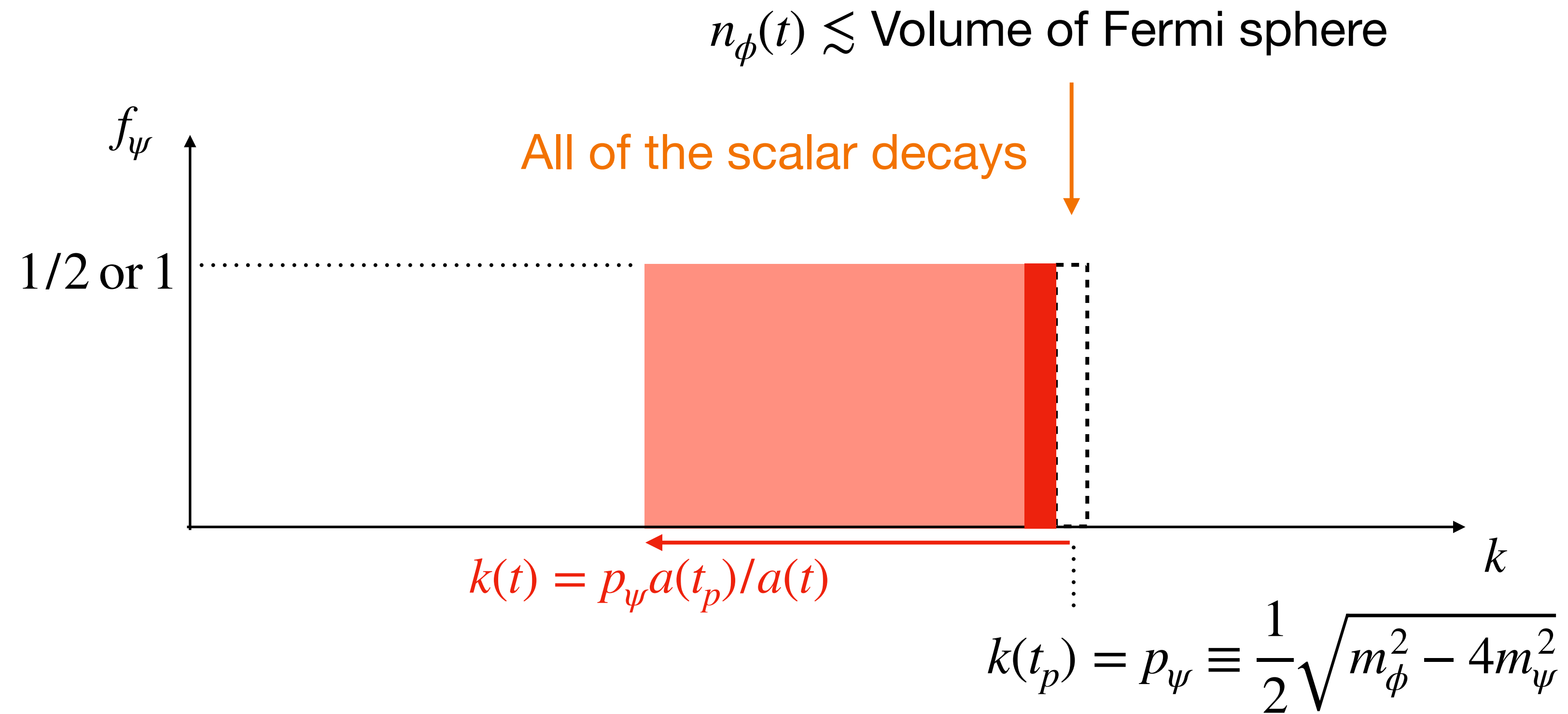


Fermi sphere
in momentum space



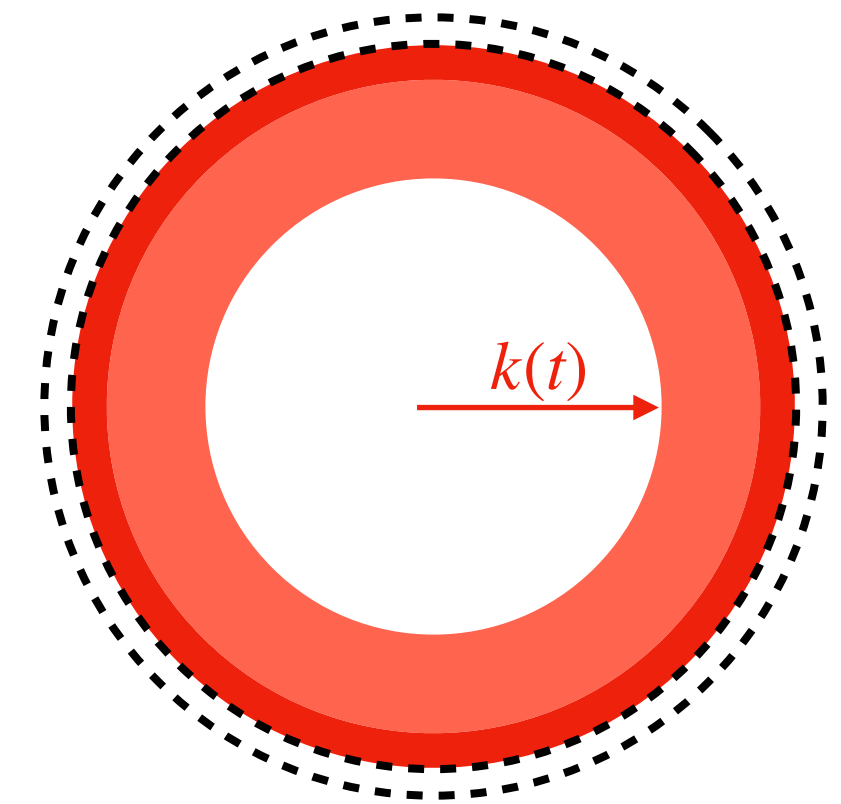
Pauli-Blocking Idea

Value of distribution function



Lifetime with Pauli blocking τ_ϕ

Fermi sphere
in momentum space



Boltzmann Equation

- Momentum distribution function:

- $$\frac{\partial f_\phi(\vec{p}, t)}{\partial t} - H\vec{k} \cdot \vec{\nabla}_{\vec{p}} f_\phi(\vec{p}, t) = f_\phi^{\text{coll}}(\vec{p}, t)$$

- $$\frac{\partial f_\psi(\vec{k}, t)}{\partial t} - H\vec{k} \cdot \vec{\nabla}_{\vec{k}} f_\psi(\vec{k}, t) = f_\psi^{\text{coll}}(\vec{k}, t)$$

- Number density:

- $$\dot{n}_\phi(t) + 3H(t)n_\phi(t) = -\Gamma_\phi n_\phi(t) \left(1 - 2f_\psi(p_\psi, t)\right)$$

- $$\dot{n}_\psi(t) + 3H(t)n_\psi(t) = \Gamma_\phi n_\phi(t) \left(1 - 2f_\psi(p_\psi, t)\right)$$

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- $$\frac{\partial f_\psi(\vec{k}, t)}{\partial t} - H\vec{k} \cdot \vec{\nabla}_{\vec{k}} f_\psi(\vec{k}, t) = f_\psi^{\text{coll}}(\vec{k}, t)$$

$$f_\phi^{\text{coll}}(\vec{p}, t) \propto \left[\underbrace{f_\psi(\vec{k}, t) f_{\bar{\psi}}(\vec{k}', t) (1 + f_\phi(\vec{p}, t))}_{\text{back-reaction term}} - \underbrace{f_\phi(\vec{p}, t) (1 - f_\psi(\vec{k}, t)) (1 - f_{\bar{\psi}}(\vec{k}', t))}_{\text{reaction term}} \right]$$

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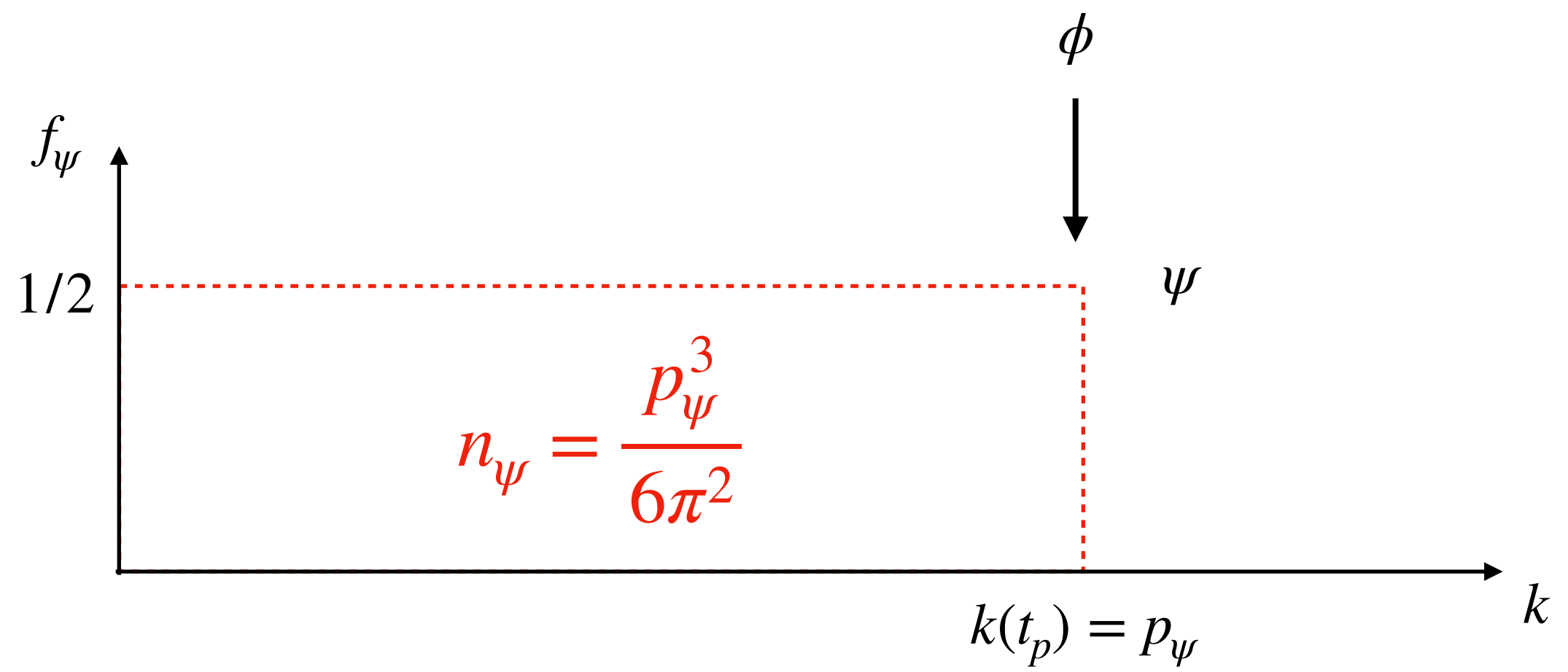
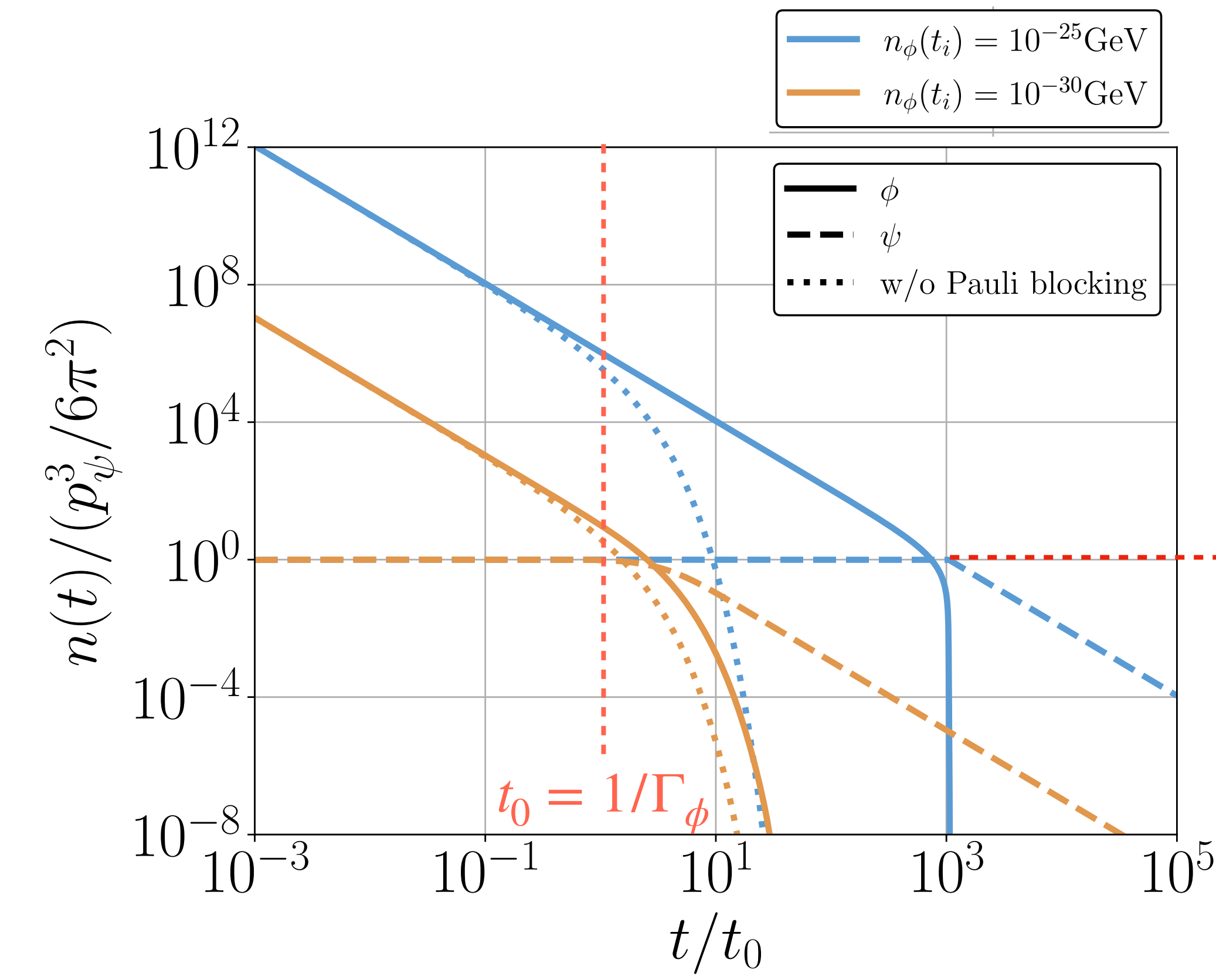
- Number density:

- $$\dot{n}_\phi(t) + 3H(t)n_\phi(t) = -\Gamma_\phi n_\phi(t) \left(1 - 2f_\psi(p_\psi, t) \right) \longrightarrow \text{Dark matter stops decaying when } f_\psi(p_\psi, t) = 1/2$$

- $$\dot{n}_\psi(t) + 3H(t)n_\psi(t) = \Gamma_\phi n_\phi(t) \left(1 - 2f_\psi(p_\psi, t) \right)$$

Result of Boltzmann Equation

W.Cho, K.Y.Choi, J.Joh, O.Seto. Phys.Dark Univ. 49 (2025)



