Cosmological Phase Transition and Gravitational Wave

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Self-introduction

Hello. My name is Kohei Fujikura (藤倉浩平).

I am currently working on the Hamiltonian formulation of lattice gauge theories with tensor network approach.

Outline

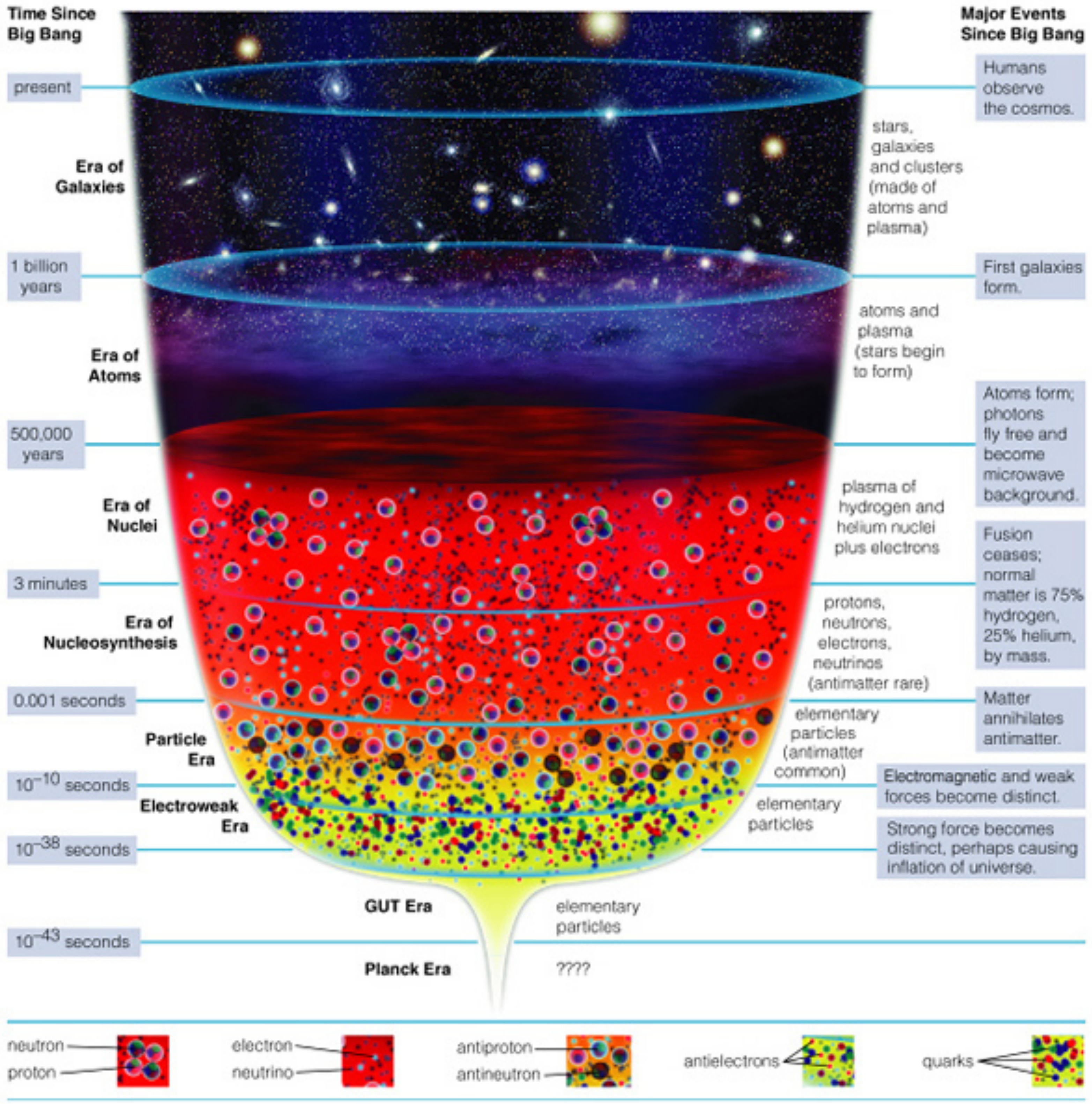
- 1: What is the order of phase transition?

- 2: Hot phase diagrams in the SM, and gravitational wave spectra from first-order phase transitions

- 3: Phase transitions in Inflationary universe, (and the Hamiltonian formulation of lattice gauge theories)

Cosmological Phase Transition

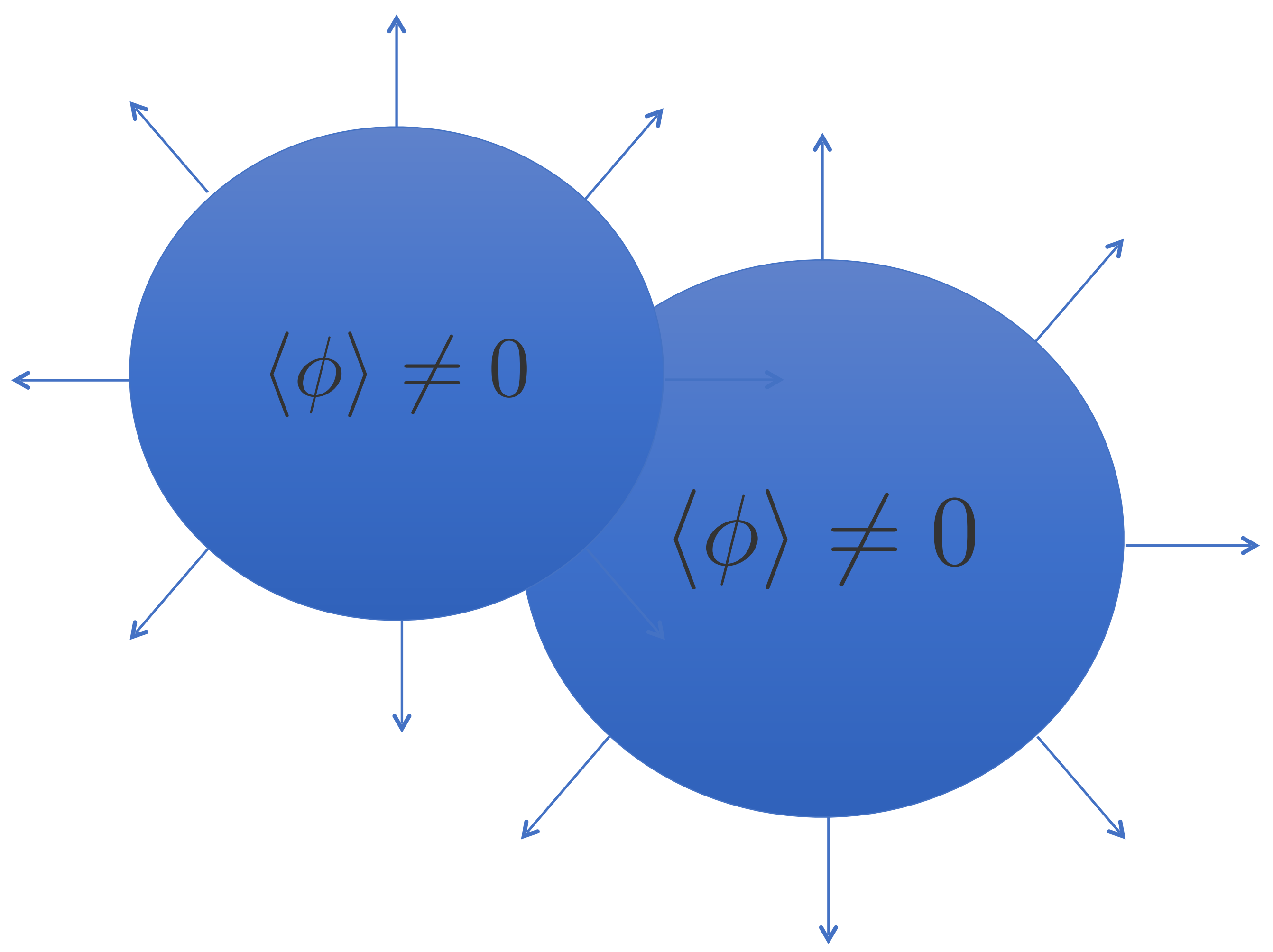
Early universe; finite temperature and (possibly finite density) system

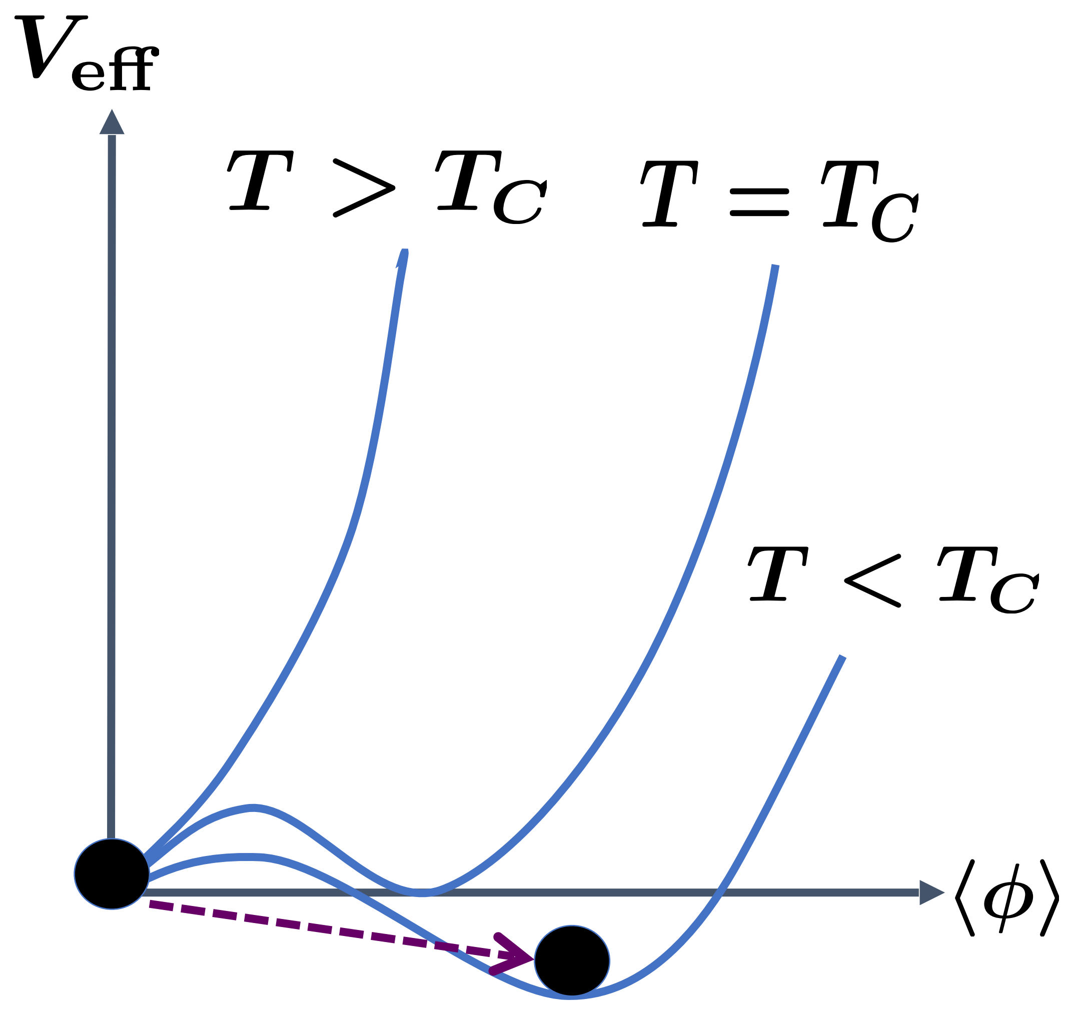


Due to the cosmic expansion, a cosmic temperature cools down.