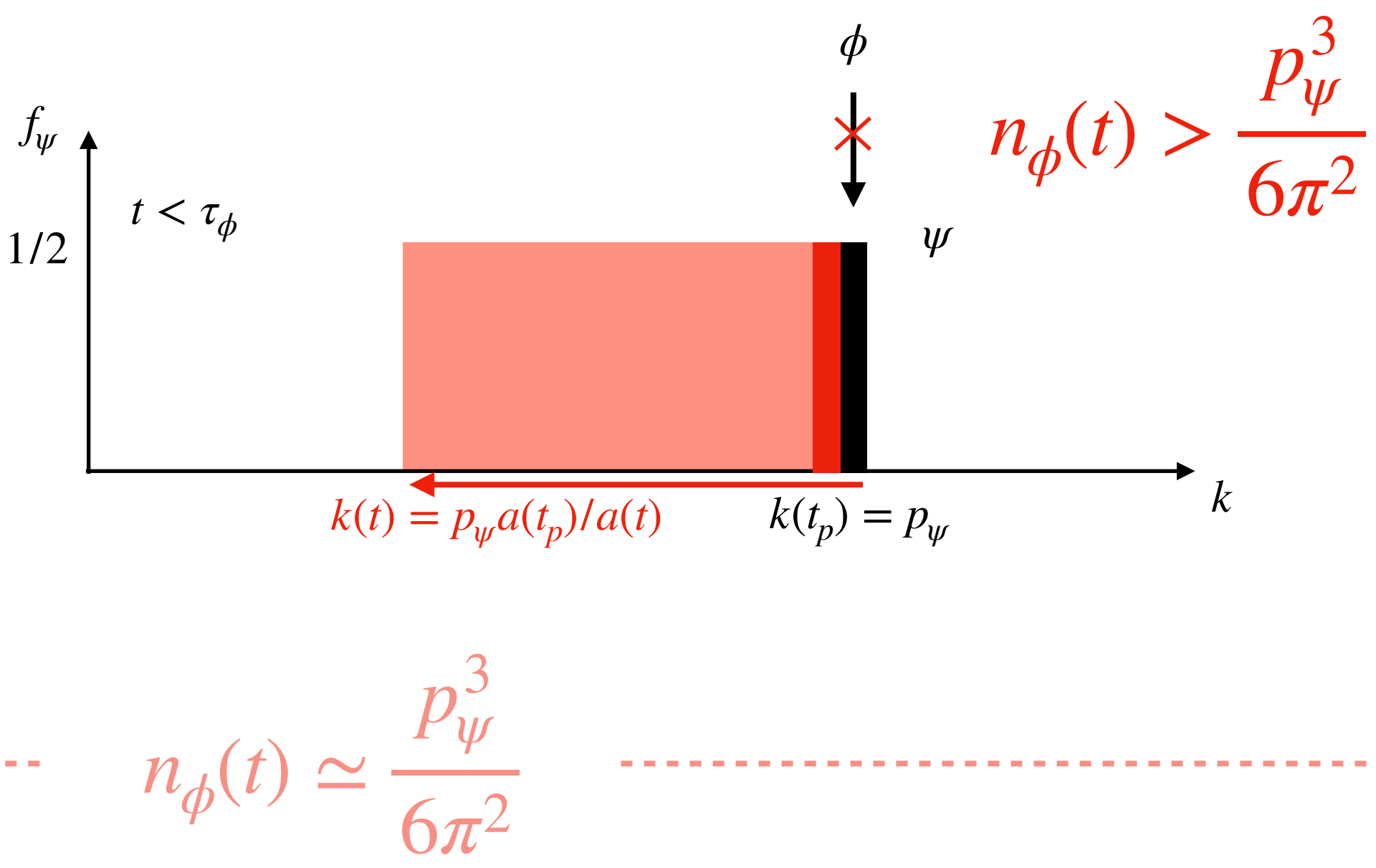
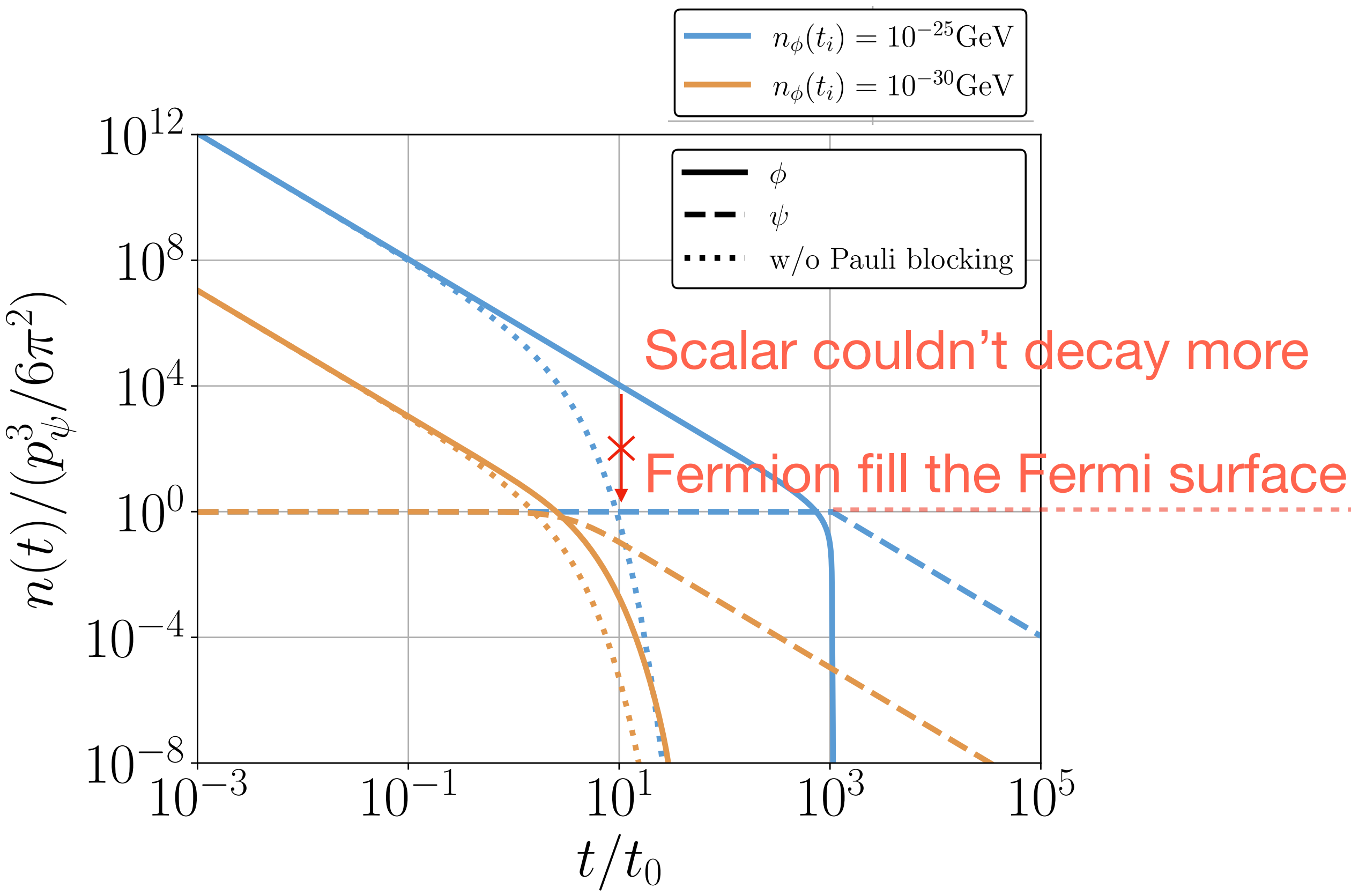
$$n_\phi(t) \simeq \frac{p_\psi^3}{6\pi^2}$$

Number density of scalar =
Volume of Fermi sphere in momentum space

$$\left(\dot{n}_\phi(t) + \dot{n}_\psi(t)\right) + 3H(t)\left(n_\phi(t) + n_\psi(t)\right) = 0$$

Result of Boltzmann Equation

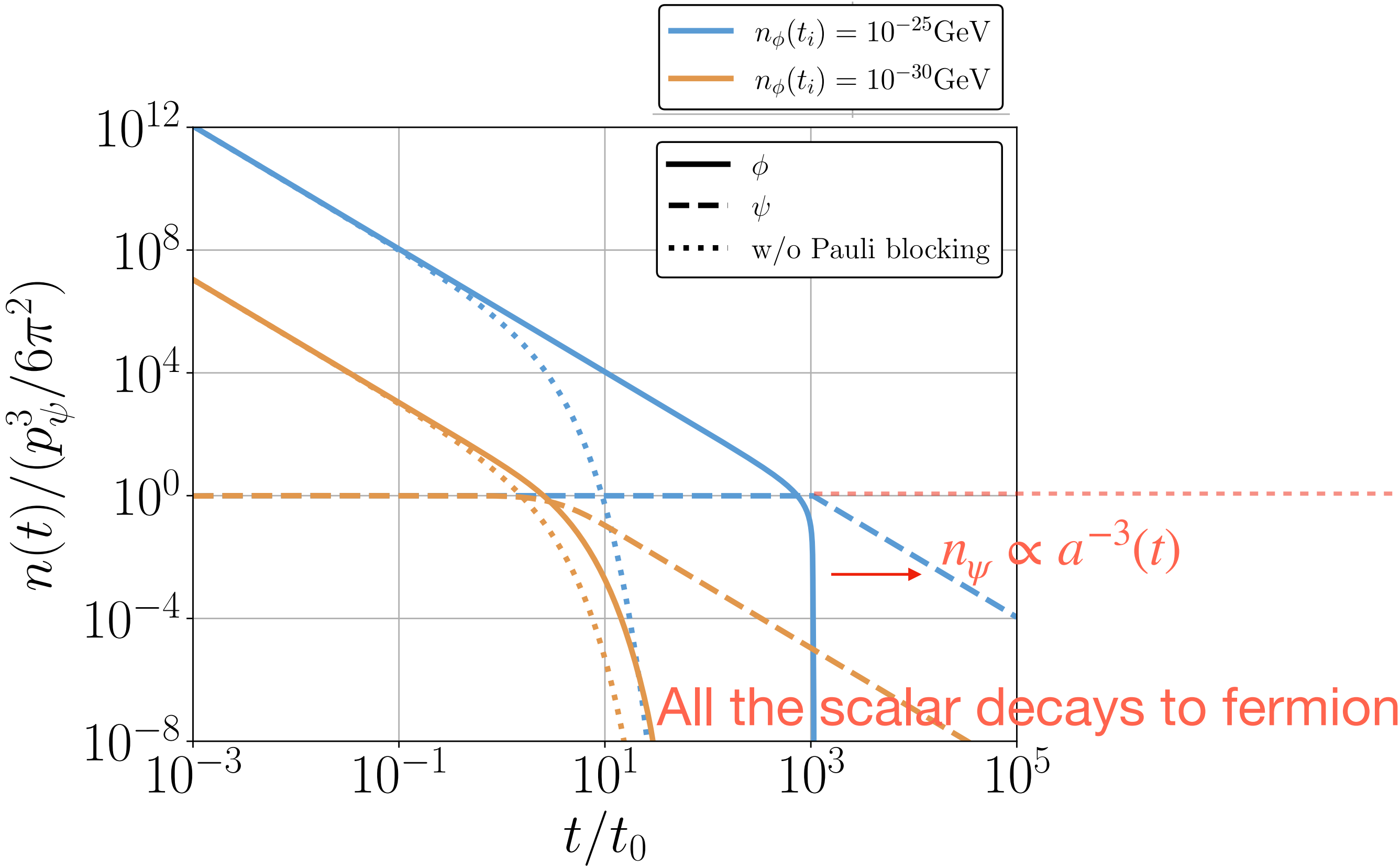
W.Cho, K.Y.Choi, J.Joh, O.Seto. Phys.Dark Univ. 49 (2025)



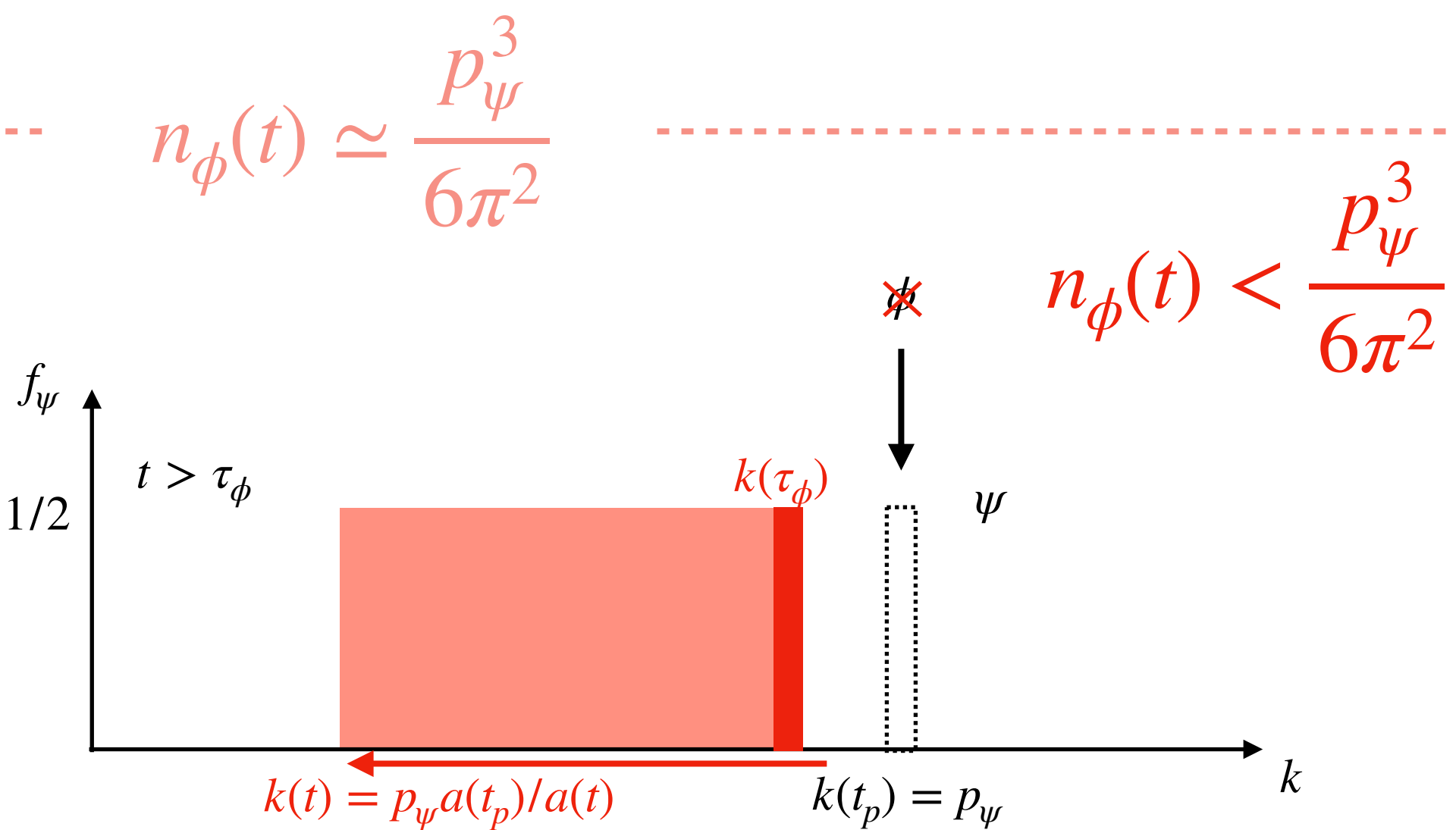
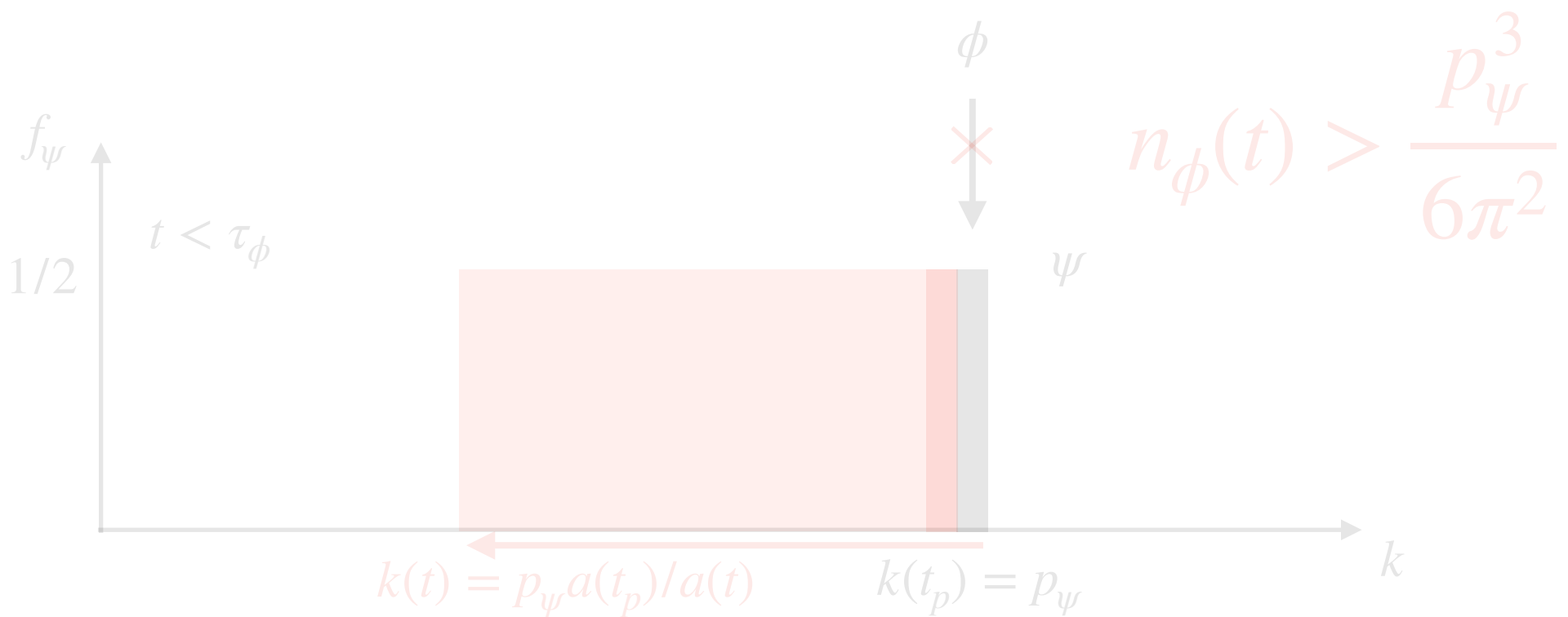
$$\left(\dot{n}_\phi(t) + \dot{n}_\psi(t)\right) + 3H(t)\left(n_\phi(t) + n_\psi(t)\right) = 0$$

Result of Boltzmann Equation

W.Cho, K.Y.Choi, J.Joh, O.Seto. Phys.Dark Univ. 49 (2025)

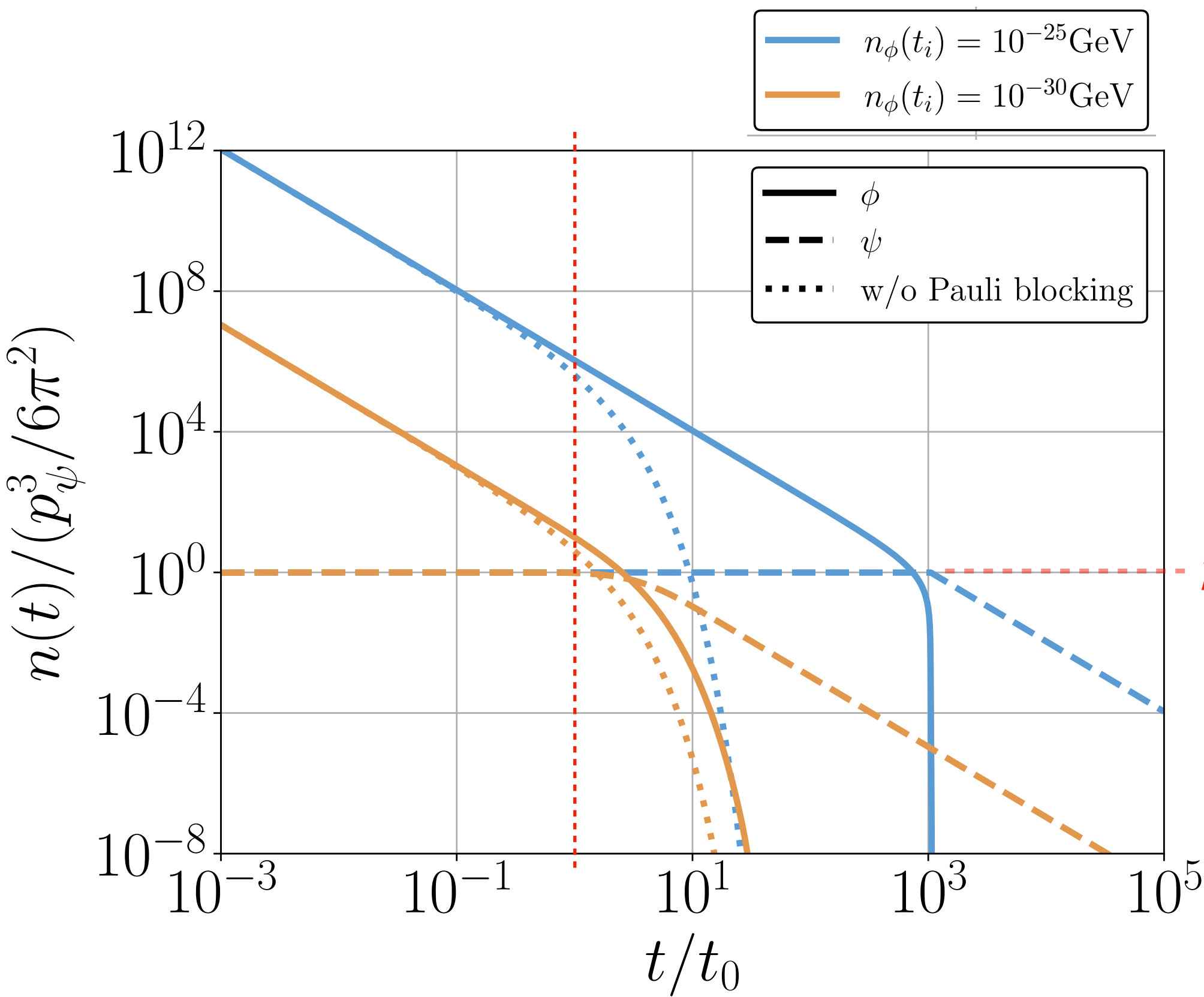


$$\left(\dot{n}_\phi(t) + \dot{n}_\psi(t)\right) + 3H(t)\left(n_\phi(t) + n_\psi(t)\right) = 0$$



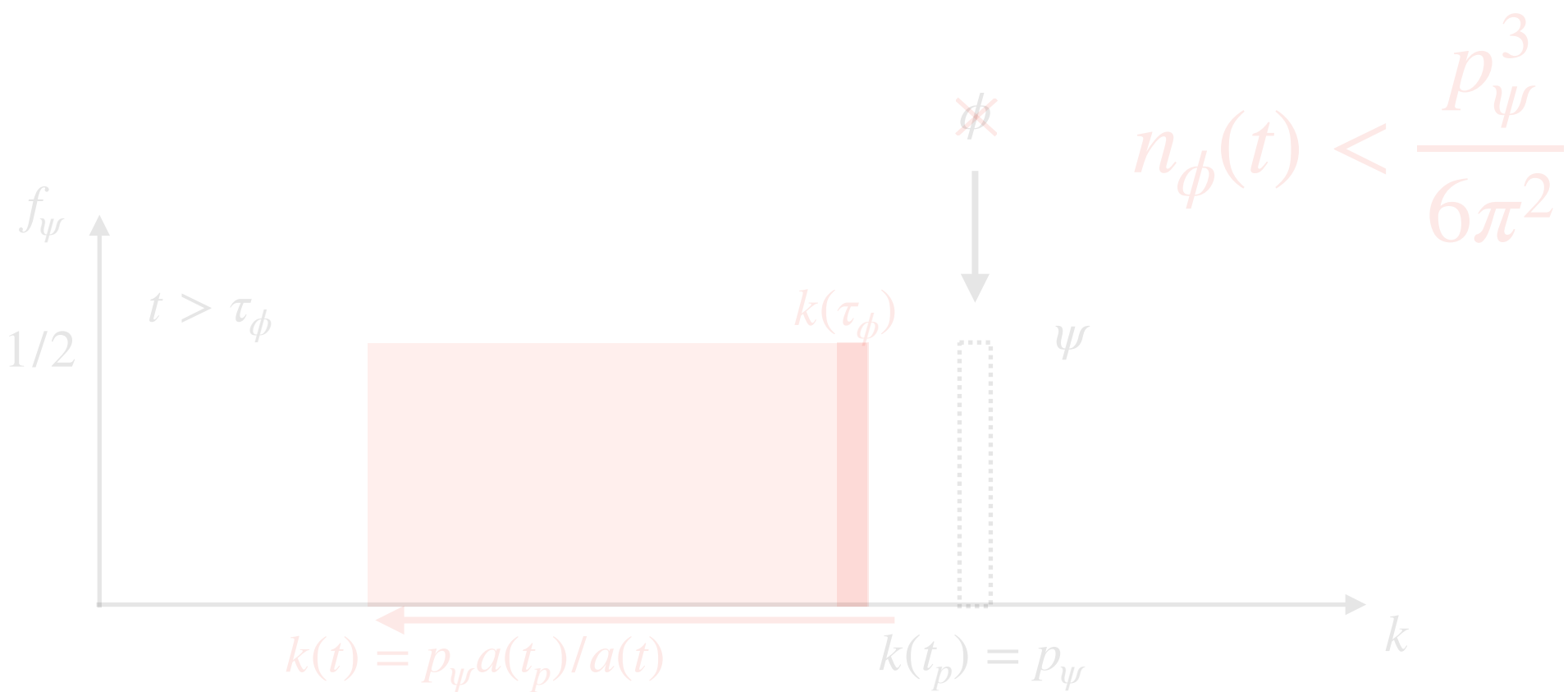
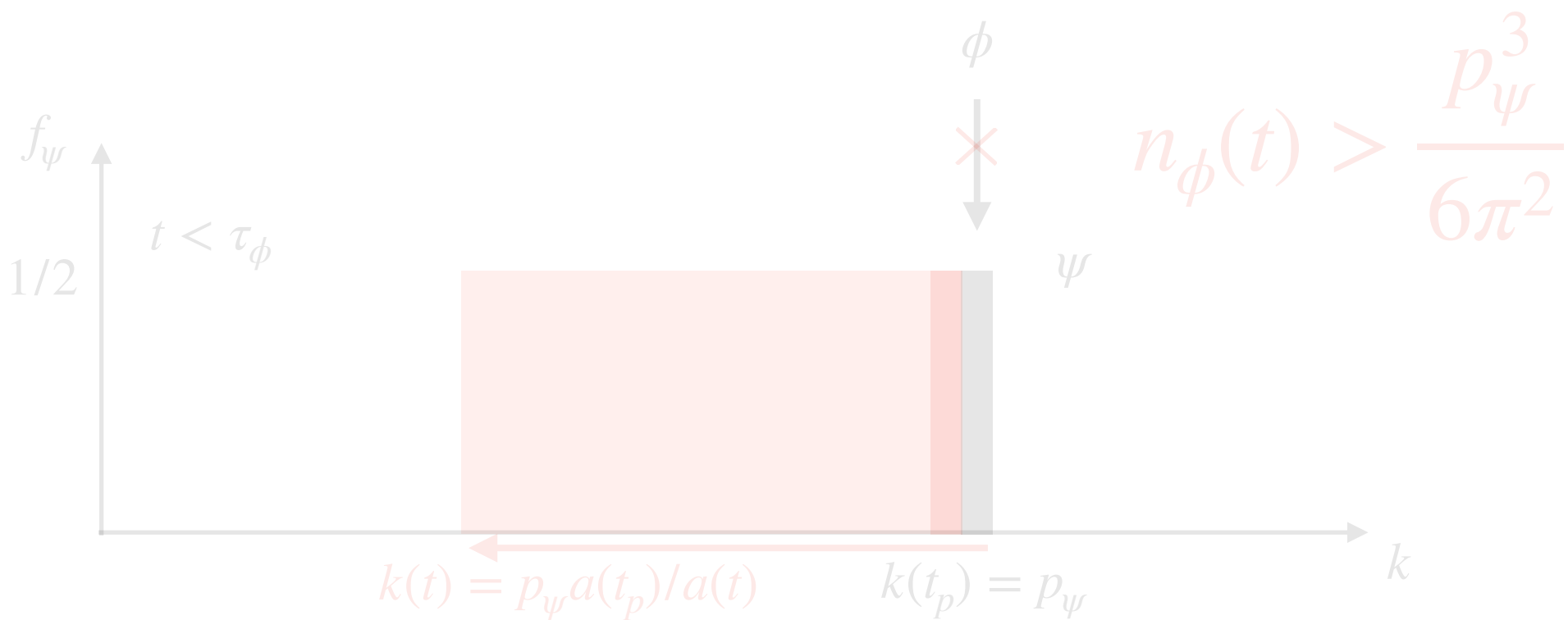
Result of Boltzmann Equation

W.Cho, K.Y.Choi, J.Joh, O.Seto. Phys.Dark Univ. 49 (2025)