Cosmological Phase Transitions

Cosmological first order phase transition ⇒ Formation of bubbles





Very roughly speaking, bubble nucleations take place when the temperature of the Universe $\simeq$ mass scale of symmetry breaking

(There exists exceptions.)

After bubble nucleations, they expand due to a pressure.

Phase Transitions in BSM

In this summer institute, many researchers give interesting presentations related to first-order cosmological phase transitions.

10 Presentations!

I hope that this lecture is helpful to understand these presentations.

1’st lecture: Order of thermal phase transition

Outline

- Phase Transition and Thermal Field Theory

- Simple model

- Application to the QCD chiral phase transition

A main claim:
Order of thermal phase transition
$\simeq$ Infrared behavior of three-dimensional field theory

Introduction: Phase Transitions

Q. What is a (definition of) phase transition?

Textbook answer: n'th order phase transition takes place when n'th derivative of a free energy with respect to the temperature (or other parameters) becomes singular.

Landau's answer: Singularities of a free energy are caused by change of underlying global symmetries of the ground state.

I would like to review Landau's answer from now on.

Landau's argument (1)

A phase transition takes place when Φ develops an expectation value.

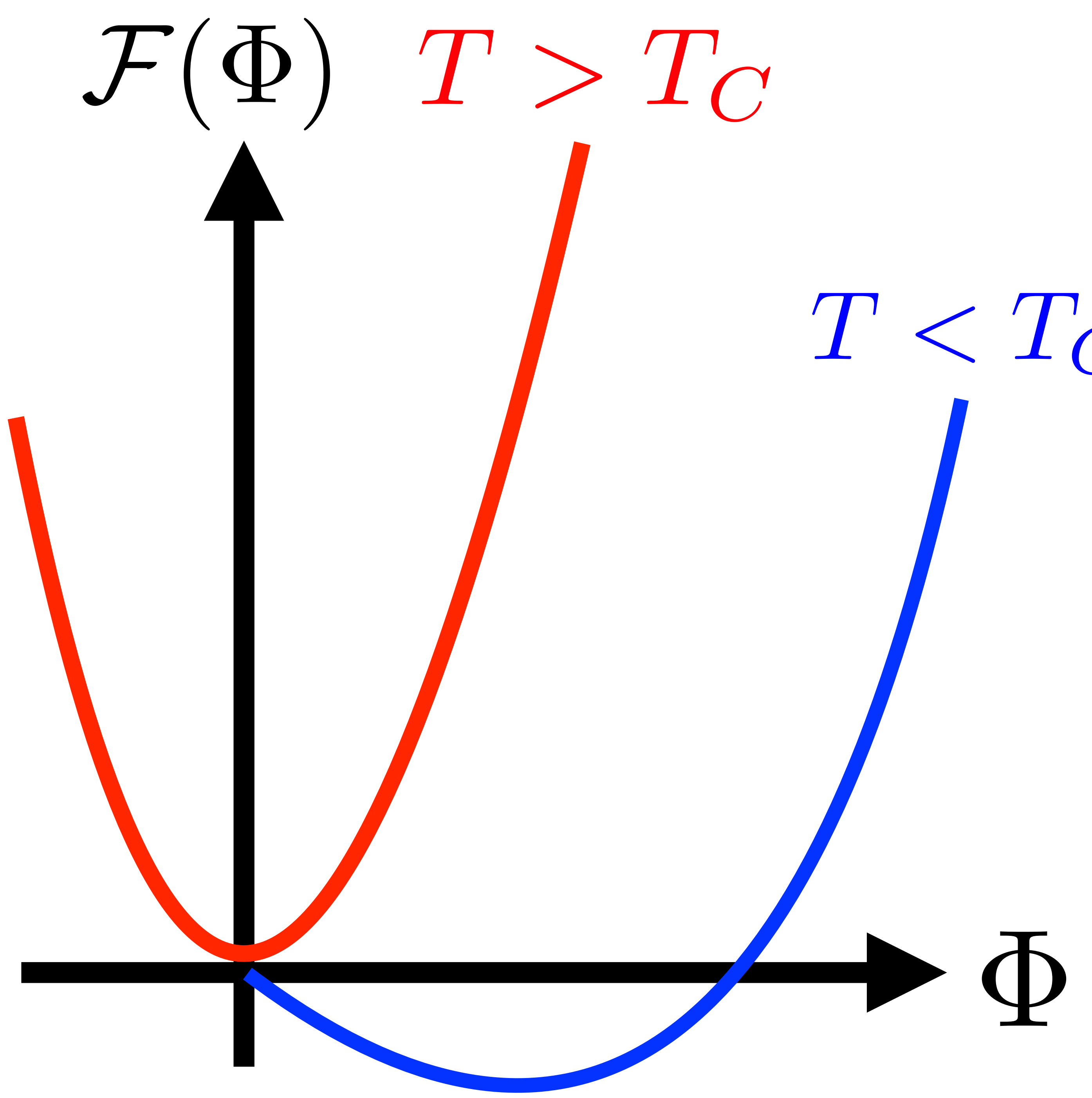
Landau assumes a simple polynomial form of a free energy:

Ex) Ferromagnetic system (Ising model)

$$\mathscr{F}(\Phi) = m^2(T)\Phi^2 + \lambda(T)\Phi^4 + \kappa(T)\Phi^6 + \cdots$$

Z_2−Symmetric phase: $m^2(T > T_C) > 0 \Rightarrow \langle\Phi\rangle = 0$

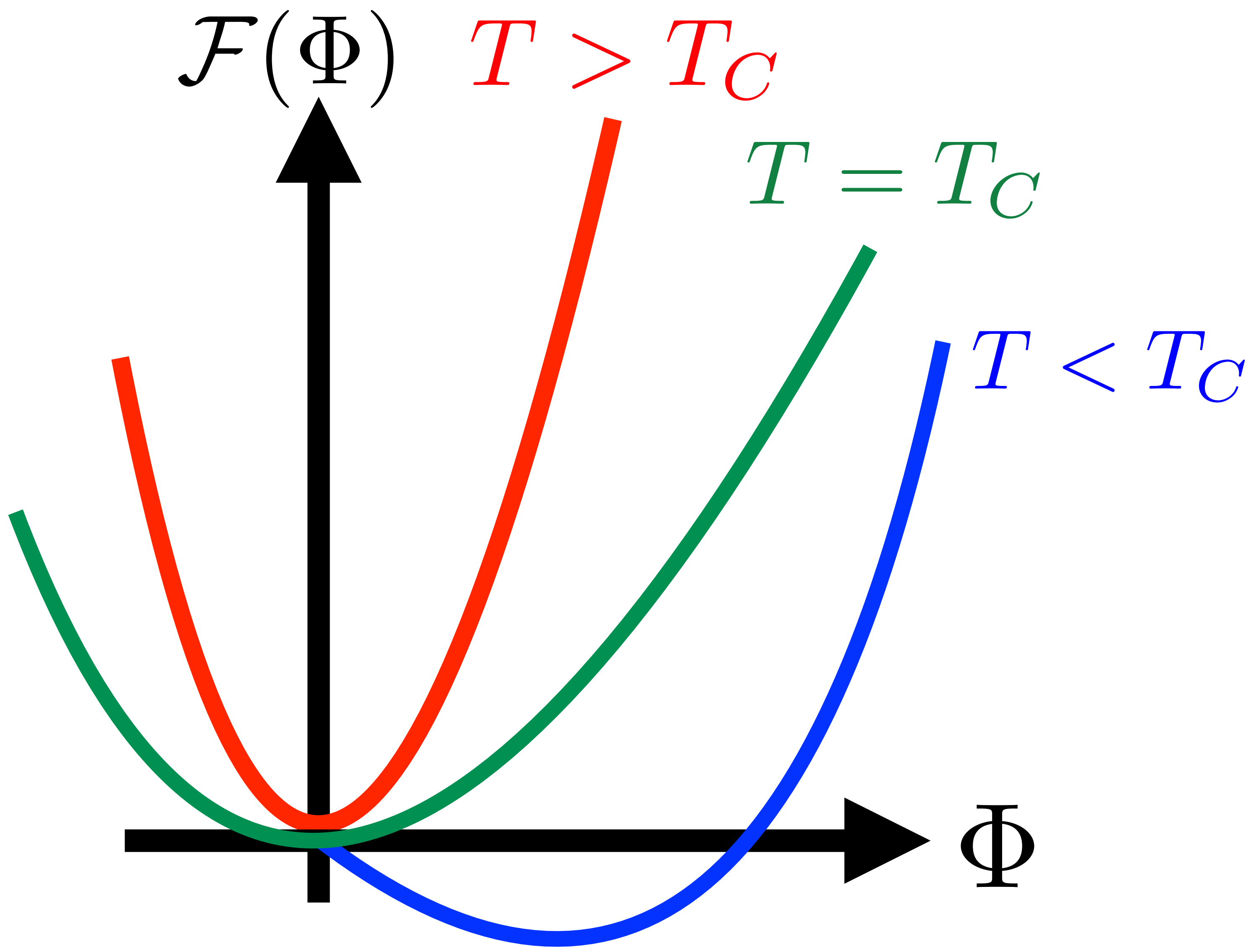
Z_2−Broken phase: $m^2(T < T_C) < 0 \Rightarrow \langle\Phi\rangle \neq 0$