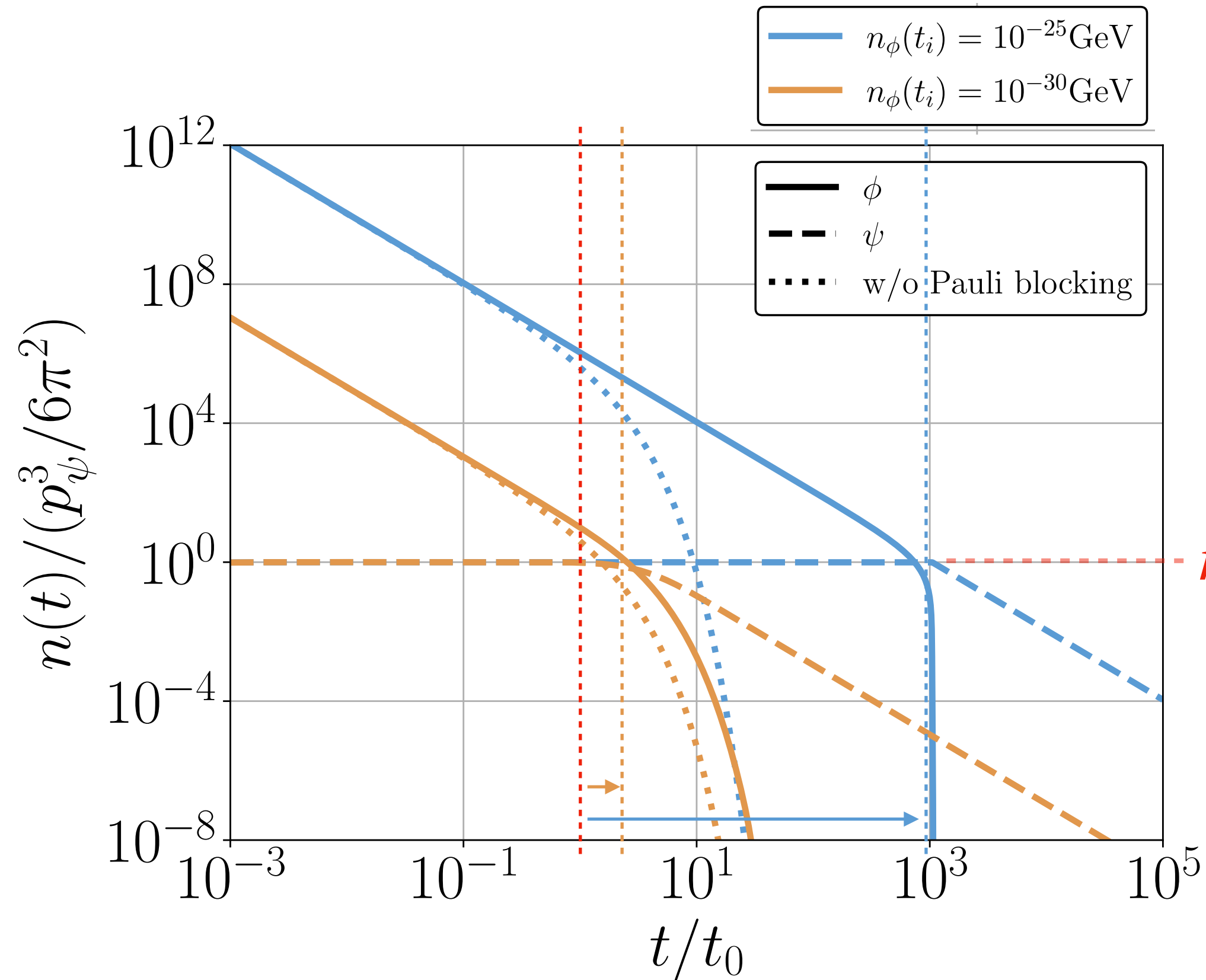
$$\tau_{\phi,0} = 1/\Gamma_\phi$$

$$n_\phi(\tau_\phi) \simeq \frac{p_\psi^3}{6\pi^2}$$



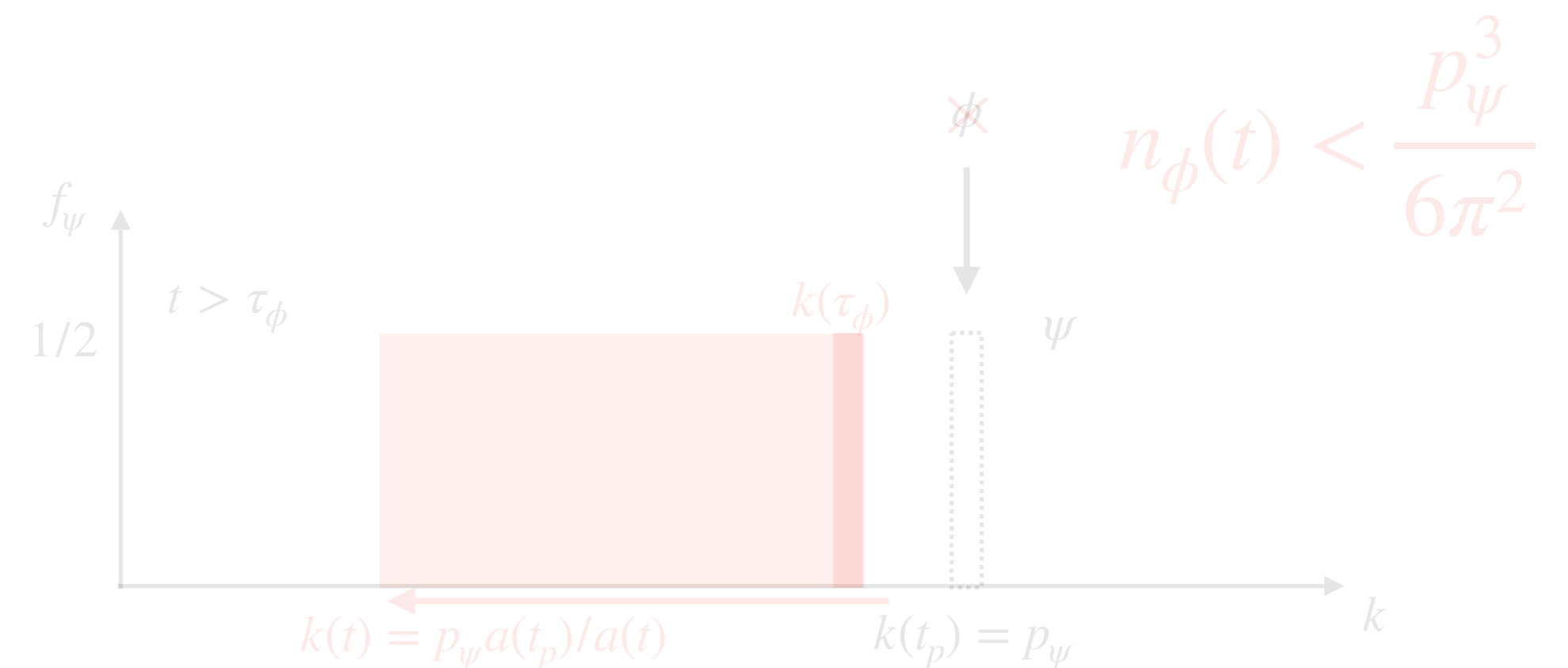
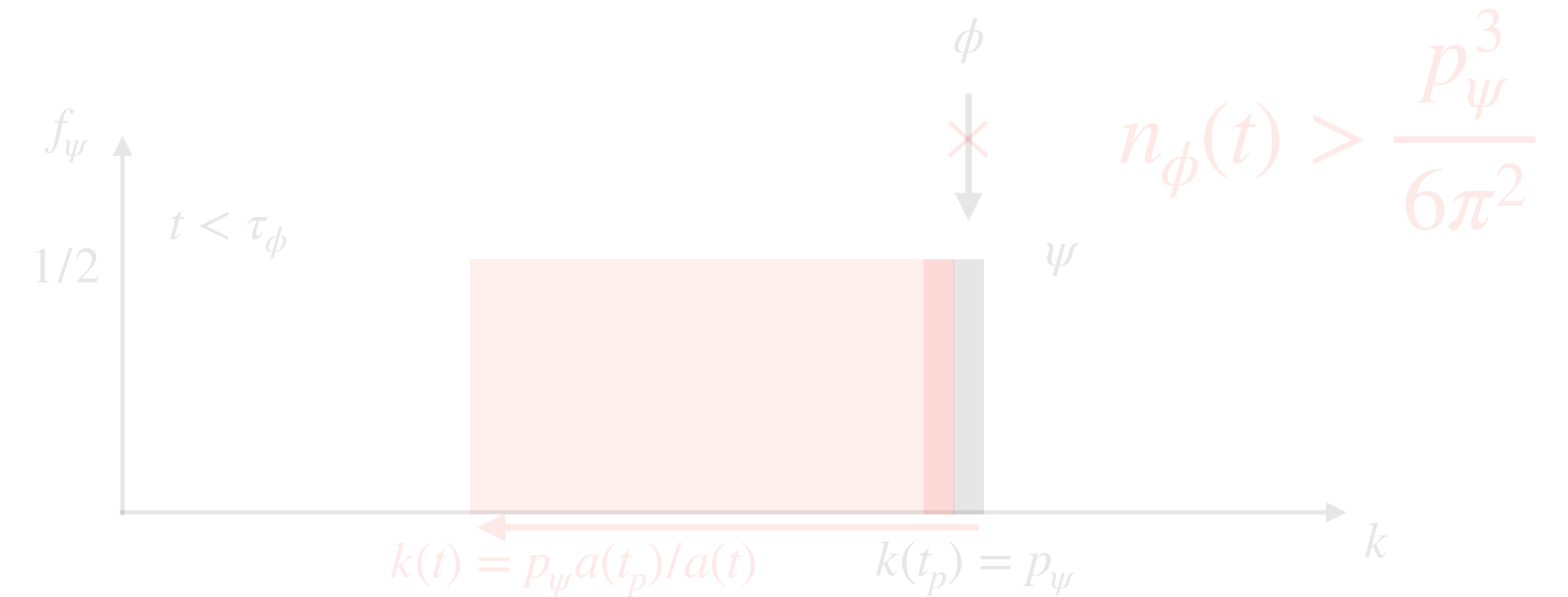
Result of Boltzmann Equation

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$$\tau_{\phi,0} = 1/\Gamma_\phi \rightarrow \tau_\phi \simeq t_i \left(\frac{\rho_\phi(t_i)}{m_\phi} \right)^{1/2} \left(\frac{p_\psi^3}{4\pi^2} \right)^{-1/2}$$

Dark matter becomes much more sustainable



Quantum Field Theory

- Distribution function:

$$f_{\psi, \vec{k}}(t) \equiv \frac{1}{2} \sum_r \frac{1}{V} \left\langle a_{\vec{k}}^{\dagger r} a_{\vec{k}}^r \right\rangle \quad \text{:fermion}$$

$$f_{\bar{\psi}, \vec{k}}(t) \equiv \frac{1}{2} \sum_r \frac{1}{V} \left\langle b_{\vec{k}}^{\dagger r} b_{\vec{k}}^r \right\rangle \quad \text{:anti-fermion}$$

- Fermion related function:

$$g_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k} \sum_r \sum_s \left\langle \underline{b_{-\vec{k}}^s a_{\vec{k}}^r} \right\rangle \bar{v}_{-\vec{k}}^s u_{\vec{k}}^r \quad \rightarrow \text{annihilation}$$

$$g_{\vec{k}}^*(t) \equiv \frac{1}{4V\omega_k} \sum_r \sum_s \left\langle \underline{a_{\vec{k}}^{r\dagger} b_{-\vec{k}}^{s\dagger}} \right\rangle \bar{u}_{\vec{k}}^r v_{-\vec{k}}^s \quad \rightarrow \text{production}$$

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$$\dot{f}_{\psi, \vec{k}} = i\lambda \bar{\phi} \left(\underline{g_{\vec{k}}} - g_{\vec{k}}^* \right) \cos m_{\phi} t \quad \rightarrow \text{|production - annihilation|}$$

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$$\dot{g}_{\vec{k}} = -i \left[2\omega_k g_{\vec{k}}(t) + \lambda \bar{\phi} \left\{ \frac{k^2}{\omega_k^2} \left(\text{*Pauli blocking term} \right) + \frac{2m_{\psi}}{\omega_k} g_{\vec{k}} \right\} \cos m_{\phi} t \right]$$

- Fermion related function:

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In boson case, $1 + f_{\chi, \vec{k}} + f_{\chi, -\vec{k}}$

Ref. T.Moroi, W.Yin, JHEP 03 (2021), 296

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→ |production - annihilation|

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*Pauli blocking term

$$\dot{g}_{\vec{k}} = -i \left[2\omega_k g_{\vec{k}}(t) + \lambda \bar{\phi} \left\{ \frac{k^2}{\omega_k^2} \left(1 - f_{\psi, \vec{k}} - f_{\bar{\psi}, -\vec{k}} \right) + \frac{2m_{\psi}}{\omega_k} g_{\vec{k}} \right\} \cos m_{\phi} t \right]$$

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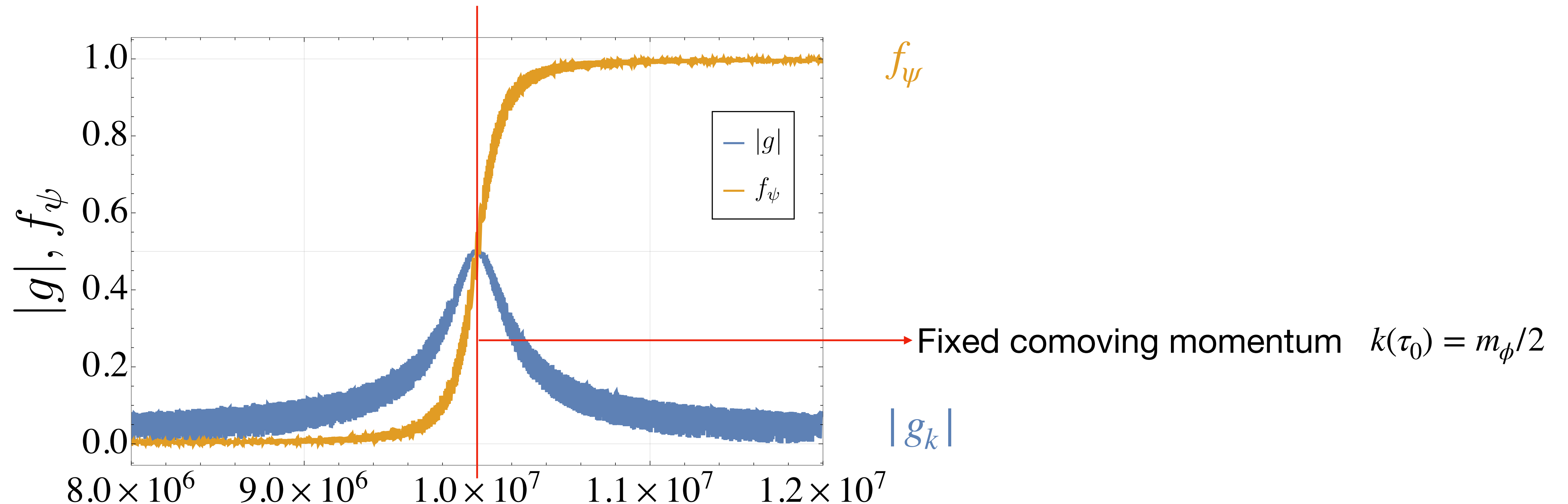
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Showing the fermion
fills all of the states

Result of Quantum Field Theory

Modified from W.Cho, K.Y.Choi, J.Joh, O.Seto. Phys.Dark Univ. 49 (2025)



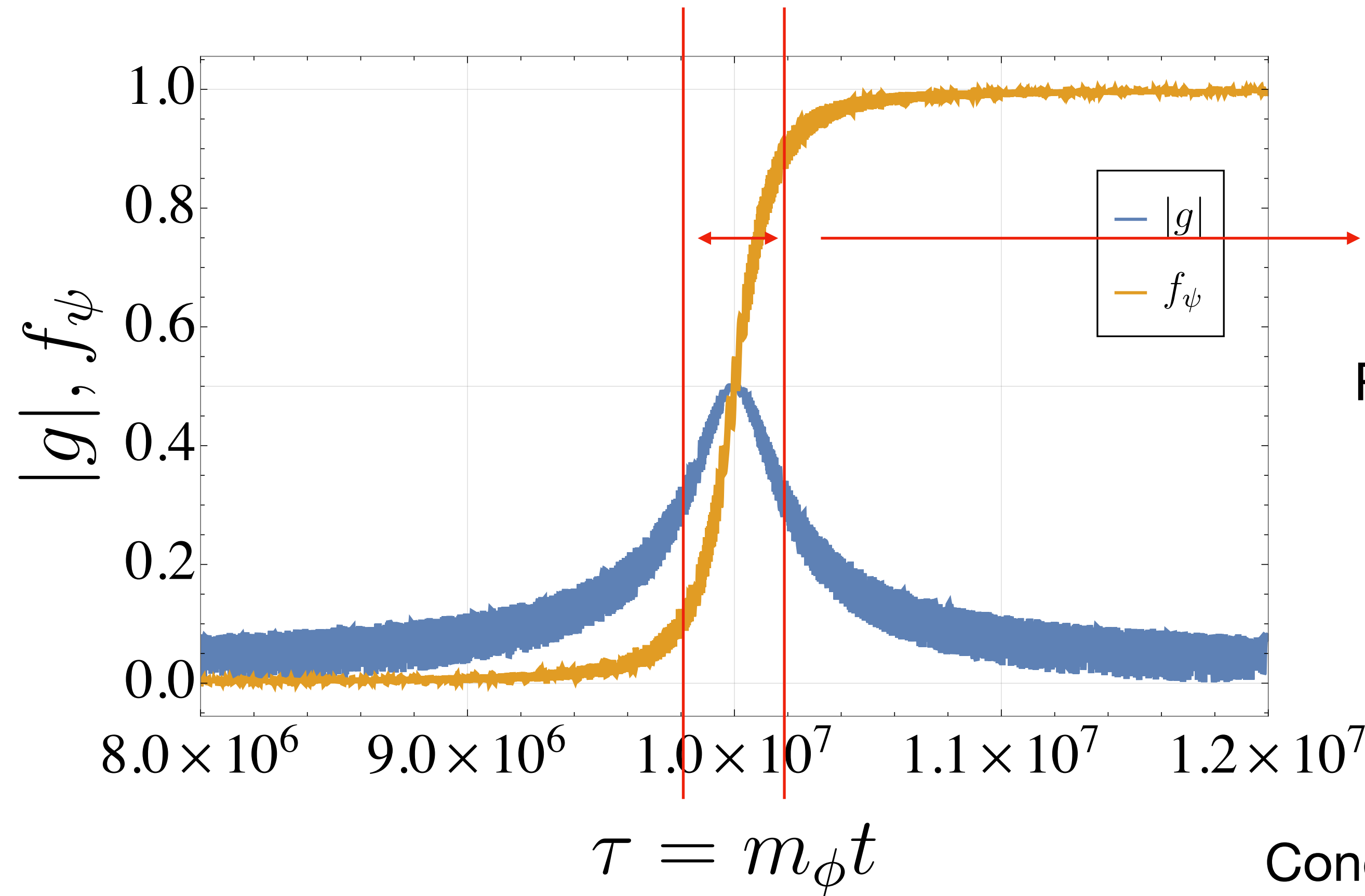
$$\tau = m_\phi t$$

Conditions: $q \equiv \lambda \bar{\phi}/m_\phi = 0.01$, $\tau_0 = 10^2 q^{-2}$

Assumption: $m_\psi \ll m_\phi$

Result of Quantum Field Theory

Modified from W.Cho, K.Y.Choi, J.Joh, O.Seto. Phys.Dark Univ. 49 (2025)



$$1 - \frac{2}{3}q \lesssim \frac{\tau}{\tau_0} \lesssim 1 + \frac{2}{3}q = \text{narrow band}$$

$$\text{Relations: } f_{\psi,k}(1 - f_{\psi,k}) = g_k g_k^* = |g_k|^2$$

Fixed comoving momentum

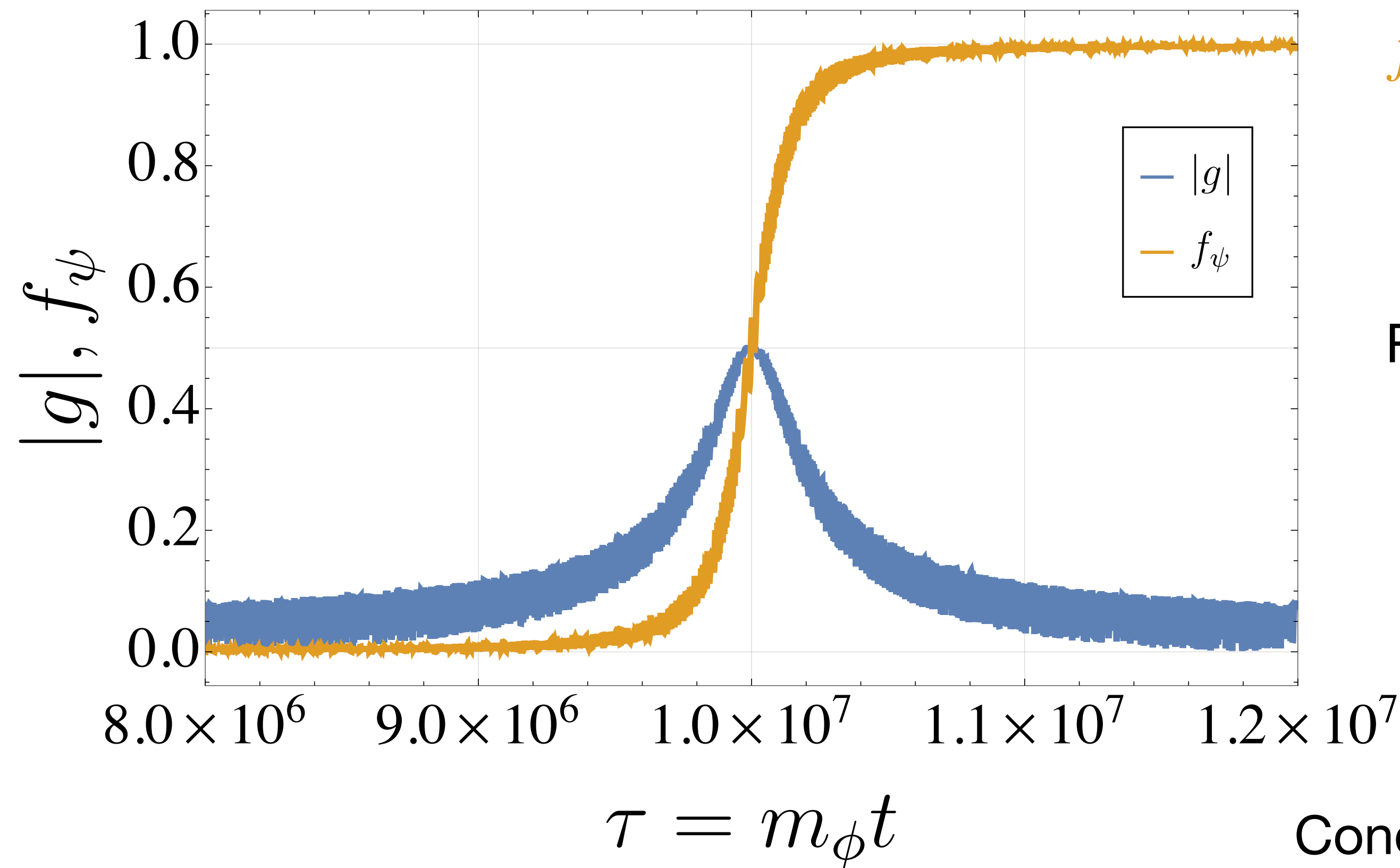
$$k(\tau_0) = m_\phi/2$$

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$$\text{Assumption: } m_\psi \ll m_\phi$$

Result of Quantum Field Theory

Modified from W.Cho, K.Y.Choi, J.Joh, O.Seto. Phys.Dark Univ. 49 (2025)



$f_\psi \rightarrow 1 \Rightarrow$ Pauli blocking is working

$$1 - \frac{2}{3}q \lesssim \frac{\tau}{\tau_0} \lesssim 1 + \frac{2}{3}q = \text{narrow band}$$

Relations: $f_{\psi,k}(1 - f_{\psi,k}) = g_k g_k^* = |g_k|^2$

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Conditions: $q \equiv \lambda \bar{\phi}/m_\phi = 0.01$, $\tau_0 = 10^2 q^{-2}$

Assumption: $m_\psi \ll m_\phi$

Example_100% Dark Matter

- Dark matter (condensed scalar): $\phi(t) = \bar{\phi} \left(\frac{a(t)}{a(t_0)} \right)^{-3/2} \cos(m_\phi t)$
- Energy density of dark matter: $\rho_\phi(t_0) = \frac{1}{2} m_\phi^2 \bar{\phi}^2 \simeq 3.7 \times 10^{-47} \text{ GeV}^4$
- Lifetime w/o Pauli blocking: $\tau_{\phi,0} \simeq 1.6 \times 10^{13} \text{ sec} \left(\frac{10^{-12}}{\lambda} \right)^2 \left(\frac{10^{-3} \text{ eV}}{m_\phi} \right)$
- Lifetime w/ Pauli blocking: $\tau_\phi \simeq 132 \times t_0 \left(\frac{10^{-3} \text{ eV}}{m_\phi} \right)^2$

$$t_0 \simeq 4.3 \times 10^{17} \text{ sec: age of the Universe}$$

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- Dark matter (condensed scalar): $\phi(t) = \bar{\phi} \left(\frac{a(t)}{a(t_0)} \right)^{-3/2} \cos(m_\phi t)$
- Energy density of dark matter: $\rho_\phi(t_0) = \frac{1}{2} m_\phi^2 \bar{\phi}^2 \simeq 3.7 \times 10^{-47} \text{ GeV}^4$
- Lifetime w/o Pauli blocking: $\tau_{\phi,0} \simeq 1.6 \times 10^{13} \text{ sec} \left(\frac{10^{-12}}{\lambda} \right)^2 \left(\frac{10^{-3} \text{ eV}}{m_\phi} \right)$

Interaction
→
Number density
- Lifetime w/ Pauli blocking: $\tau_\phi \simeq 132 \times t_0 \left(\frac{10^{-3} \text{ eV}}{m_\phi} \right)^2$ λ-independent

$$t_0 \simeq 4.3 \times 10^{17} \text{ sec: age of the Universe}$$

Conclusion

- When the scalar field can decay only to the fermions, the Pauli blocking can delay the decay of the DM
→ the lifetime getting longer than the age of the Universe.
- The results from using Boltzmann equations and QFT shows the Pauli blocking is working.
- In the example, as the scalar is 100% dark matter, the lifetime becomes 100 times the age of the Universe where $m_\phi \simeq 10^{-3}$ eV, and independent of the coupling λ

Appendix

Isotropic Fermion

- Scalar DM is condensate when the mass is small
- Lots of dark matter decays into fermion almost at the same time
- Since the direction of the fermion is determined randomly, the fermion becomes isotropic $\Rightarrow f_{\psi}(\vec{k}, t) = f_{\psi}(k, t)$

Boltzmann Equation without Expansion

- Momentum distribution function:

- $\frac{\partial f_\phi(\vec{p}, t)}{\partial t} = f_\phi^{\text{coll}}(\vec{p}, t)$

- $\frac{\partial f_\psi(\vec{k}, t)}{\partial t} = f_\psi^{\text{coll}}(\vec{k}, t)$

- Number density:

- $\dot{n}_\phi(t) = -\Gamma_\phi n_\phi(t) \left(1 - 2f_\psi(p_\psi, t) \right)$ → Dark matter stops decaying when $f_\psi(p_\psi, t) = 1/2$

- $\dot{n}_\psi(t) = \Gamma_\phi n_\phi(t) \left(1 - 2f_\psi(p_\psi, t) \right)$

$$\Rightarrow n_\phi(t) + n_\psi(t) = n_\phi(t) + \frac{p_\psi^3}{\pi^2} f_\psi(p_\psi, t) = C_0 \quad : \text{Total number density}$$

Boltzmann Equation without Expansion

- Solution:

$$\frac{1}{n_\phi(t)} = \frac{B}{A} + \left(\frac{1}{n_\phi(0)} - \frac{B}{A} \right) e^{A(t-t_0)}$$

$$= \left(C_0 - \frac{p_\psi^3}{4\pi^2} \right)^{-1} + \left(\frac{1}{n_\phi(t_0)} - \left(C_0 - \frac{p_\psi^3}{4\pi^2} \right)^{-1} \right) \text{Exp} \left(\Gamma_\phi \left(1 - C_0 \frac{4\pi^2}{p_\psi^3} \right) (t - t_0) \right)$$

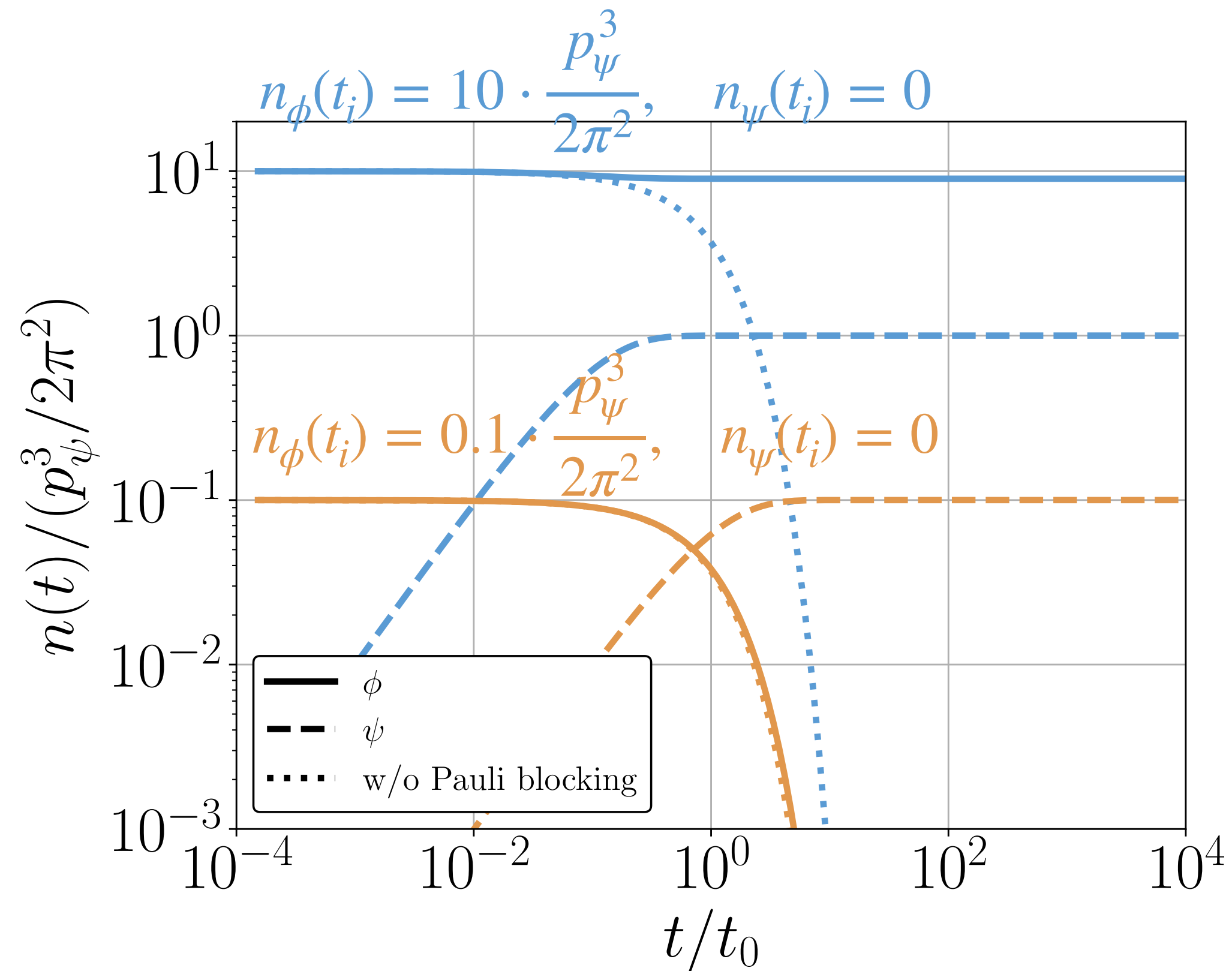
$$\bullet \quad C_0 - \frac{p_\psi^3}{4\pi^2} = n_\phi(t) + \frac{p_\psi^3}{2\pi^2} \left(f_\psi(p_\psi, t) - \frac{1}{2} \right) < 0 \quad \Rightarrow n_\phi(t) \rightarrow 0, \quad f_\psi \rightarrow \frac{2\pi^2}{p_\psi^3} C_0$$

$$\bullet \quad C_0 - \frac{p_\psi^3}{4\pi^2} = n_\phi(t) + \frac{p_\psi^3}{2\pi^2} \left(f_\psi(p_\psi, t) - \frac{1}{2} \right) > 0$$

$$\Rightarrow n_\phi(t) \rightarrow n_\phi(t_0) - \frac{p_\psi^3}{2\pi^2} \left(\frac{1}{2} - f_\psi(p_\psi, t_0) \right), \quad f_\psi \rightarrow \frac{1}{2}$$

Boltzmann Equation without Expansion

Analytical results



$$n_\phi(t) + n_\psi(t) = n_\phi(t) + \frac{p_\psi^3}{\pi^2} f_\psi(p_\psi, t) = C_0$$

$$n_\phi(t) + n_\psi(t) > \frac{p_\psi^3}{2\pi^2}$$

$$\Rightarrow n_\phi(t) \rightarrow n_\phi(t_0) - \frac{p_\psi^3}{\pi^2} \left(\frac{1}{2} - f_\psi(p_\psi, t_0) \right),$$

$$n_\psi(t) \rightarrow \frac{p_\psi^3}{2\pi^2}, \quad f_\psi \rightarrow \frac{1}{2}$$

$$n_\phi(t) + n_\psi(t) < \frac{p_\psi^3}{2\pi^2}$$

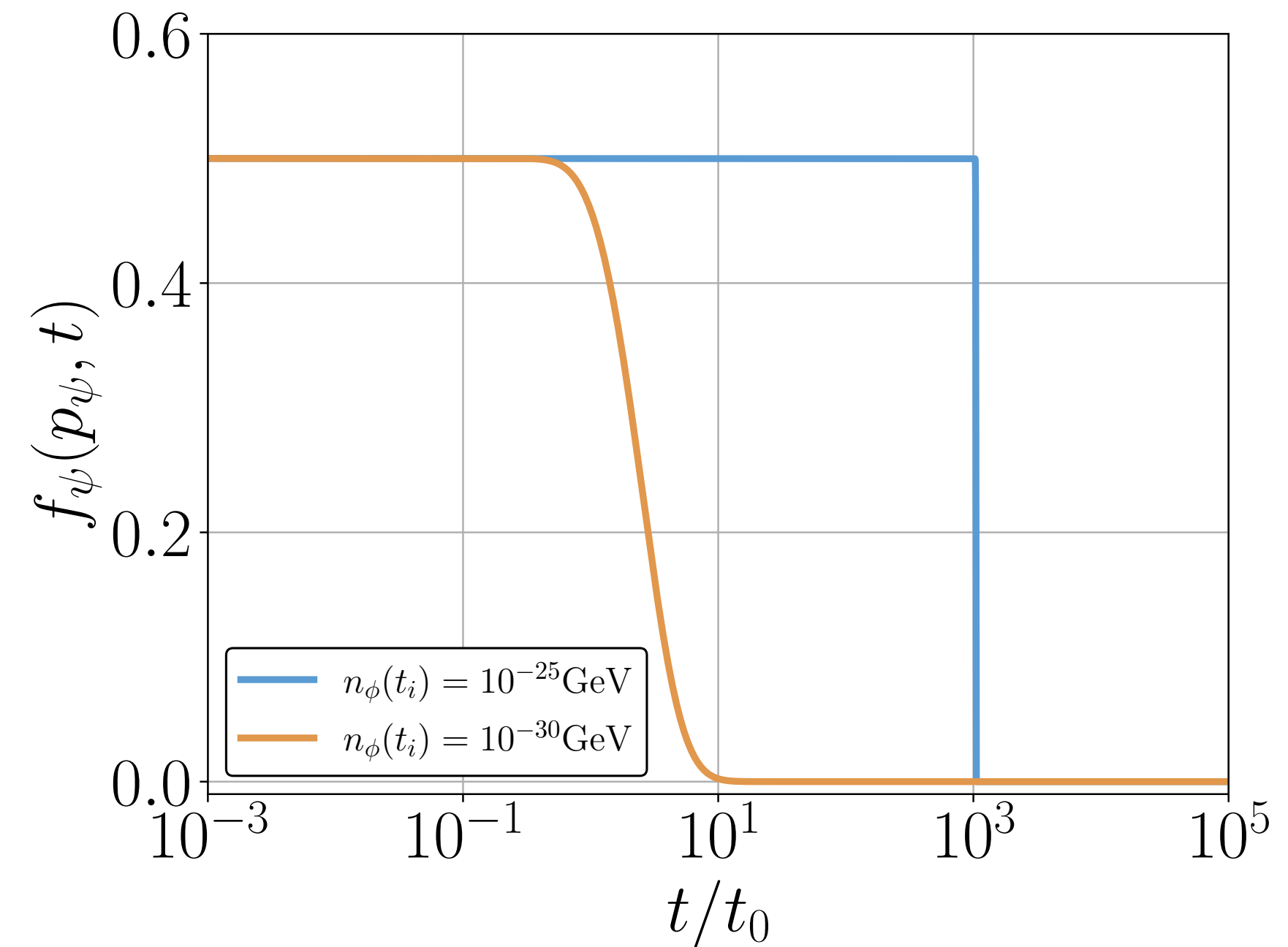
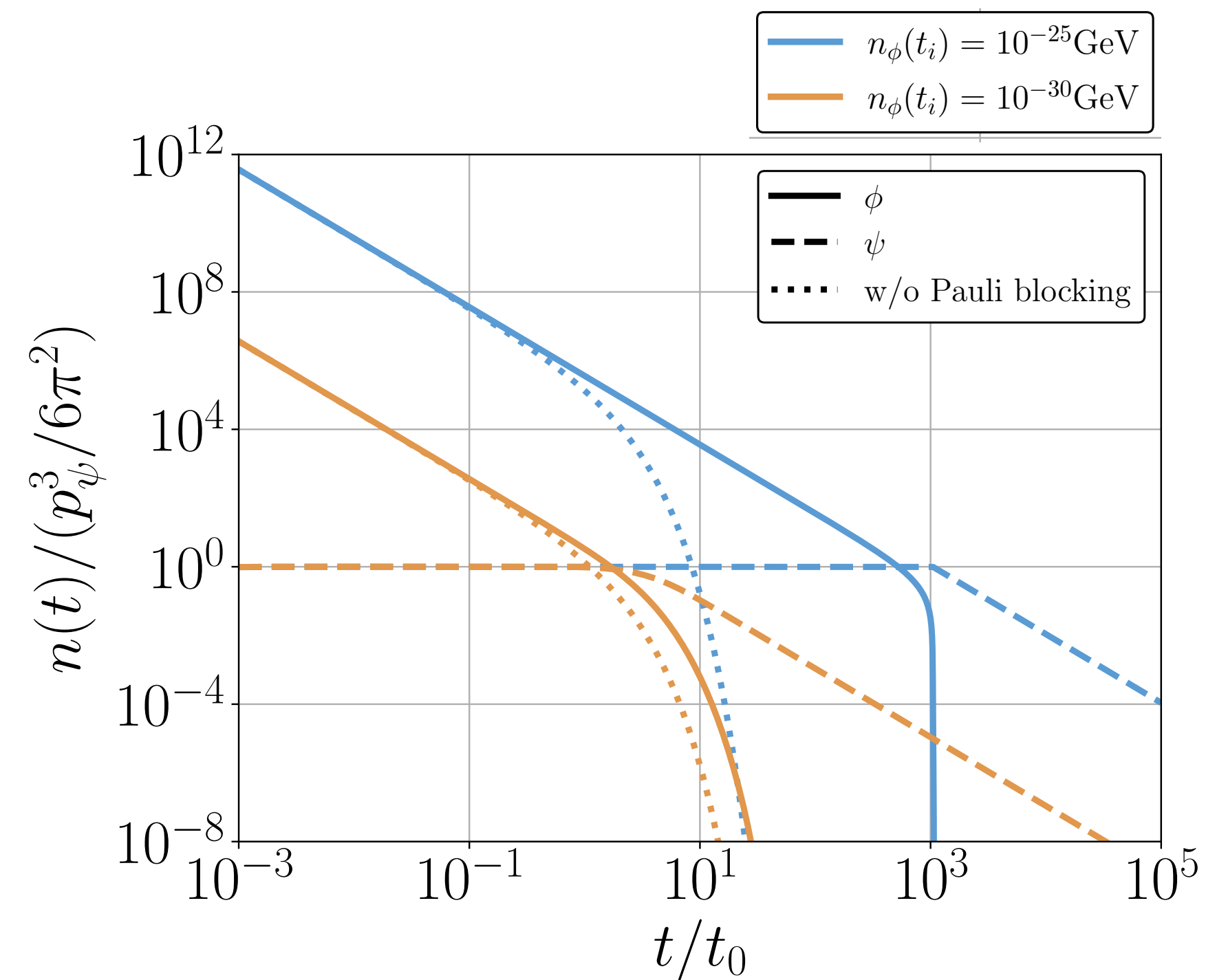
$$\Rightarrow n_\phi(t) \rightarrow 0$$

$$n_\psi(t) \rightarrow n_\phi(t_i), \quad f_\psi \rightarrow \frac{2\pi^2}{p_\psi^3} n_\phi(t_i)$$