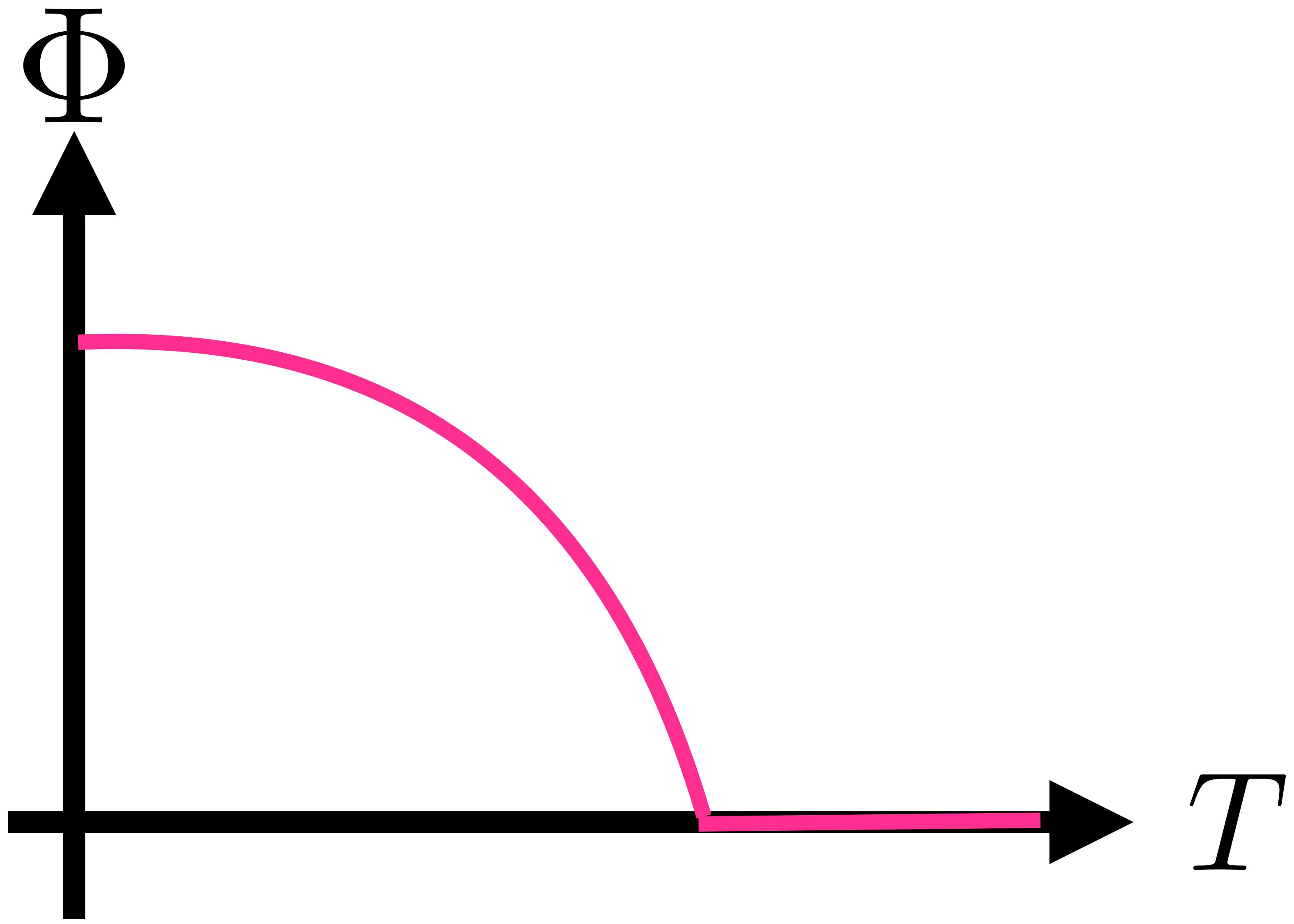


Landau's argument (2)

$$\mathscr{F}(\Phi) = m^2(T)\Phi^2 + \lambda(T)\Phi^4 + \kappa(T)\Phi^6 + \cdots$$

$\lambda(T) > 0$: 2'nd order phase transition takes place.



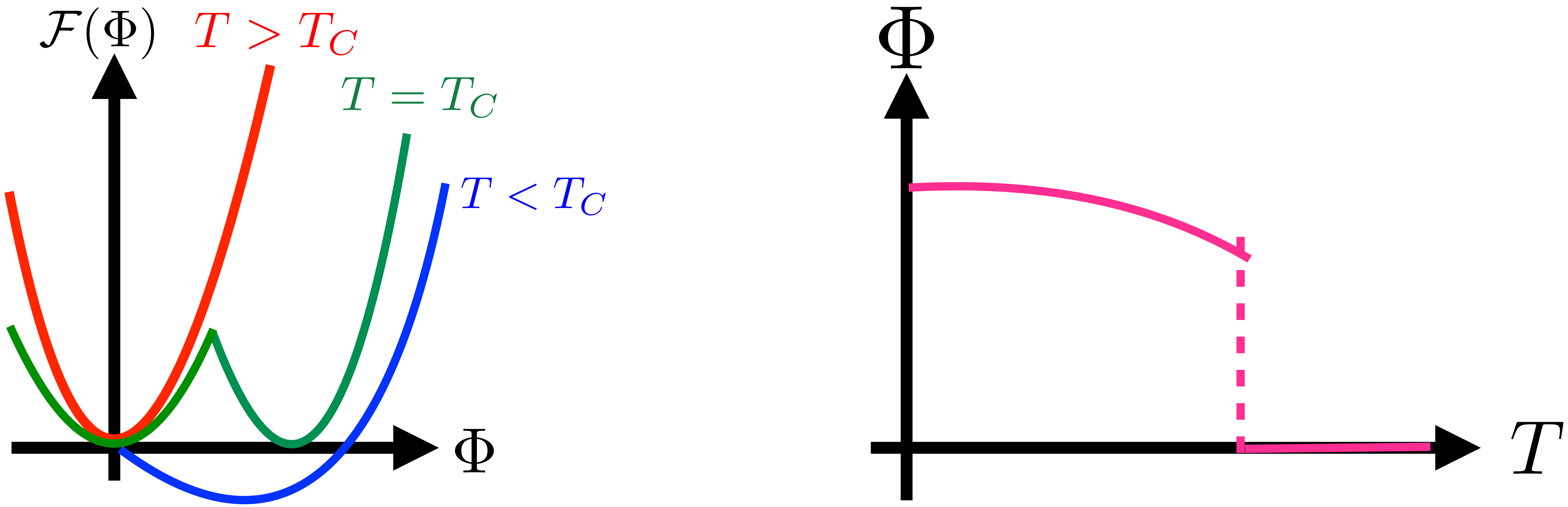


A salient feature of the 2'nd phase transition:
A correlation length diverges $1/m(T \to T_c) \to \infty$ at critical point.
(Order parameter is continuous, but its derivative is not for T.)

Landau's argument (3)

$$\mathscr{F}(\Phi) = m^2(T)\Phi^2 + \lambda(T)\Phi^4 + \kappa(T)\Phi^6 + \cdots$$

$\lambda(T) < 0,\ \kappa(T) > 0$: 1'st order phase transition takes place.