Boltzmann Equation with Expansion

$$\dot{n}_\phi(t) + 3H(t)n_\phi(t) = -\Gamma_\phi n_\phi(t) \left(1 - 2f_\psi(p_\psi, t)\right)$$

$$\dot{n}_\psi(t) + 3H(t)n_\psi(t) = \Gamma_\phi n_\phi(t) \left(1 - 2f_\psi(p_\psi, t)\right)$$



Quantum Field Theory

- Hamiltonian:

$$H_{\text{int}}(t) = \lambda \int d^3x \phi(x) \bar{\psi}(x) \psi(x)$$

$$H_{\text{free}} = \int \frac{d^3p}{(2\pi)^3} \sum_s \omega_{\vec{p}} (a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s + b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s)$$

- Particles/Fields:

$$\phi(t) = \bar{\phi} \cos(m_{\phi} t) \quad \text{*assumption: oscillation, homogeneous \& classical}$$

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\vec{p}}^s(t) u_{\vec{p}}^s e^{-i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^{s\dagger}(t) v_{\vec{p}}^s e^{i\vec{p}\cdot\vec{x}} \right)$$

$$\bar{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\vec{p}}^{s\dagger}(t) \bar{u}_{\vec{p}}^s e^{i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^s(t) \bar{v}_{\vec{p}}^s e^{-i\vec{p}\cdot\vec{x}} \right)$$

- Distribution function:

$$f_{\psi, \vec{k}}(t) \equiv \frac{1}{2} \sum_r \frac{1}{V} \left\langle a_{\vec{k}}^{r\dagger} a_{\vec{k}}^r \right\rangle$$

$$f_{\bar{\psi}, \vec{k}}(t) \equiv \frac{1}{2} \sum_r \frac{1}{V} \left\langle b_{\vec{k}}^{r\dagger} b_{\vec{k}}^r \right\rangle$$

- Fermion related function:

$$g_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k} \sum_r \sum_s \left\langle \underline{b_{-\vec{k}}^s a_{\vec{k}}^r} \right\rangle \bar{v}_{-\vec{k}}^s u_{\vec{k}}^r \quad \rightarrow \text{annihilation}$$

$$g_{\vec{k}}^*(t) \equiv \frac{1}{4V\omega_k} \sum_r \sum_s \left\langle \underline{a_{\vec{k}}^{r\dagger} b_{-\vec{k}}^{s\dagger}} \right\rangle \bar{u}_{\vec{k}}^r v_{-\vec{k}}^s \quad \rightarrow \text{production}$$

Quantum Field Theory without Expansion

- Differential equations

- $\phi(x, t) \rightarrow \phi(t) = \bar{\phi} \cos(m_\phi t)$

- $\dot{f}_{\psi, \vec{k}} = i\lambda \bar{\phi} \left(g_{\vec{k}} - g_{\vec{k}}^* \right) \cos m_\phi t$

- $\dot{f}_{\bar{\psi}, \vec{k}} = i\lambda \bar{\phi} \left(g_{-\vec{k}} - g_{-\vec{k}}^* \right) \cos m_\phi t$

- $\dot{g}_{\vec{k}} = -i \left[2\omega_k g_{\vec{k}}(t) + \lambda \bar{\phi} \left\{ \frac{k^2}{\omega_k^2} \left(1 - f_{\psi, \vec{k}} - f_{\bar{\psi}, -\vec{k}} \right) + \frac{2m_\psi}{\omega_k} g_{\vec{k}} \right\} \cos m_\phi t \right]$

- Differential equations

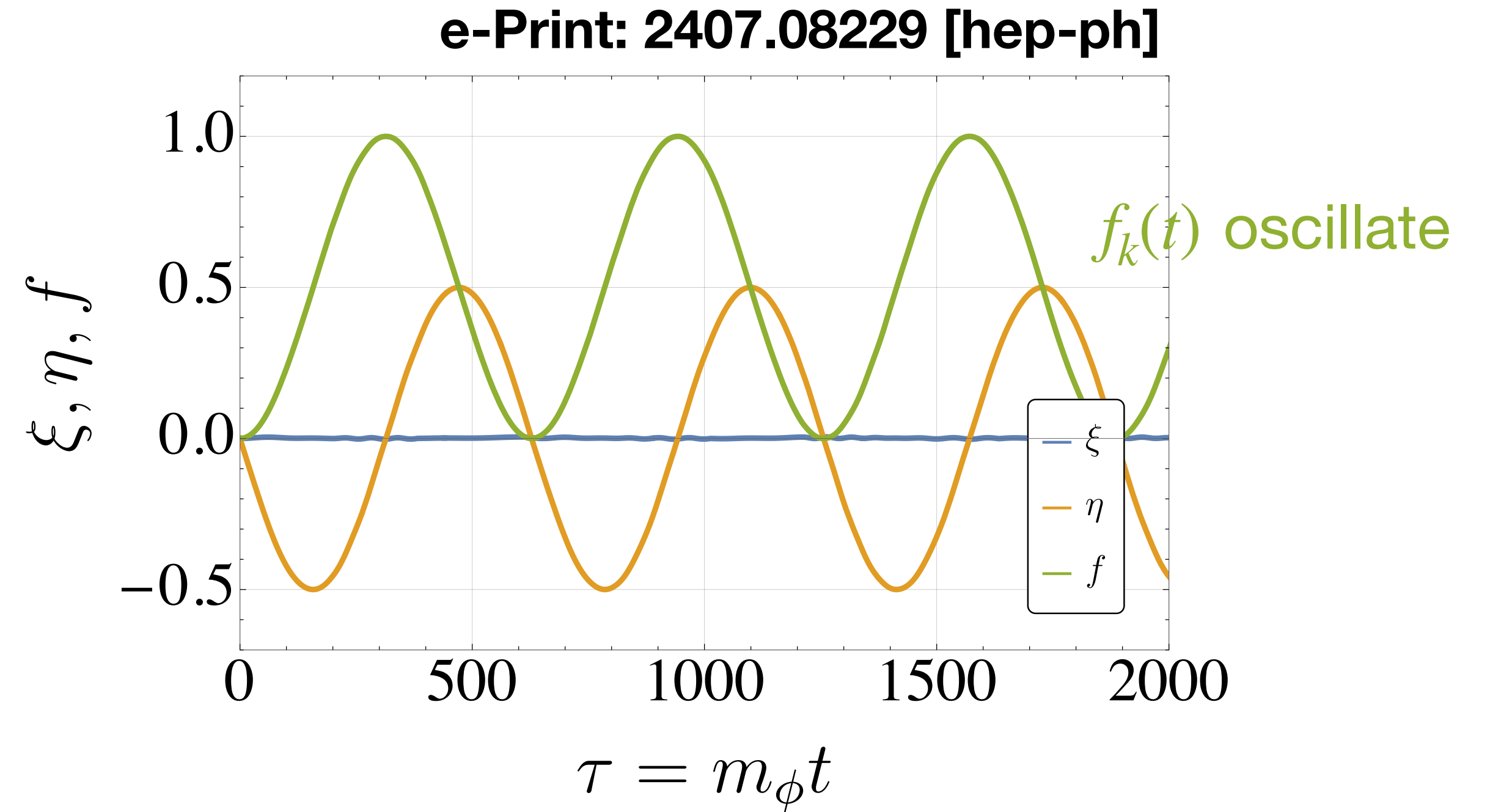
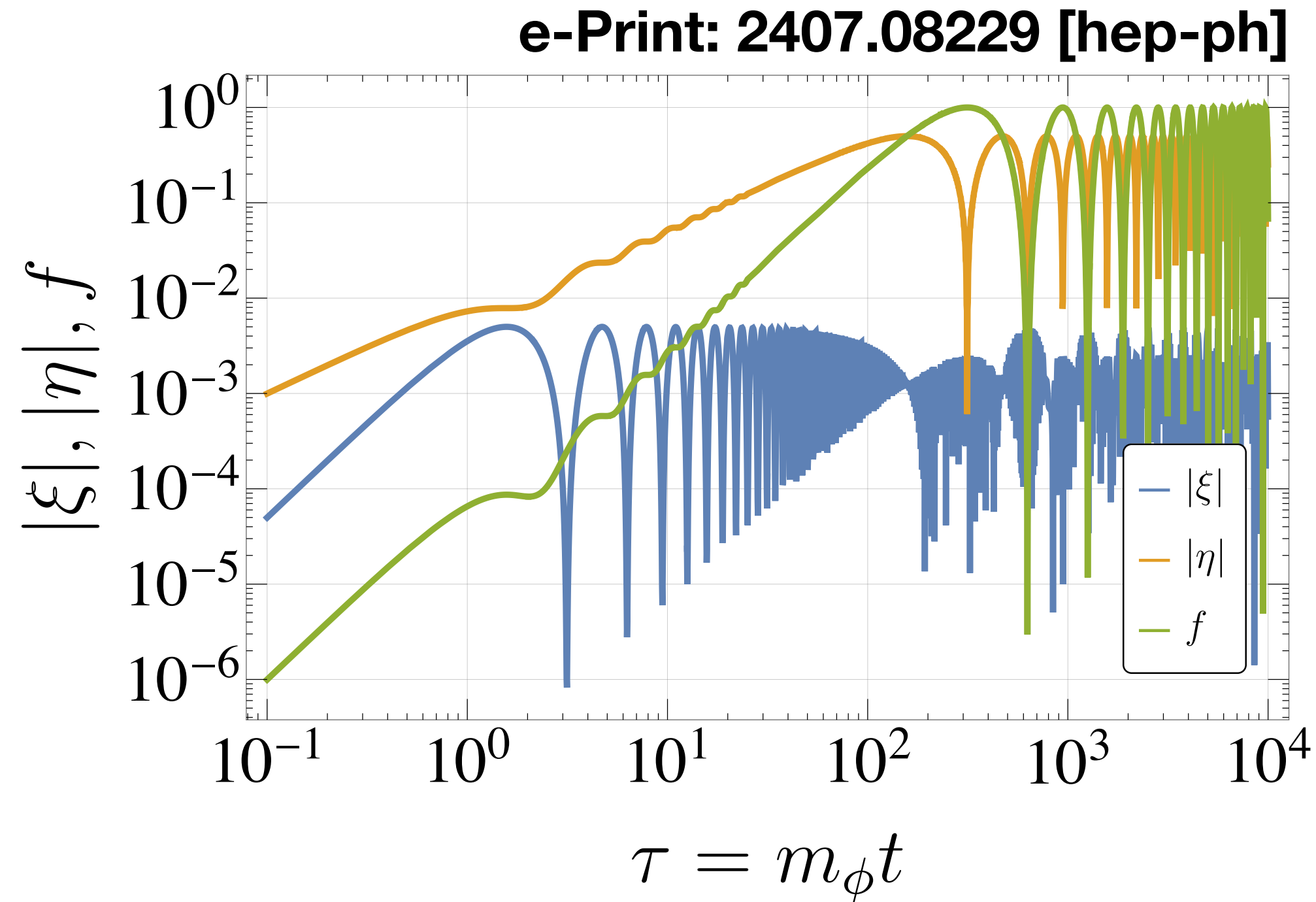
- $\frac{df_k}{d\tau} = q \left[2\xi_k \sin(2\tilde{\omega}_k \tau) - 2\eta_k \cos(2\tilde{\omega}_k \tau) \right] \cos \tau$

- $\frac{d\xi_k}{d\tau} = q \left[(1 - 2f_k) \sin(2\tilde{\omega}_k \tau) \right] \cos \tau$

- $\frac{d\eta_k}{d\tau} = -q \left[(1 - 2f_k) \cos(2\tilde{\omega}_k \tau) \right] \cos \tau$

- where $g_k(t) = e^{-2i\omega_k t} [\xi_k(t) + i\eta_k(t)]$, $\tau = m_\phi t$, $q = \lambda \bar{\phi} / m_\phi$, and $\tilde{\omega}_k = \omega_k / m_\phi$

Quantum Field Theory without Expansion



- Approximate solutions: ^{*assumption: $m_\psi = 0$}
^{*approximation: $\omega_k \sim m_\phi/2 \Rightarrow \xi_k(\tau) \simeq 0$}

$$f_k(\tau) \simeq \sin^2 \left[\frac{1}{2} q (\tau + \cos \tau \sin \tau) \right] \sim \sin^2(q\tau/2) \rightarrow \text{average value is } 1/2$$

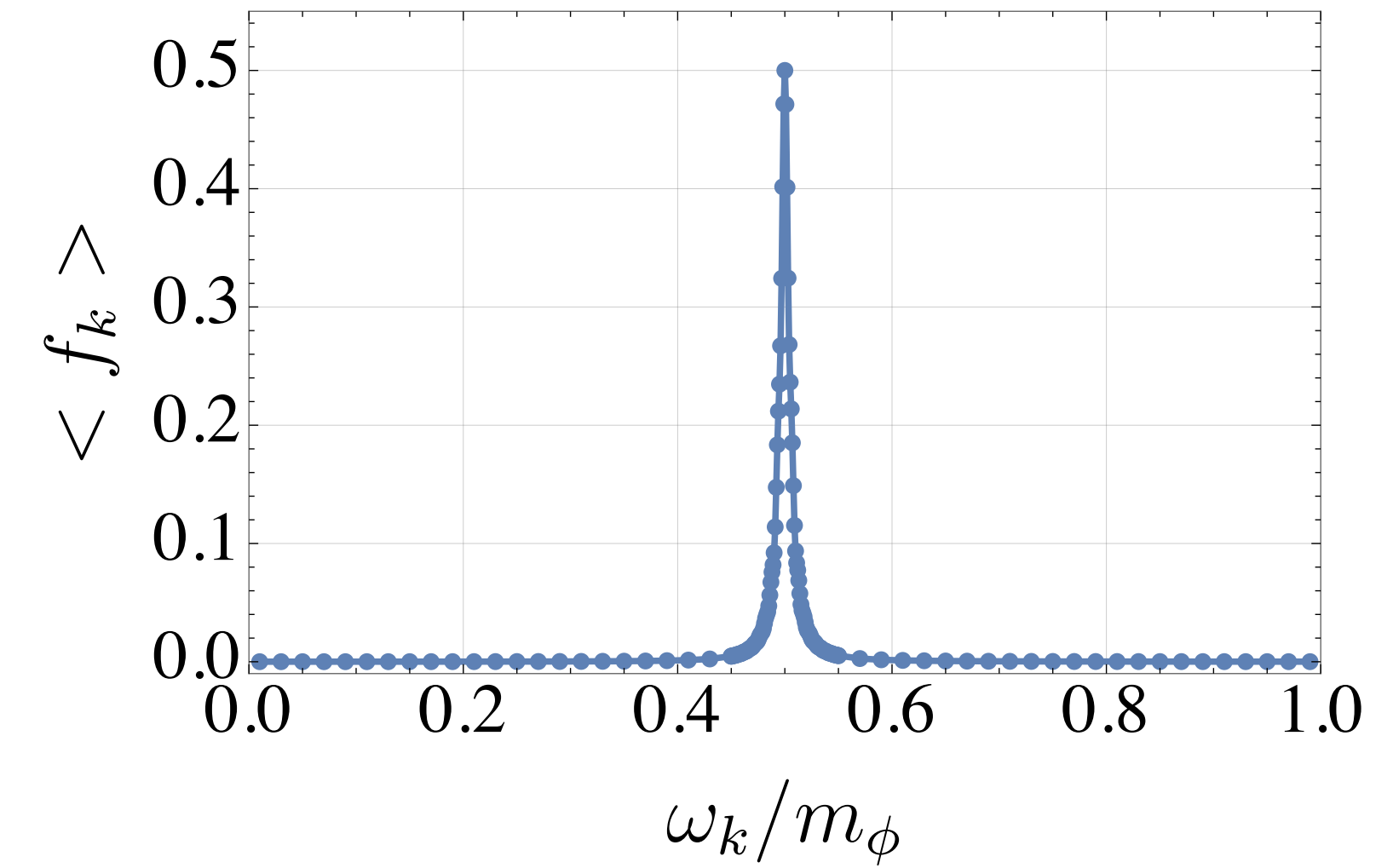
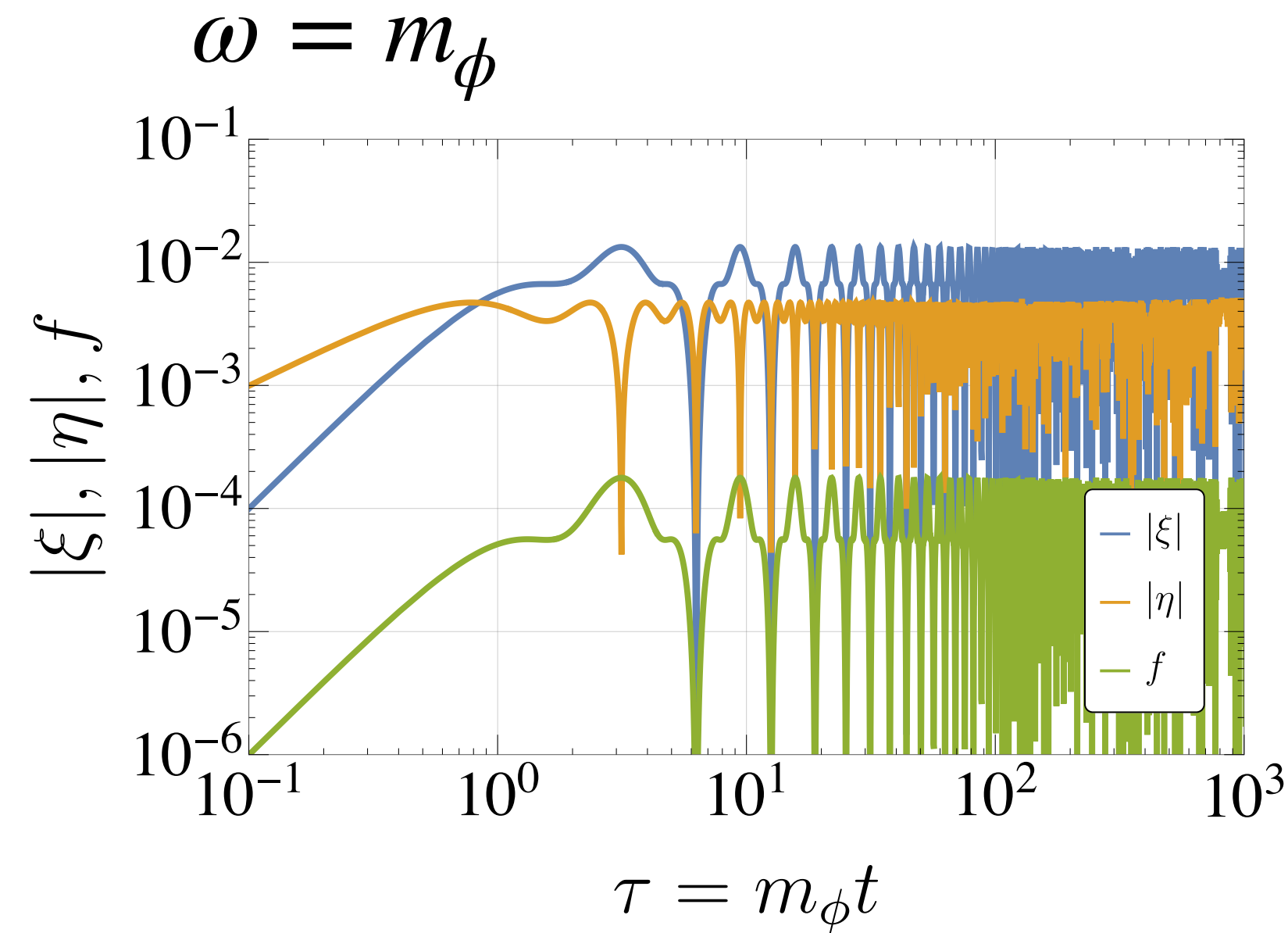
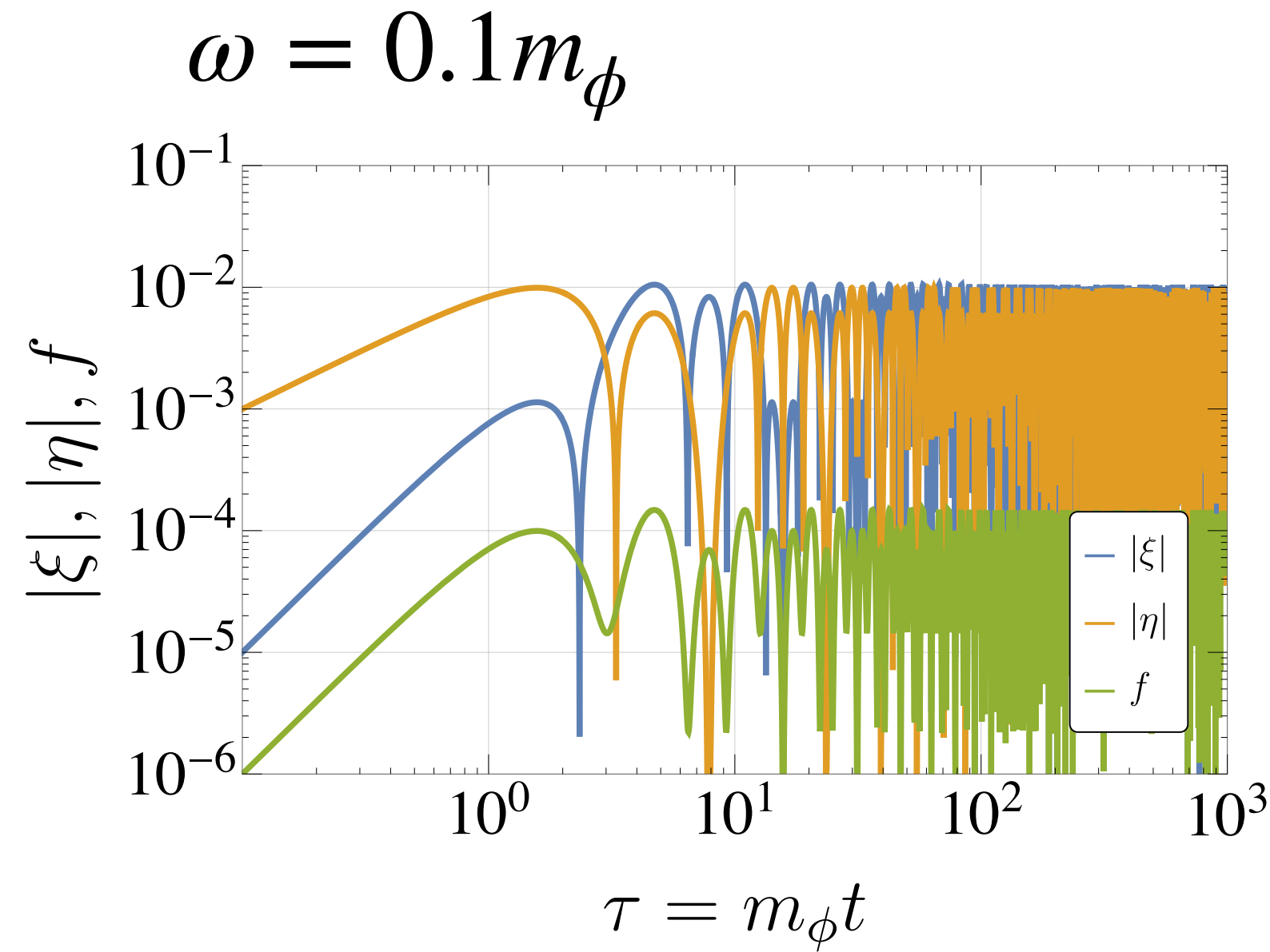
$$\eta_k(\tau) \simeq -\frac{1}{2} \sin [q(\tau + \cos \tau \sin \tau)] \sim -\frac{1}{2} \sin(q\tau)$$

$$g_k(t) = e^{-2i\omega_k t} [\xi_k(t) + i\eta_k(t)]$$

$$\tau = m_\phi t, \quad q = \lambda \bar{\phi} / m_\phi, \quad \text{and} \quad \tilde{\omega}_k = \omega_k / m_\phi$$

Quantum Field Theory without Expansion

When ω_k is far from $m_\phi/2$



$$f_k(\tau) \simeq \frac{q^2 (1 + 12\tilde{\omega}_k^2 + (-1 + 4\tilde{\omega}_k^2)\cos(2\tau) - 8\tilde{\omega}_k(2\tilde{\omega}_k \cos(\tau)\cos(2\tilde{\omega}_k\tau) + \sin(\tau)\sin(2\tilde{\omega}_k\tau)))}{2(1 - 4\tilde{\omega}_k^2)^2}$$

$$\xi_k(\tau) \simeq \frac{-q (-2\tilde{\omega}_k + 2\tilde{\omega}_k \cos(\tau)\cos(2\tilde{\omega}_k\tau) + \sin(\tau)\sin(2\tilde{\omega}_k\tau))}{-1 + 4\tilde{\omega}_k^2}$$

$$\eta_k(\tau) \simeq \frac{q (\cos(2\tilde{\omega}_k\tau)\sin(\tau) - 2\tilde{\omega}_k \cos(\tau)\sin(2\tilde{\omega}_k\tau))}{-1 + 4\tilde{\omega}_k^2}$$

Quantum Field Theory with Expansion

- Differential equations:

$$\dot{f}_k = kH \frac{\partial f_k}{\partial k} + i\lambda\phi(t)(g_k - g_k^*)$$

$$\dot{g}_k = kH \frac{\partial g_k}{\partial k} - i \left[2\Omega_k(t)g_k(t) + \lambda\phi(t) \left\{ \frac{K(t)^2}{\Omega(t)^2} (1 - 2f_k) + \frac{2m_\psi}{\Omega(t)} g_k \right\} \right]$$

- Differential equations:

$$\frac{df_*(\tau)}{d\tau} = q \left(\frac{\tau}{\tau_0} \right)^{-1} (2\xi_*(\tau)\sin 2\Theta(\tau) - 2\eta_*(\tau)\cos 2\Theta(\tau)) \cos \tau$$

$$\frac{d\xi_*(\tau)}{d\tau} = q \left(\frac{\tau}{\tau_0} \right)^{-1} \left[(1 - 2f_*(\tau)) \sin 2\Theta(\tau) \right] \cos \tau$$

$$\frac{d\eta_*(\tau)}{d\tau} = -q \left(\frac{\tau}{\tau_0} \right)^{-1} \left[(1 - 2f_*(\tau)) \cos 2\Theta(\tau) \right] \cos \tau$$

$$\Omega(\tau) = \left(\frac{\tau_0}{\tau} \right)^{2/3} \omega_k$$

$$\Theta(\tau) = \frac{3\omega_k}{m_\phi} \tau_0 \left(\frac{\tau}{\tau_0} \right)^{1/3}$$

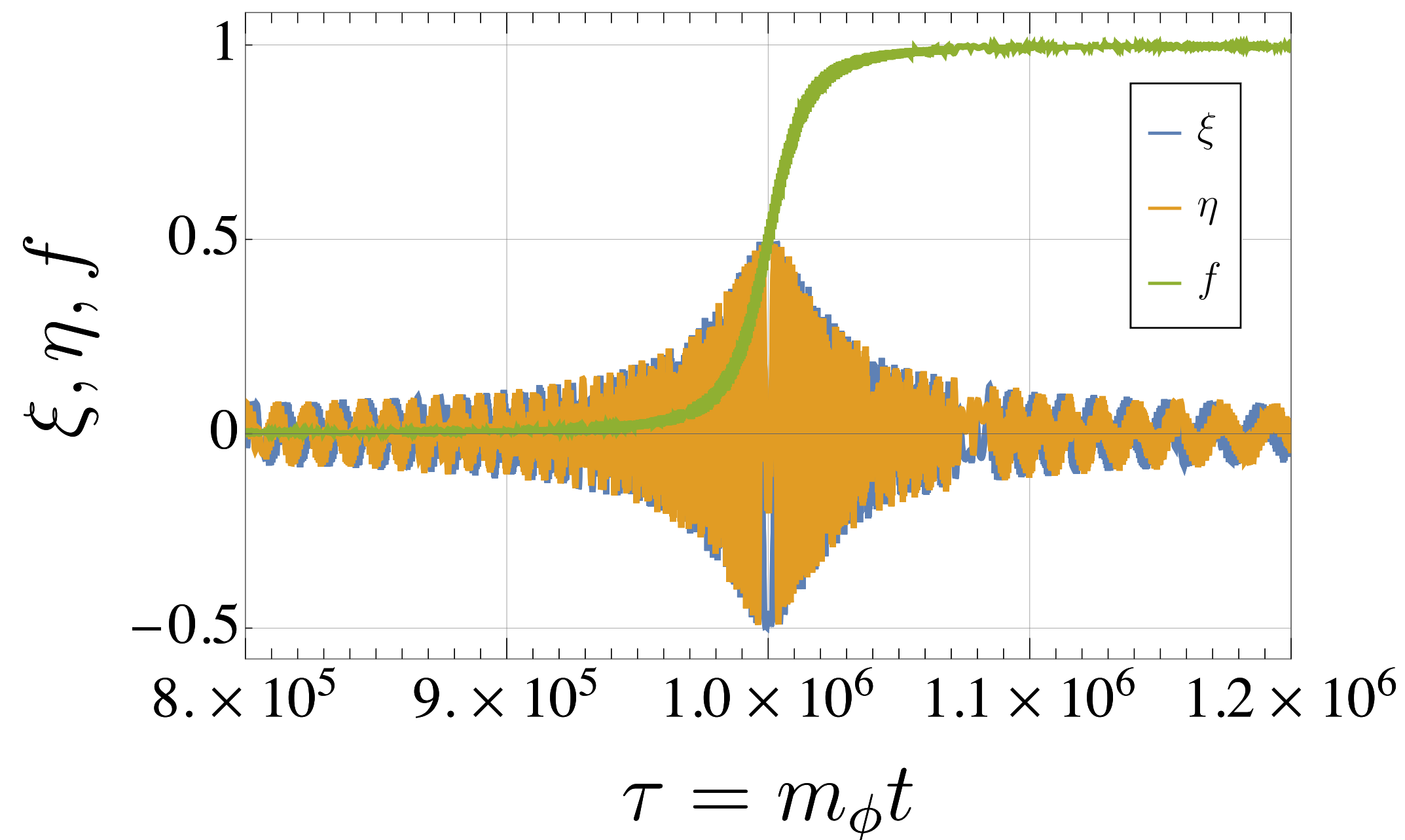
$$K(t) = \frac{a_0}{a(t)} \hat{k} = \frac{m_\phi}{2}$$

$$q = \frac{\lambda\bar{\phi}}{m_\phi}, \quad \tilde{K}(t) = \frac{K(t)}{m_\phi}, \quad \tilde{\Omega}(t) = \frac{\Omega(t)}{m_\phi}$$

$$f_k(1 - f_k) = \xi_k^2 + \eta_k^2 = g_k g_k^* \geq 0$$

Quantum Field Theory with Expansion

e-Print: 2407.08229 [hep-ph]