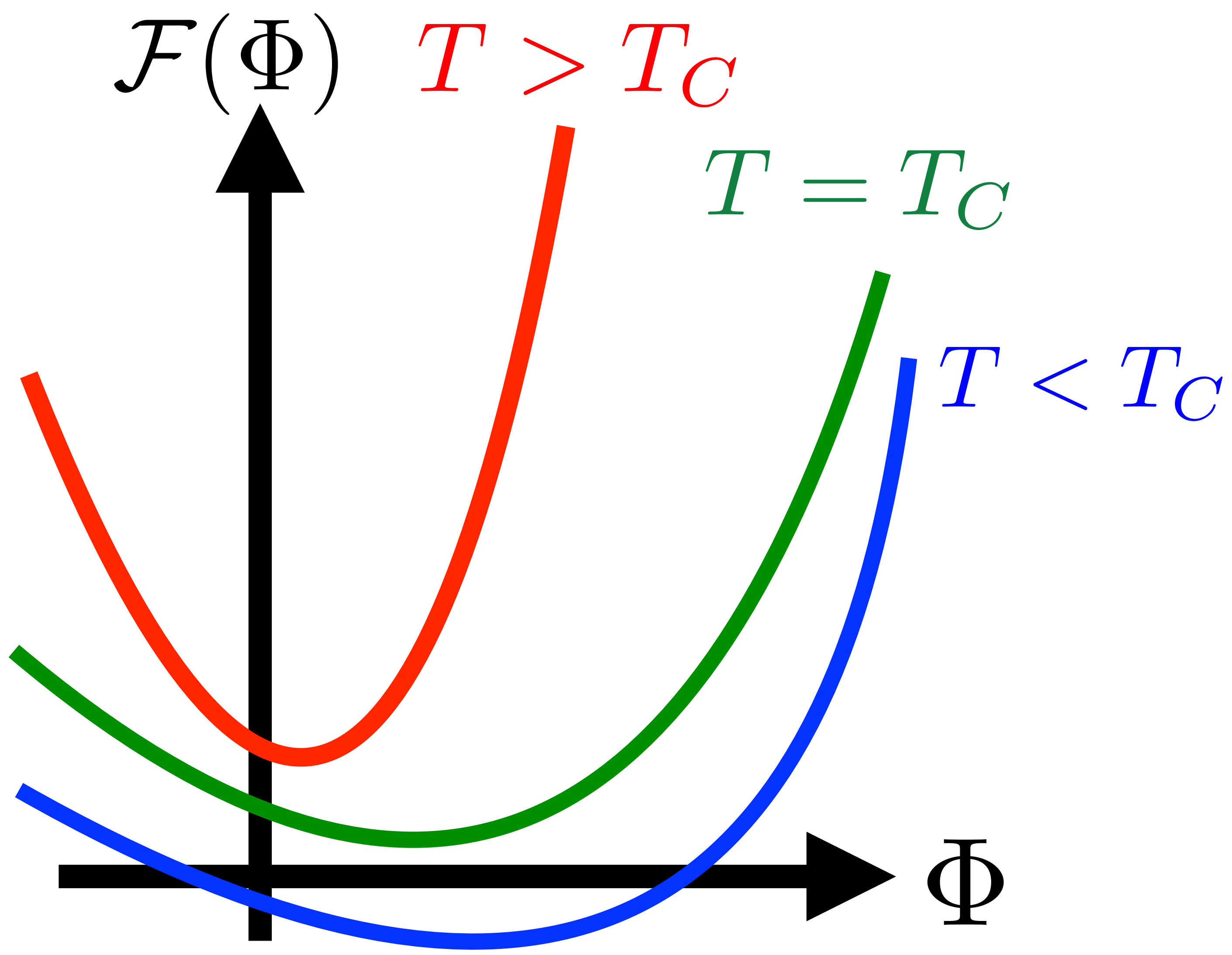


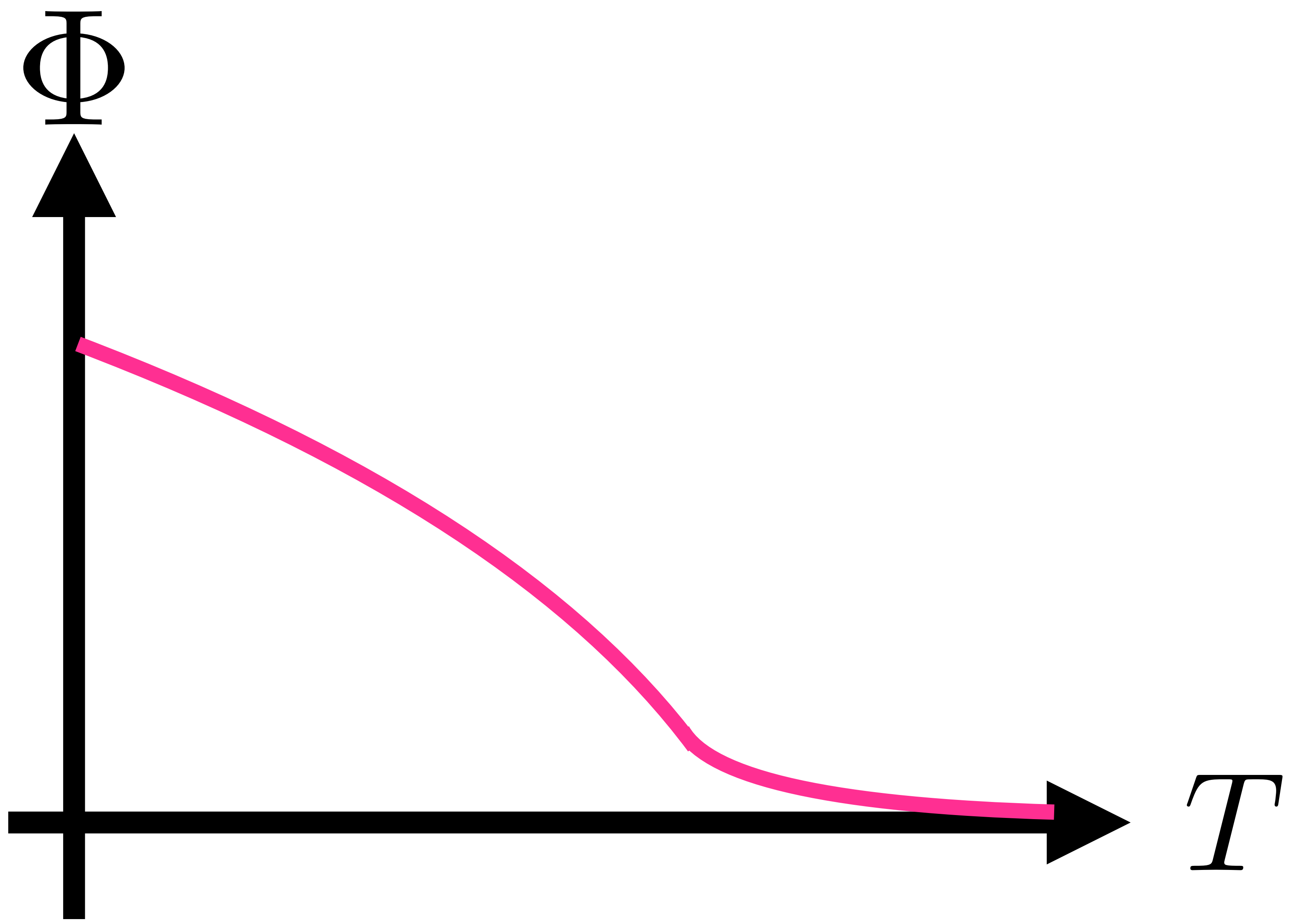
A salient feature of the 1'st phase transition:
A correlation length is always finite, but it jumps!
(Order parameter also jumps!)

A smooth crossover

$$\mathscr{F}(\Phi) = J\Phi + m^2(T)\Phi^2 + \lambda(T)\Phi^4$$

When there exists the operator which explicitly breaks the symmetry, there is no definite phase transition, or 1'st order phase transition takes place (depending on the parameter).





No global symmetry ~ crossover or 1'st order

Landau Free Energy

In principle, we can compute the Landau free energy from the first-principle approach for a given theory (action).

$$S[\phi, A_\mu, \psi, \cdots]$$

↓ Thermal effect

$$F[\phi, A_\mu, \psi, \cdots]$$

Method: imaginary time (Matsubara) formalism

Thermal effect (1)

Thermal effect ~ Gibbs state

Gibbs state is described by the following density matrix:

$$\rho = \sum_i e^{-H/T} |i\rangle \langle i|$$

H: The total Hamiltonian of the theory

i labels the (all possible) physical state.

An expectation value is understood as the canonical ensemble:

$$\langle \hat{O} \rangle = \mathrm{tr}\left(\hat{\rho}\, \hat{O} \right)$$

Thermal effect (2)

In an infinite temperature limit $T \to \infty$, the density matrix becomes

$$\rho = \mathbf{1} = \sum_i |i\rangle\langle i|$$

Any states is equally appears.

Expectation values are invariant under any unitary (internal) transformations.

$$\langle \hat{O} \rangle \to \langle \hat{U}^\dagger \hat{O} \hat{U} \rangle = \mathrm{Tr}(\hat{U}^\dagger \hat{O} \hat{U} \mathbf{1}) = \mathrm{Tr}(\hat{O} \mathbf{1}) = \langle \hat{O} \rangle.$$

This is the intuition of symmetry restoration at very high-temperature.

Thermal effect (3)

In the zero temperature limit $T \to 0$, the density matrix becomes

$$\rho = \sum_i e^{-H/T} |i\rangle\langle i| \to e^{-E_0/T} |0\rangle\langle 0| \ (T \to 0)$$

Only the ground state $|0\rangle$ appears.

Therefore, $\langle \dots \rangle_{T\to 0}$ means an expectation value with respect to the ground state. ($i\epsilon$-prescription)

(From this consideration, it is obvious that the finite temperature theory is identical to the ordinary QFT at $T \to 0$.)

Liouville equation

In quantum theory, time evolution of the density matrix is described by the quantum Liouville equation:

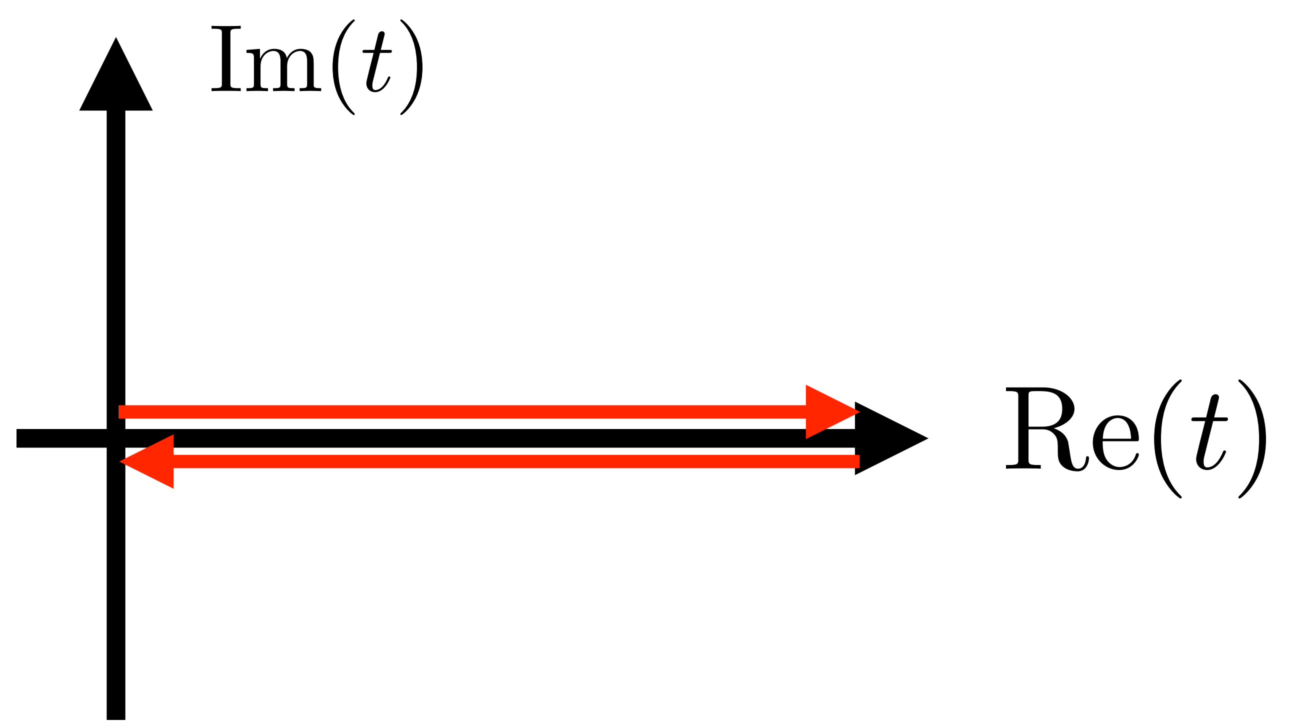
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = -i[\hat{\rho}(t), \hat{H}].$$