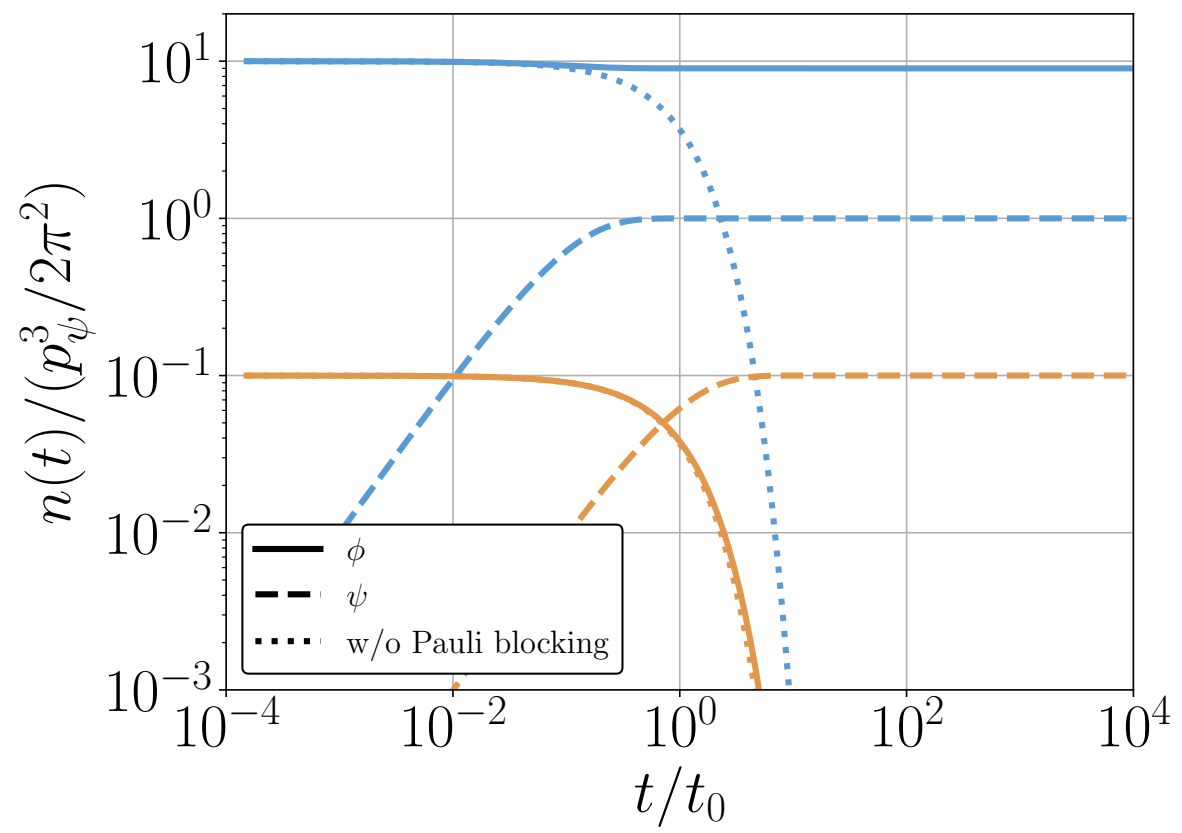
$f_\psi \rightarrow 1$ instead of $f_\psi \rightarrow 1/2$

“The difference can be cured if the factor $1 \pm 2f_\psi$ in the Boltzmann equation is modified to $1 \pm f_\psi$ ”

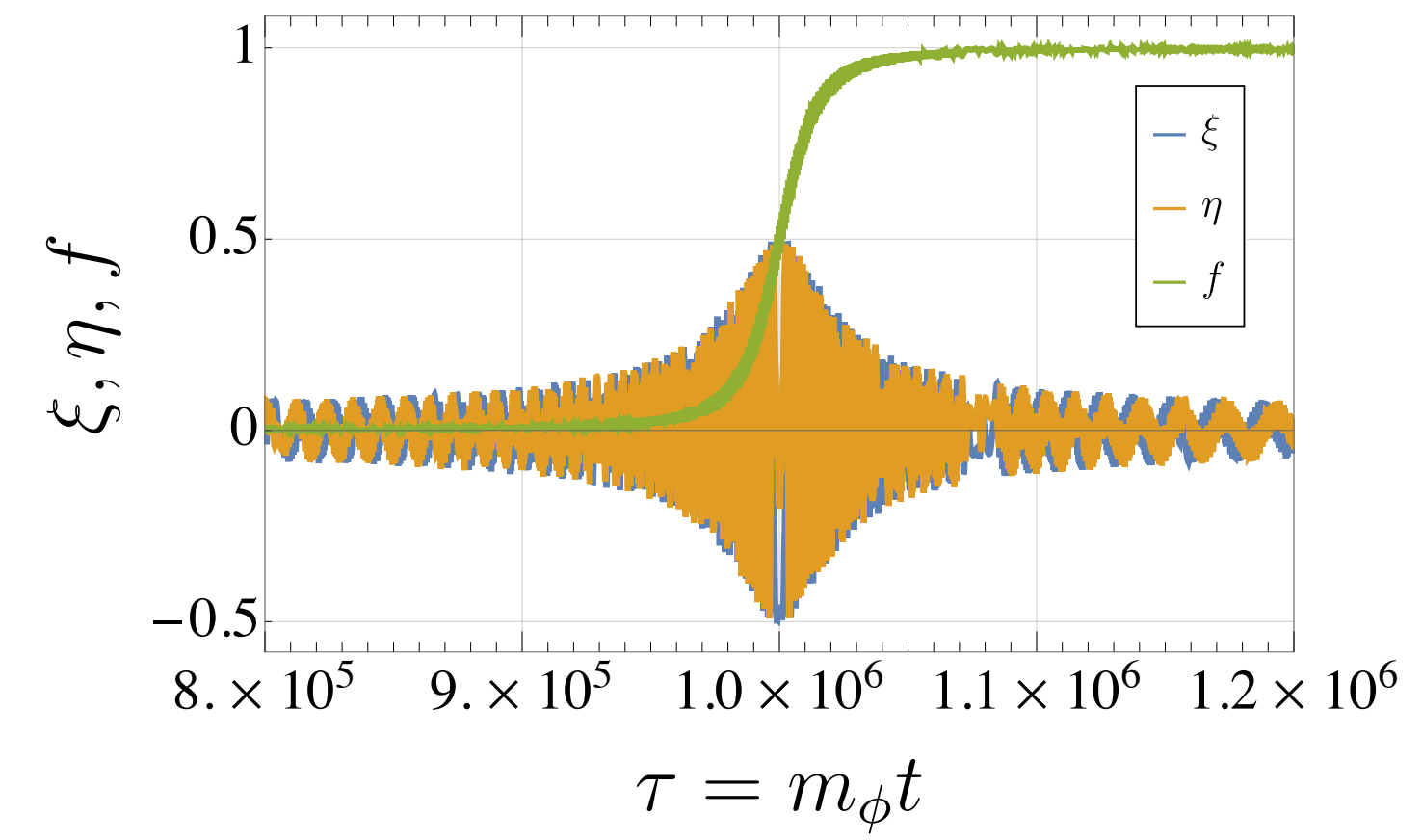
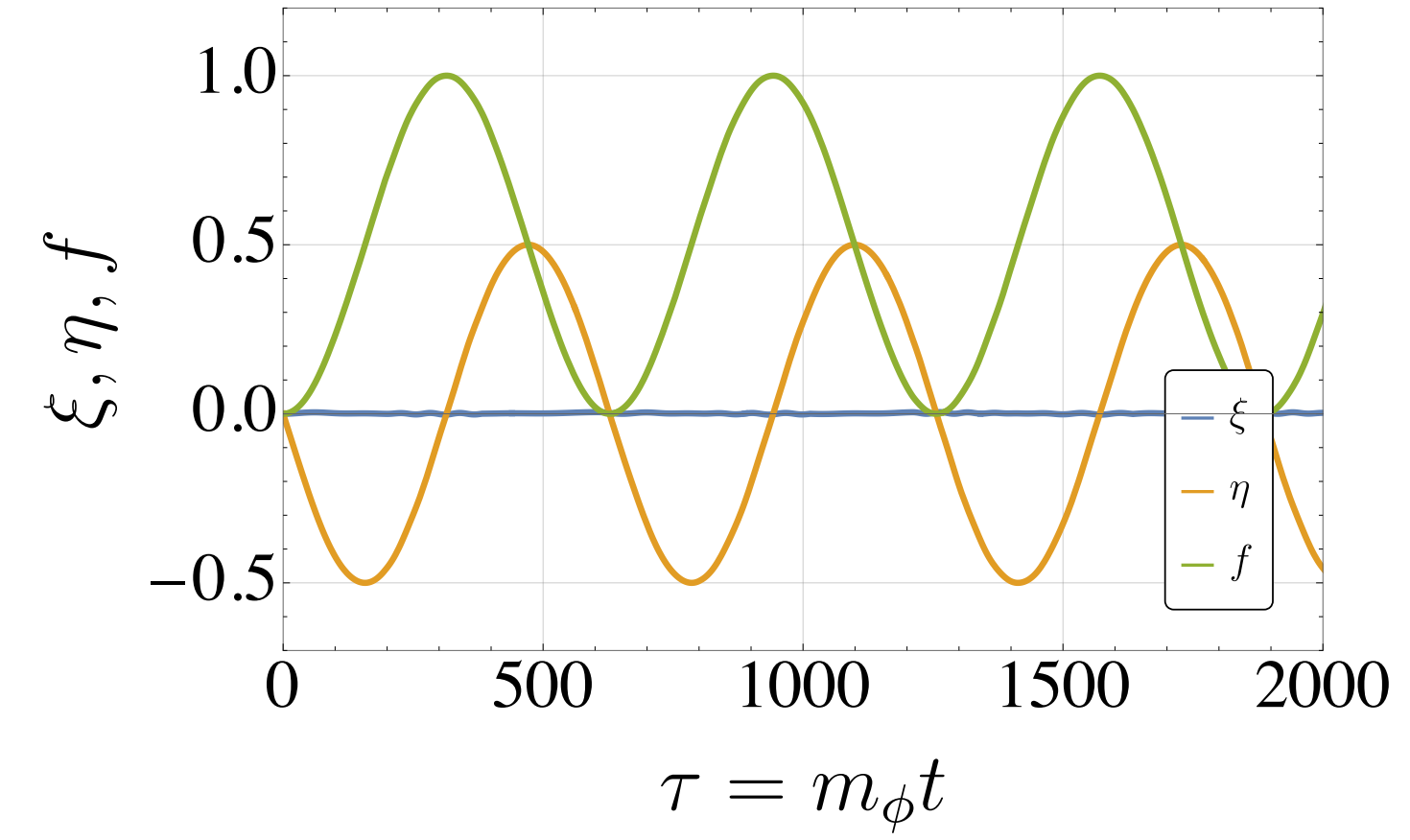
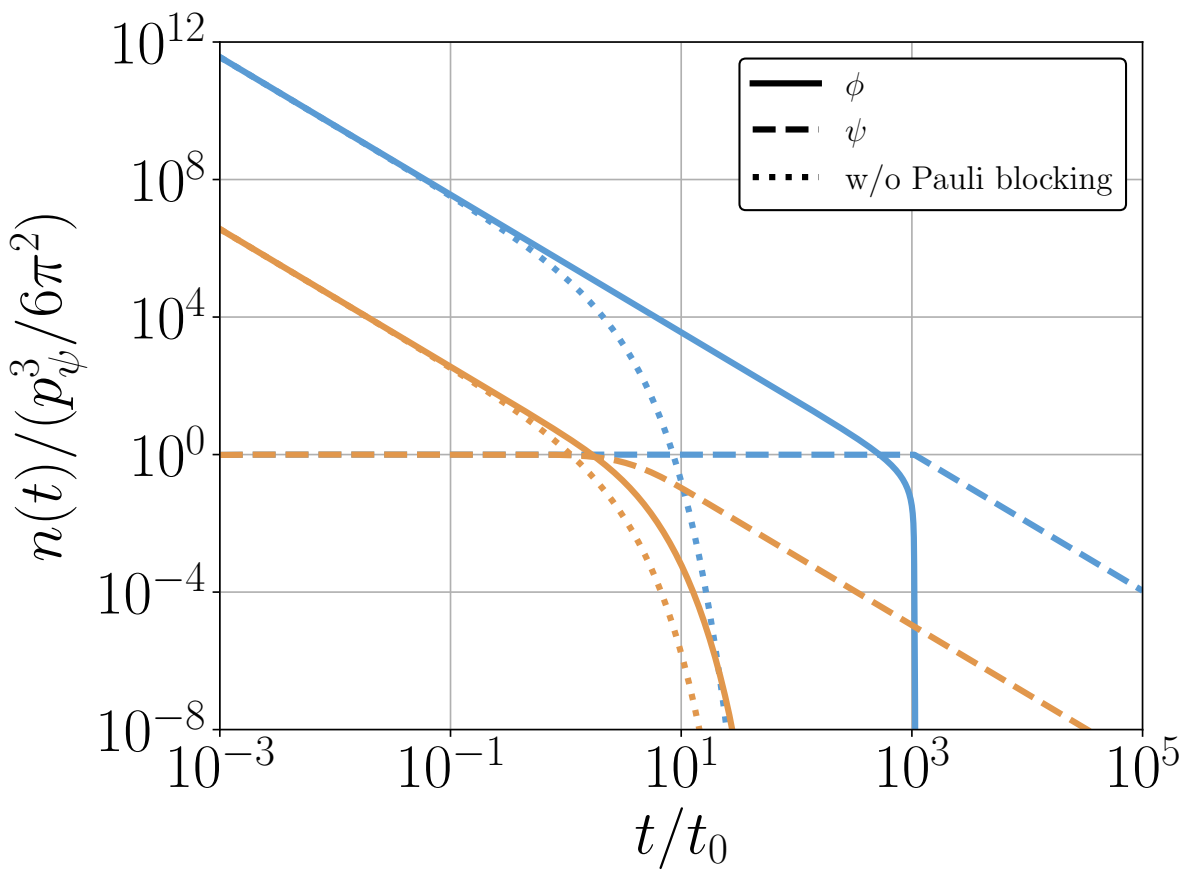
Boson case: Ref. Takeo Moroi, et al. JHEP 03 (2021), 296

*assumption: $m_\psi = 0$

Comparison between Boltzmann & QFT



	Boltzmann	QFT
w/o Expansion	$f_\psi \rightarrow 1/2$	Oscillate & $\langle f_\psi \rangle \rightarrow 1/2$
w/ Expansion	$f_\psi \rightarrow 1/2$	$f_\psi \rightarrow 1$



Example_Type-II seesaw mechanism

- Lagrangian of Yukawa interaction:

$$\mathcal{L}_{\text{Yukawa}} \supset -\frac{1}{\sqrt{2}} Y_{\Delta}^{ij} \bar{l}_L^{iC} \cdot \Delta l_L^j + h.c.$$

l_L : left-handed lepton doublet

Δ : triplet Higgs field

Φ : SM doublet Higgs

- Left-handed neutrino mass matrix:

$$(\mathcal{M}_{\nu})_{ij} = Y_{\Delta}^{ij} v_{\Delta}$$

S : Real singlet scalar
(dark matter candidate)

- The scalar potential:

$$V = V_1 + V_2$$

$$\begin{aligned} V_1 = & -\hat{\mu}_1^2 |\Phi|^2 + \frac{1}{2} \lambda_1 |\Phi|^4 \\ & + \hat{\mu}_3^2 \text{Tr}(\Delta^\dagger \Delta) + \frac{1}{2} \Lambda_1 \left(\text{Tr}(\Delta^\dagger \Delta) \right)^2 + \frac{1}{2} \Lambda_2 \left(\left(\text{Tr}(\Delta^\dagger \Delta) \right)^2 - \text{Tr}[(\Delta^\dagger \Delta)^2] \right) \\ & + \Lambda_4 |\Phi|^2 \text{Tr}(\Delta^\dagger \Delta) + \Lambda_5 \Phi^\dagger [\Delta^\dagger, \Delta] \Phi - \frac{\Lambda_6}{\sqrt{2}} (\Phi^T \cdot \Delta \Phi + \text{H.c.}) \\ V_2 = & \frac{1}{2} \mu_S^2 S^2 + \frac{1}{4} \lambda_S S^4 + \frac{1}{2} \lambda_{S1} S^2 |\Phi|^2 + \frac{1}{2} \lambda_{S\Delta} S^2 \text{Tr}(\Delta^\dagger \Delta) - \frac{\lambda_6}{\sqrt{2}} S (\Phi^T \cdot \Delta \Phi + \text{H.c.}) \end{aligned}$$

Example_Type-II seesaw mechanism

- Neutral component: S, Φ and $\Delta \rightarrow s, \phi^0$ and δ^0
- The unitary matrix diagonalizing the mass-squared matrix:

$$U = \begin{pmatrix} U_{s1} & U_{s2} & U_{s3} \\ U_{\phi^0 1} & U_{\phi^0 2} & U_{\phi^0 3} \\ U_{\delta^0 1} & U_{\delta^0 2} & U_{\delta^0 3} \end{pmatrix}$$

- Decay rate:

$$\Gamma(\phi_1 \rightarrow \nu\nu) \sim \frac{1}{4\pi} |Y_\Delta U_{\delta^0 1}|^2 m_{\phi_1}$$

$$\sim \frac{m_{\phi_1}^2 m_\nu^2}{4\pi v_\Delta^2} |U_{\delta^0 1}|^2$$

$$\simeq 8 \times 10^{-43} \text{ GeV} \left(\frac{m_{\phi_1}}{10^{-3} \text{ eV}} \right) \left(\frac{m_\nu}{10^{-6} \text{ eV}} \right)^2 \left(\frac{1 \text{ keV}}{v_\Delta} \right)^2 \left(\frac{|U_{\delta^0 1}|^2}{10^{-11}} \right)$$

$$\text{where } U_{\delta^0 1} \sim \frac{M_{\delta^0}^2}{M_{\delta^0 \delta^0}^2} \simeq \frac{\lambda_6 v^2}{\hat{\mu}_3^2} \simeq 10^{-5} \left(\frac{\lambda_6}{0.1} \right) \left(\frac{10^4 \text{ GeV}}{\hat{\mu}_3} \right)^2$$

$$\Rightarrow \Gamma^{-1} > \mathcal{O}(10)t_0$$

Example_100% Dark Matter

$$\tau_\phi \simeq 132 \times t_0 \left(\frac{10^{-3} \text{ eV}}{m_\phi} \right)^2 (\text{MD}), \quad \tau_\phi \simeq 3.8 \times t_0 \left(\log \left(\frac{m_\phi}{10^{-3} \text{ eV}} \right)^4 \right) (\text{DE})$$