A formal solution of this equation:

$$\hat{\rho}(t) = \bar{T}\left[e^{-i\int_{t_i}^{t}\mathrm{d}t\,\hat{H}}\right]\hat{\rho}(t_i)T\left[e^{-i\int_{t_i}^{t}\mathrm{d}t\,\hat{H}}\right].$$

$$T\left[e^{i\int_{t_i}^{t}\mathrm{d}t\,\hat{H}}\right] = \lim_{n\to\infty}\prod_{n,\,t_{n+1}=t,\,t_0=t_i}^{t_{n-1}<t_n} e^{i\hat{H}(t_n-t_{n-1})},$$

$$\bar{T}\left[e^{i\int_{t_i}^{t}\mathrm{d}t\,\hat{H}}\right] = \lim_{n\to\infty}\prod_{n,\,t_{n+1}=t,\,t_0=t_i}^{t_{n-1}>t_n} e^{i\hat{H}(t_n-t_{n-1})}.$$

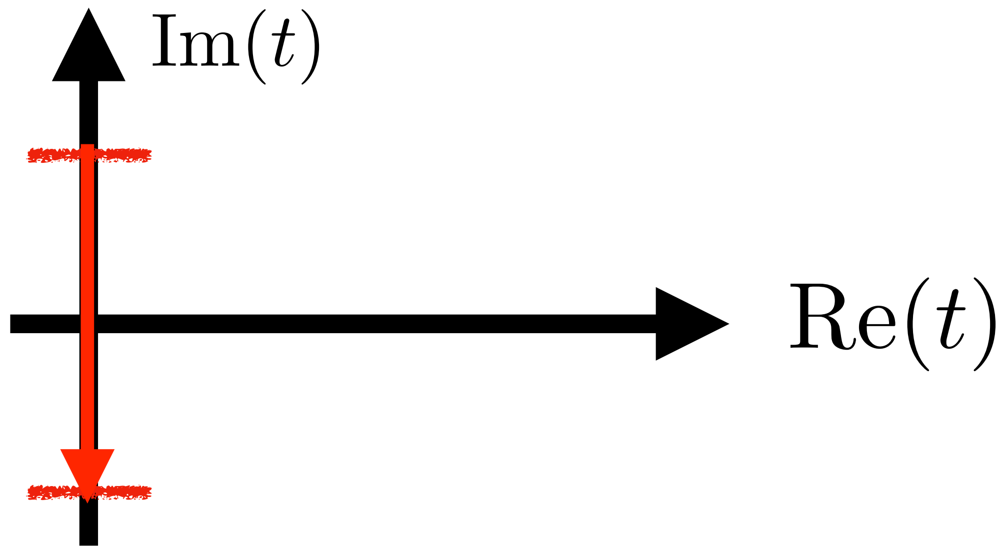


Gibbs state as imaginary time evolution

$$\hat{\rho}(t) = \bar{T}\left[e^{-i\int_{t_i}^{t} \mathrm{d}t\, \hat{H}}\right]\hat{\rho}(t_i)T\left[e^{-i\int_{t_i}^{t} \mathrm{d}t\, \hat{H}}\right].$$

Assumption: The Hamiltonian does not depend on time.

Let us set $\hat{\rho}(t_i) = \mathbf{1}$ and evolve the *imaginary time* from $t_i = 0$ to $t = -i/(2T)$ (for a time-ordered product).



$$\hat{\rho}(t = i/(2T)) = e^{-\hat{H}/(2T)}\mathbf{1}e^{-\hat{H}/(2T)} = e^{-\hat{H}/T}$$

A main claim: Gibbs state can be interpreted as the imaginary time evolution from the maximally entangled state $\hat{\rho} = \mathbf{1}$.

Path-integral Approach (I)

A generic expression is too formal to compute some physical quantities.

For an illustrative purpose, let us consider one-dimensional quantum mechanics.

$$\hat{H}_{(\hat{\pi}_q,\hat{q})} = \frac{1}{2}\hat{\pi}_q^2 + V(\hat{q}).$$

Canonical commutation relation:

$$[\hat{\pi}_q(t), \hat{q}(t)] = -i$$

Coherent state, $\hat{q}\,|q\rangle = q\,|q\rangle$.

Path-integral Approach (II)

Inserting the complete set and perform the integration w.r.t π_q:

$$T\left[e^{-i\int_{t_i}^{t}\mathrm{d}t\,\hat{H}}\right] = \lim_{n\to\infty}\prod_{n,\,t_{n+1}=t,\,t_0=t_i}^{t_{n-1}<t_n} e^{-i\hat{H}(t_n-t_{n-1})} = \#\int Dq\,|q(t)\rangle e^{iS}\langle q(t_i)|\ ,\ S=\int_{t_i}^{t}\mathrm{d}t\left[\frac{\dot{q}^2}{2}-V(q)\right]$$

The density matrix has the following path-integral representation:

$$\hat{\rho}(t) = \#\int Dq_+Dq_-\,|q_-(t_i)\rangle e^{iS_-}\langle q_-(t)|\hat{\rho}(t_i)|q_+(t)\rangle e^{iS_+}\langle q_+(t_i)|,$$

$$S_\pm = \pm\int_{t_i}^{t}\mathrm{d}t\left[\frac{\dot{q}_\pm^2}{2}-V(q_\pm)\right]$$

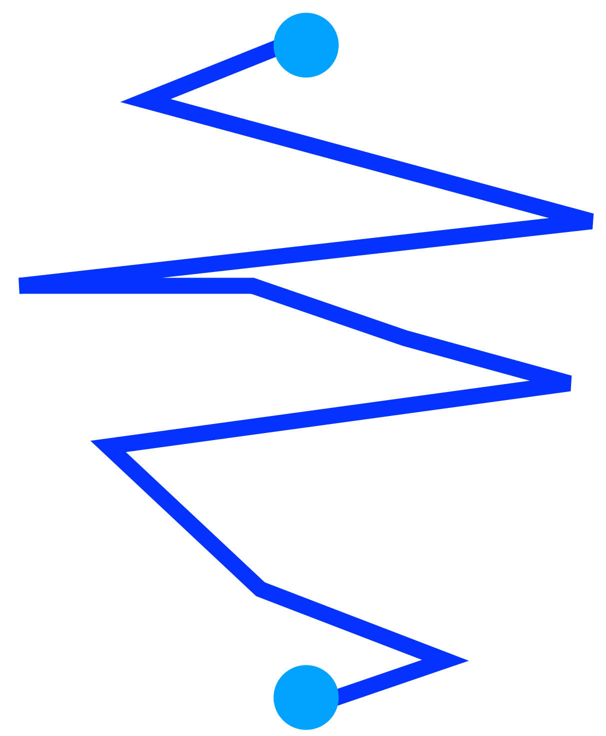
Periodic boundary condition

Putting $\hat{\rho}(t_i) = \mathbf{1}$ and imaginary time evolution:

$$e^{-\hat{H}/T} = \# \int_{q(\tau=-\beta/2)=q(\tau=\beta/2)} Dq\, |q_-(0)\rangle e^{-S_T} \langle q_+(0)|, \;\; S_T = \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \left[\frac{1}{2} \left(\frac{\mathrm{d}q}{\mathrm{d}\tau} \right)^2 + V(q) \right]$$

Hence canonical ensemble can be computed by the standard path integral approach with a periodic boundary condition.

$$Z = \int Dq\, e^{-S_T}$$



●:same point

Caution: Anti-periodic boundary condition is taken for a fermion field.

Field theory at finite-temperature

Quantum field theory ~ each space points has a harmonic oscillator.

$$\hat{q}(t) \to \hat{\phi}(t, \mathbf{x})$$

The thermal partition function is now defined on Euclidean $\mathbf{R}^3 \times \mathbf{S}^1$:

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int D\phi\, e^{-S[\phi]}$$

The summation is taken over all possible field configurations under the boundary condition $\phi(\tau, \mathbf{x}) = \phi(\tau + 1/T, \mathbf{x})$ for bosons.

Outline

- Phase Transition and Thermal Field Theory
- Simple model
- Application to the QCD chiral phase transition

Example: (continuum) Ising model

Let us consider the following example.

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int D\phi \, e^{-S[\phi]}$$

$$S = \int_{-1/(2T)}^{1/(2T)} \mathrm{d}\tau \int \mathrm{d}^3 x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4}\phi^4 \right]$$

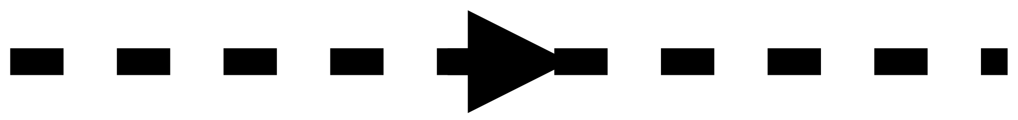
There exists a Z_2 symmetry: $\phi \to -\phi$.

At zero-temperature with $m^2 < 0$ leads to spontaneous symmetry breaking, $\langle \phi \rangle \neq 0$.

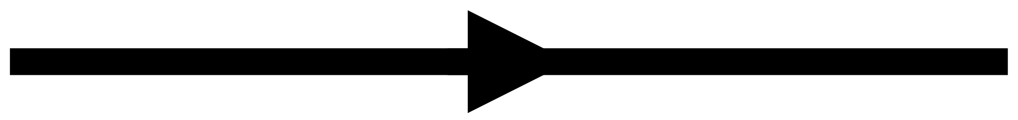
Feynman rule at finite temperature

You can perturbatively evaluate the path integral using the standard Feynman diagram technique.

Because of the periodic (or anti-periodic boundary) condition, propagator is replace by the sum of discrete frequencies.



$$\frac{1}{\omega_{B,n}^2 + \mathbf{p}^2 + m_B^2}$$



$$\frac{\gamma^\mu p_\mu + m_F}{\omega_{F,n}^2 + \mathbf{p}^2 + m_F^2}$$

$$\omega_{B,n} = 2\pi n T$$

$$\omega_{F,n} = 2\pi \left(n + \frac{1}{2}\right) T$$

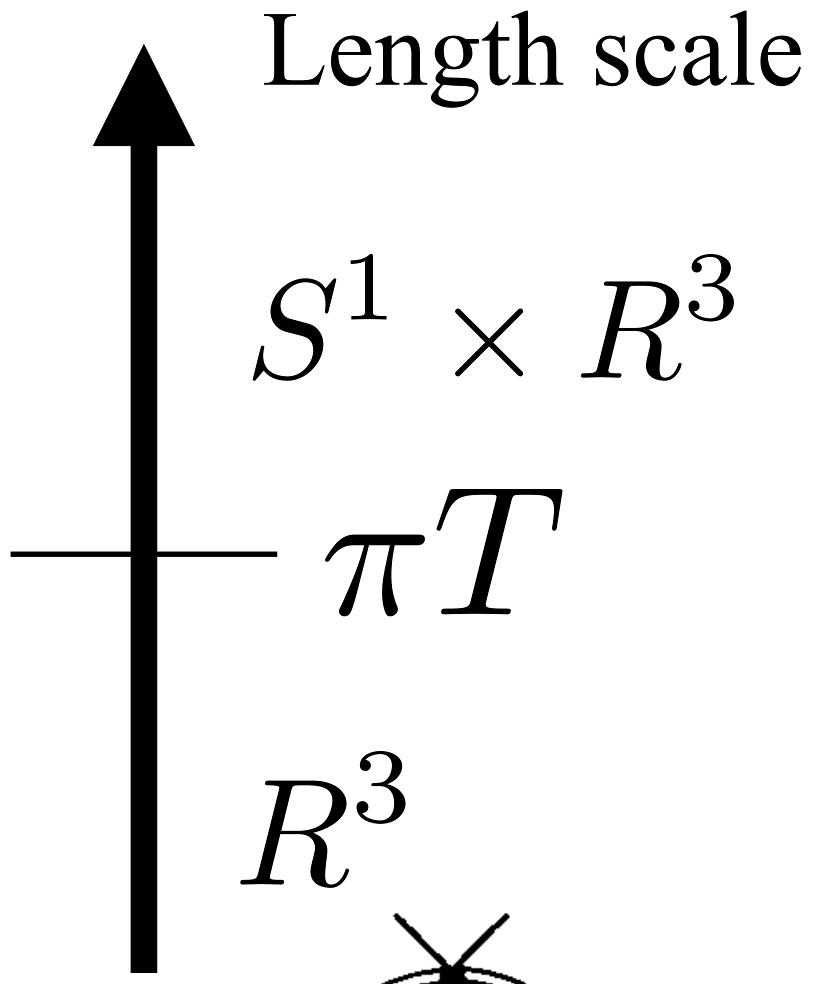
Loop integral: $T \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3 p}{(2\pi)^3}$

There are no zero-modes for fermion fields (anti-periodic condition).

Effective action for zero-mode

Let us focus on the lightest field and integrating out all massive modes.
(Kaluza Klein modes)

[P. Ginsparg (1980)]



$$\phi(\tau, \mathbf{x}) = \sum_{n=-\infty}^{\infty} \left(\phi_n(\mathbf{x}) e^{i\omega_n \tau} + \phi_n^*(\mathbf{x}) e^{-i\omega_n \tau} \right)$$

$\phi_{n=0}$: zero mode (External legs)

$\phi_{n\neq 0}$: Integrated out (Internal lines)

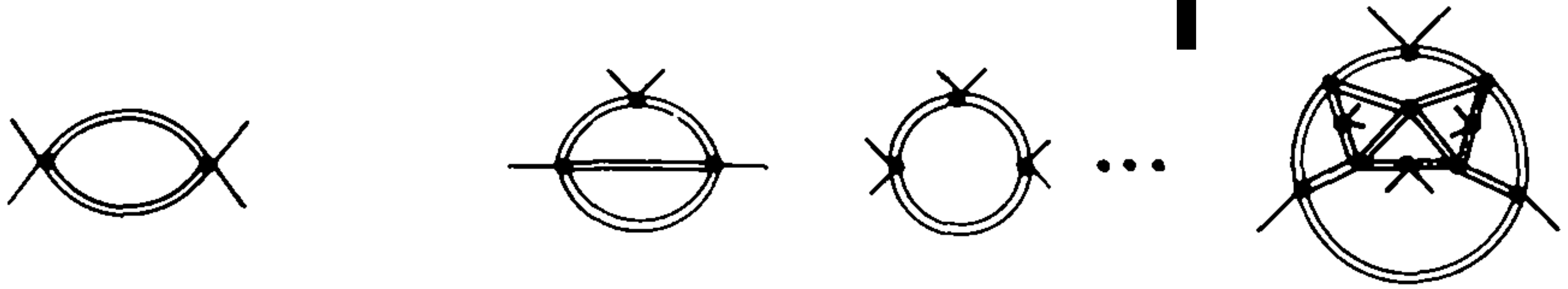


Fig. 4. Some higher-order contributions to $\mathfrak{L}_{eff}$.

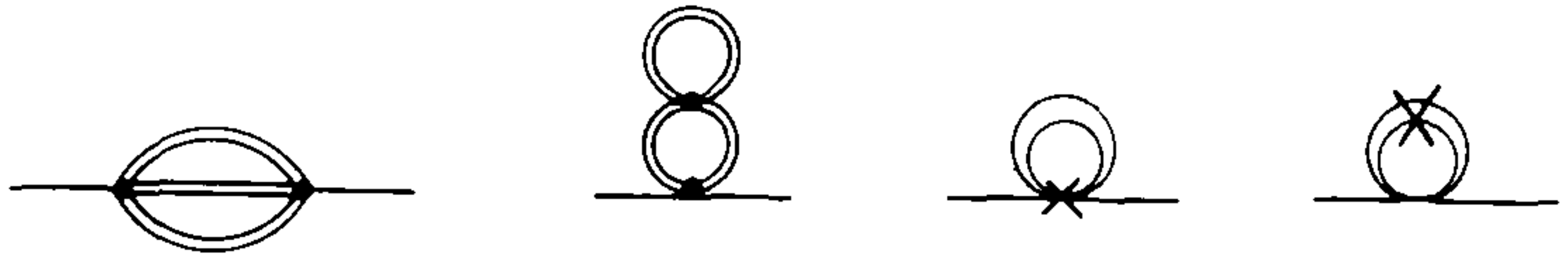


Fig. 5. Two-loop contributions to tree approximation mass term in $\mathfrak{L}_{eff}$.

Landau free energy

The resultant effective action at finite temperature with one-loop:

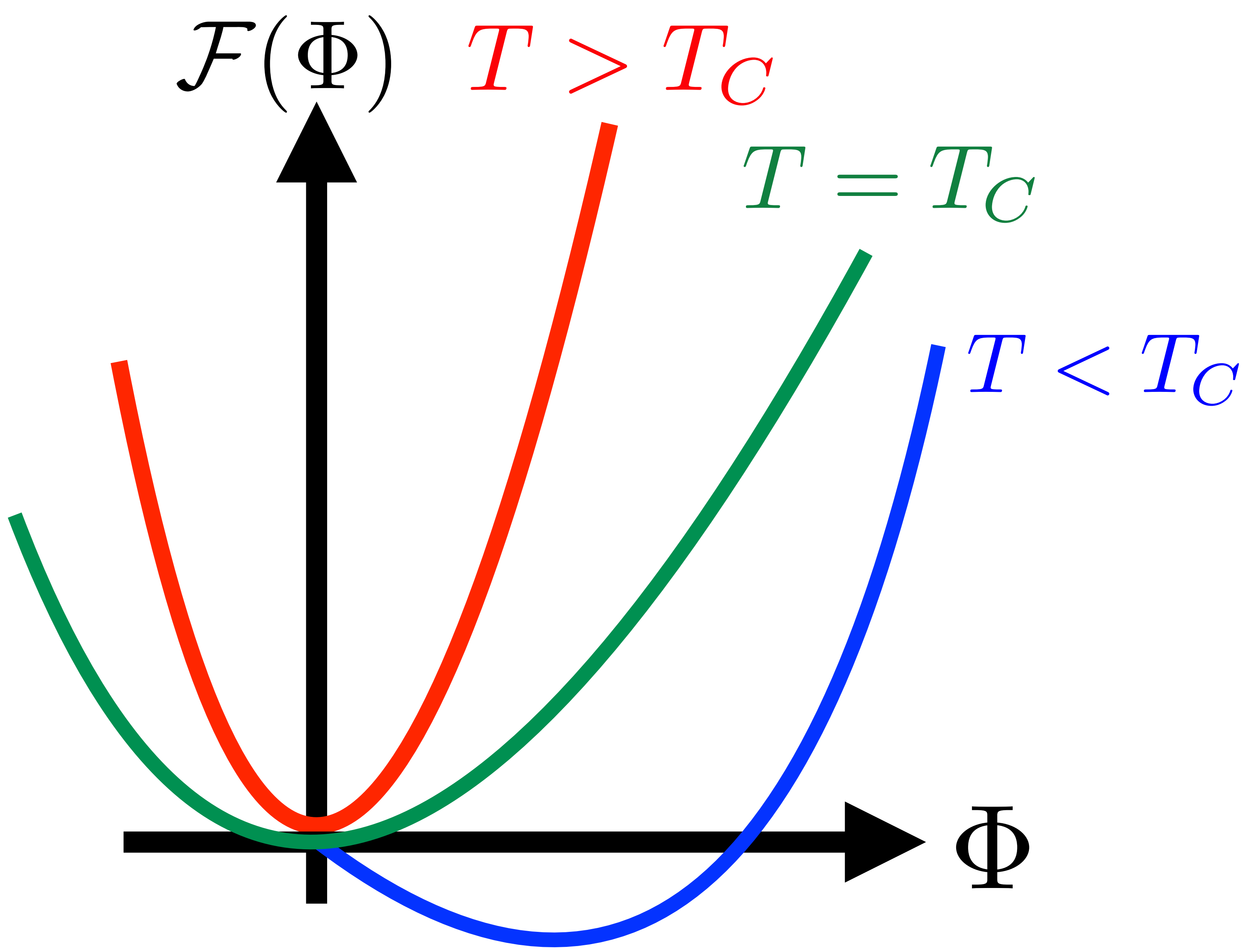
$$Z = \int D\phi_{n=0}\, e^{-S_{\rm eff}[\phi_{n=0}]}$$

$$S_{\rm eff} = \int \mathrm{d}^3\mathbf{x} \left[\frac{1}{2}(\partial\phi_{n=0})^2 + \frac{m_3^2(T)}{2}\phi_{n=0}^2 + \frac{\lambda_3(T)}{4}\phi_{n=0}^4\right]$$

$$m_3^2(T) \sim m^2 + \lambda T^2, \quad (m^2 < 0)$$

$$\lambda_3(T) \simeq \lambda T$$

A mean field approximation:



However, you cannot believe the analysis based on the mean field analysis at $m^2(T_C) = 0$.

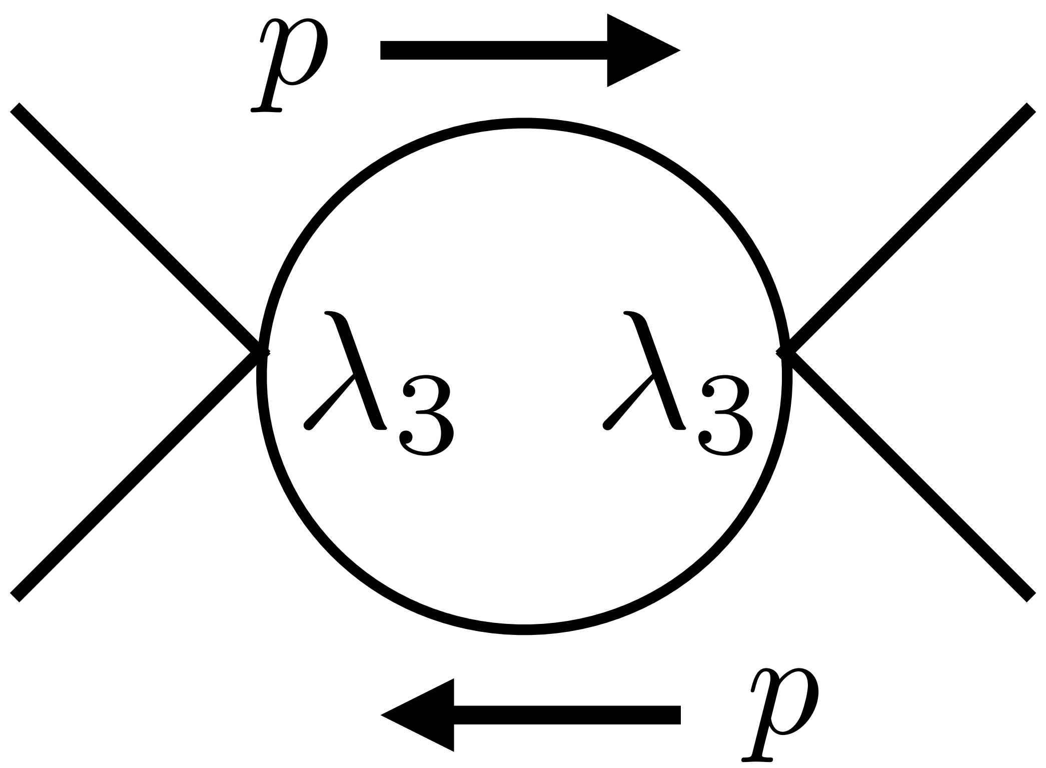
Infrared divergence

There is a non-perturbatively large correction at $m_3^2(T) = 0$.

Perturbative expansion is justified when $\alpha \simeq \lambda_3^2/m_3 < 1$

λ_3 : mass dimension 1/2, m_3 : mass dimension 1.

Example)



$$\sim \int d^3p \, \frac{\lambda_3^2}{(p^2 + m_3^2)^2} \sim \lambda_3^2/m_3$$

A three-dimensional field theory is superenomalizable and is strongly coupled at infrared.

What can we learn from this analysis?

Let us summarize the situation again.

- A finite-temperature field theory is a field theory on $\mathbf{R}^3 \times \mathbf{S}^1$.
- We can obtain effective action by integrating out all massive modes.
- The resultant zero-modes effective action is defined in three spatial dimension and is strongly coupled at infrared (at a transition point).
- (Roughly speaking, $\mathbf{S}^1$ direction shrinks at long distance physics.)

Order of phase transition

An effect of non-perturbative fluctuation should be included.

- Renormalization group (RG) analysis(ϵ-expansion, functional RG method)

- Lattice simulation

- Conformal bootstrap

Renormalization group analysis

A non-perturbative effect is included as the renormalization group effect.

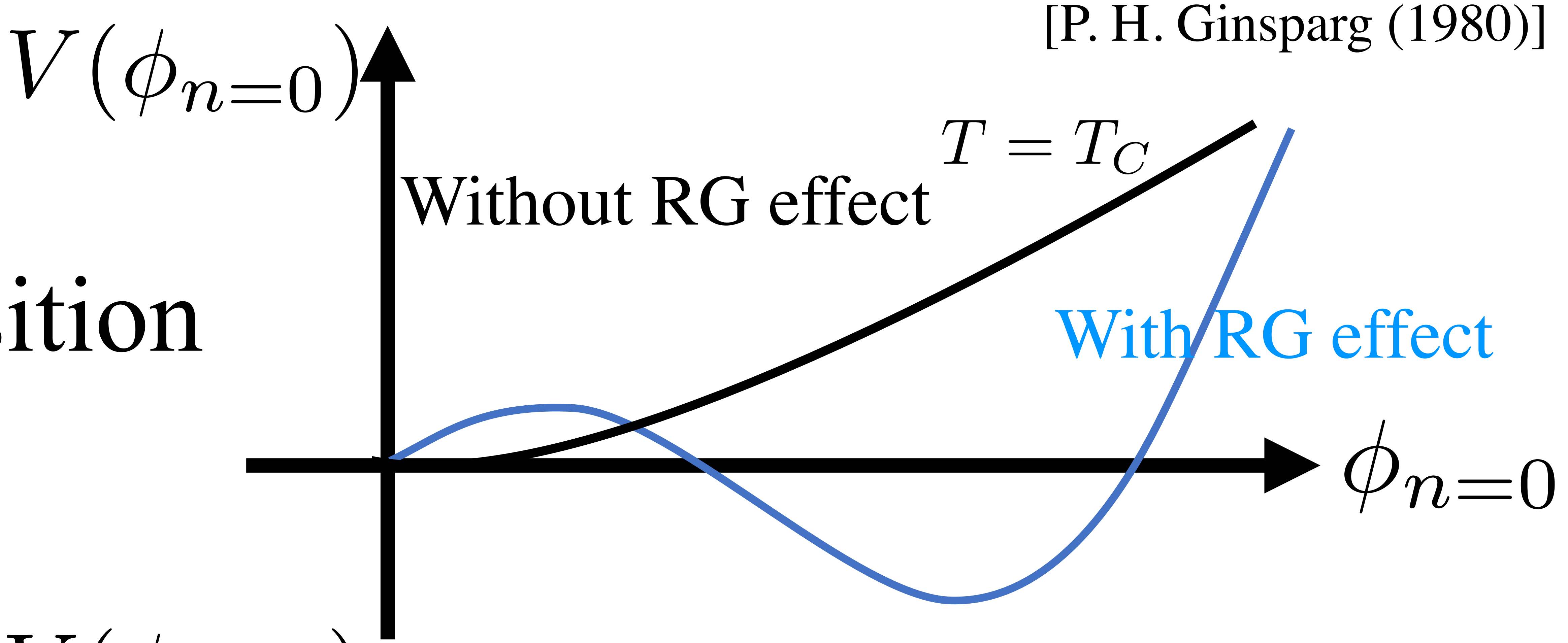
- First-order phase transition

No stable infrared fixed point

Fluctuation-induced first-order phase transition

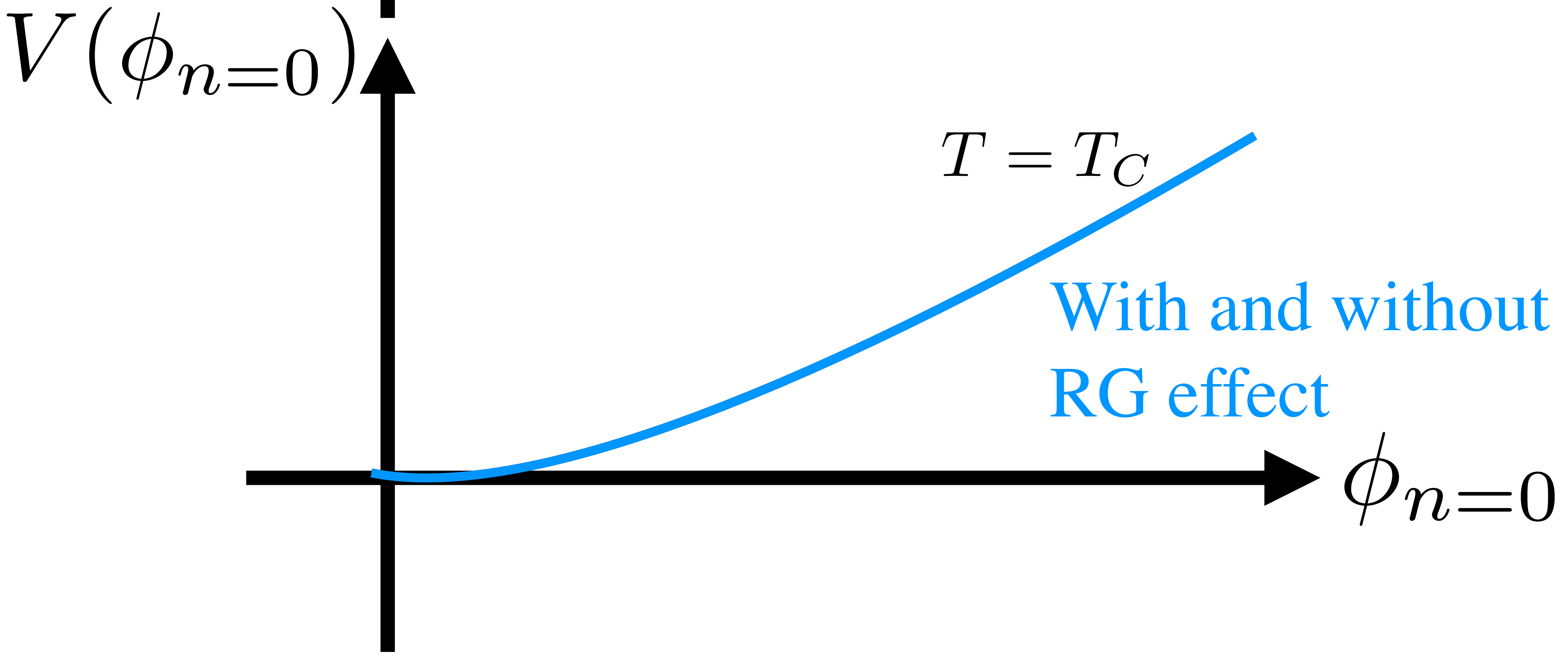
New minimum appears

[P. H. Ginsparg (1980)]



- Second-order phase transition

Attract to the infrared fixed point



Question

Now the problem is the following.
Starting from the three-dimensional effective theory:

$$Z = \int D\phi_{n=0}\, e^{-S_{\rm eff}[\phi_{n=0}]}$$

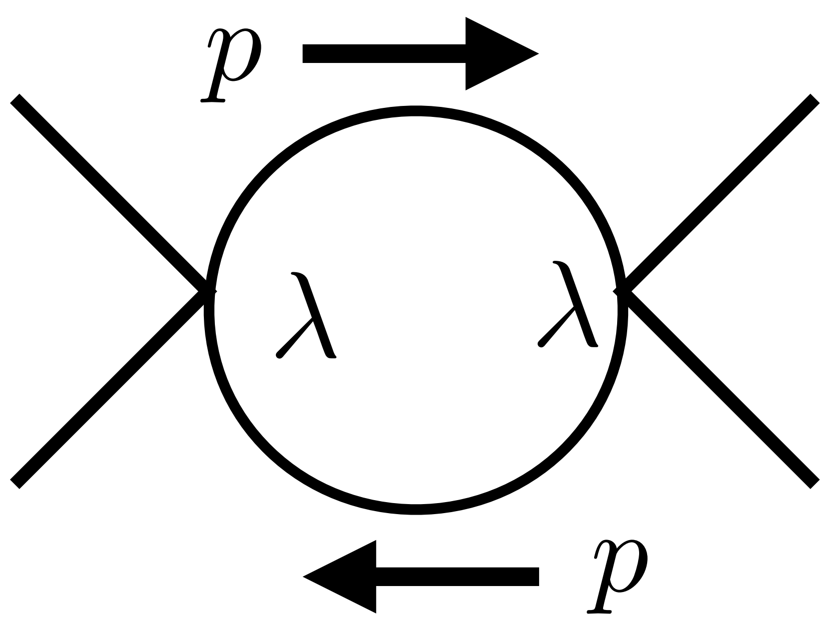
$$S_{\rm eff} = \int \mathrm{d}^3\mathbf{x} \left[\frac{1}{2}(\partial\phi_{n=0})^2 + \frac{m_3^2(T)}{2}\phi_{n=0}^2 + \frac{\lambda_3(T)}{4}\phi_{n=0}^4\right]$$

Is there stable IR fixed point at $m_3^2(T_C) = 0$?

Wilson Fisher fixed point

$\varepsilon-$expansion: compute Feynman diagram in $(4-\varepsilon)$-dim with $\epsilon \ll 1$
(and finally takes the extrapolation $\varepsilon \to 1$).

[K.G. Wilson and M.E. Fisher (1972)]

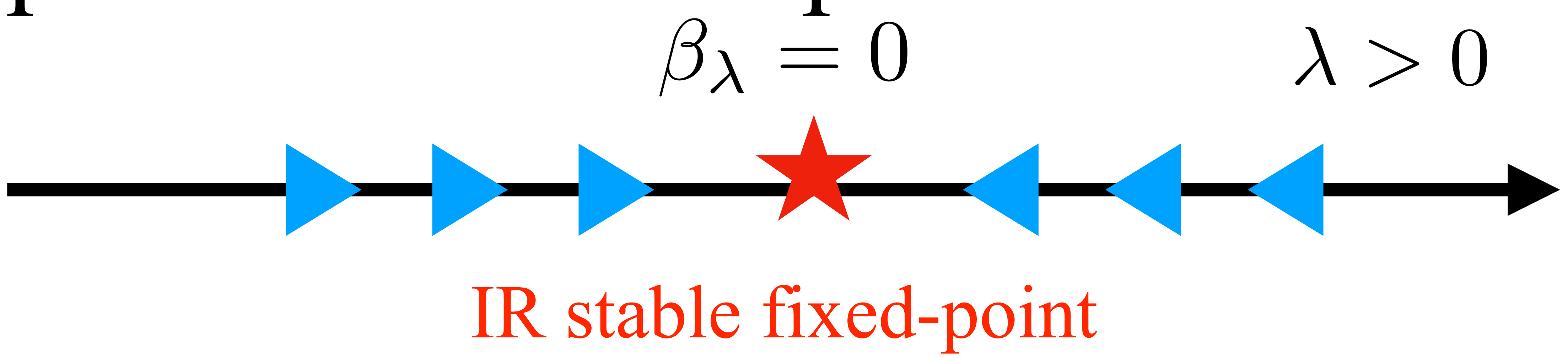


$$\sim \mu^{\epsilon} \int \mathrm{d}^{4-\epsilon} p \, \frac{\lambda^2}{(p^2+m^2)^2}$$

μ : renormalization scale

Renormalization group equation at one-loop:

$$\beta_\lambda \equiv \mu \frac{\partial \lambda}{\partial \mu} = -\epsilon\lambda + 3\frac{\lambda^2}{16\pi^2}$$



At the fixed-point, correlation length diverges.
⇒ 2'nd order transition! (Also confirmed by other methods)

Application: QCD chiral phase transition

Pisarski and Wilczek (1984) applied this analysis to the QCD chiral phase transition. $SU(N_F)_L \times SU(N_F)_R \times U(1)_V \times U_A(1) \to SU(N_F)_V \times U(1)_V$

Order parameter (chiral condensate): $\bar{q}_i q_j \sim \Phi \to U\Phi V^\dagger \; (U, V \in U(N_F))$

Effective action:

$$S_{\rm eff} = \int \mathrm{d}^3\mathbf{x} \left[\mathrm{Tr}(\partial\Phi\partial\Phi^\dagger) + M^2(T)\Phi\Phi^\dagger + u(\mathrm{Tr}\Phi\Phi^\dagger)^2 + v\mathrm{Tr}(\Phi\Phi^\dagger\Phi\Phi^\dagger) + \underline{c_0 \det\Phi + \mathrm{c,c.}}\right]$$

Axial anomaly

(~~$U(1)_A$~~)

Is there infrared stable fixed point? (Order of phase transitions)

QCD phase chiral phase transition

Results (based on the ϵ-expansion with one-loop analysis)

$$S_{\rm eff} = \int \mathrm{d}^3\mathbf{x} \left[\mathrm{Tr}(\partial\Phi\partial\Phi^\dagger) + M^2(T)\Phi\Phi^\dagger + u(\mathrm{Tr}\Phi\Phi^\dagger)^2 + v\mathrm{Tr}(\Phi\Phi^\dagger\Phi\Phi^\dagger) + c_0 \det\Phi + \mathrm{c.c.}\right]$$

	$N_{\rm f} = 2$	$N_{\rm f} \geq 3$
$\mathrm{U(1)_A}$ symmetric ($c_0 = 0$)	Fluctuation-induced 1st order	1st order
$\mathrm{U(1)_A}$ broken ($c_0 \neq 0$)	2nd order [O(4) universality]	1st order

However, this conclusion is based on the extrapolation, $\epsilon \to 1$.

Application to BSM physics

QCD-like dynamics is well-motivated in the physics beyond the standard model. (Composite Higgs model)

List of chiral symmetry breaking pattern $\mathscr{G} \to \mathscr{H}$ in composite Higgs model

$\mathcal{G}$	$\mathcal{H}$	C	N_G	$\mathbf{r}_{\mathcal{H}} = \mathbf{r}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}\ (\mathbf{r}_{\mathrm{SU}(2)\times\mathrm{U}(1)})$	Ref.
SO(5)	SO(4)	✓	4	$\mathbf{4} = (\mathbf{2}, \mathbf{2})$	[11]
SU(3) × U(1)	SU(2) × U(1)		5	$\mathbf{2}_{\pm 1/2} + \mathbf{1}_{\mathbf{0}}$	[10, 35]
SU(4)	Sp(4)	✓	5	$\mathbf{5} = (\mathbf{1}, \mathbf{1}) + (\mathbf{2}, \mathbf{2})$	[29, 47, 64]
SU(4)	$[\mathrm{SU}(2)]^2 \times \mathrm{U}(1)$	✓*	8	$(\mathbf{2}, \mathbf{2})_{\pm \mathbf{2}} = 2 \cdot (\mathbf{2}, \mathbf{2})$	[65]
SO(7)	SO(6)	✓	6	$\mathbf{6} = 2 \cdot (\mathbf{1}, \mathbf{1}) + (\mathbf{2}, \mathbf{2})$	–
SO(7)	G_2	✓*	7	$\mathbf{7} = (\mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2})$	[66]
SO(7)	SO(5) × U(1)	✓*	10	$\mathbf{10}_{\mathbf{0}} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2})$	–
SO(7)	$[\mathrm{SU}(2)]^3$	✓*	12	$(\mathbf{2}, \mathbf{2}, \mathbf{3}) = 3 \cdot (\mathbf{2}, \mathbf{2})$	–
Sp(6)	Sp(4) × SU(2)	✓	8	$(\mathbf{4}, \mathbf{2}) = 2 \cdot (\mathbf{2}, \mathbf{2})$	[65]
SU(5)	SU(4) × U(1)	✓*	8	$\mathbf{4}_{-5} + \bar{\mathbf{4}}_{\mathbf{+5}} = 2 \cdot (\mathbf{2}, \mathbf{2})$	[67]
SU(5)	SO(5)	✓*	14	$\mathbf{14} = (\mathbf{3}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{1})$	[9, 47, 49]
SO(8)	SO(7)	✓	7	$\mathbf{7} = 3 \cdot (\mathbf{1}, \mathbf{1}) + (\mathbf{2}, \mathbf{2})$	–
SO(9)	SO(8)	✓	8	$\mathbf{8} = 2 \cdot (\mathbf{2}, \mathbf{2})$	[67]
SO(9)	SO(5) × SO(4)	✓*	20	$(\mathbf{5}, \mathbf{4}) = (\mathbf{2}, \mathbf{2}) + (\mathbf{1} + \mathbf{3}, \mathbf{1} + \mathbf{3})$	[34]
$[\mathrm{SU}(3)]^2$	SU(3)		8	$\mathbf{8} = \mathbf{1}_{\mathbf{0}} + \mathbf{2}_{\pm 1/2} + \mathbf{3}_{\mathbf{0}}$	[8]
$[\mathrm{SO}(5)]^2$	SO(5)	✓*	10	$\mathbf{10} = (\mathbf{1}, \mathbf{3}) + (\mathbf{3}, \mathbf{1}) + (\mathbf{2}, \mathbf{2})$	[32]
SU(4) × U(1)	SU(3) × U(1)		7	$\mathbf{3}_{-\mathbf{1/3}} + \bar{\mathbf{3}}_{+\mathbf{1/3}} + \mathbf{1}_{\mathbf{0}} = 3 \cdot \mathbf{1}_{\mathbf{0}} + \mathbf{2}_{\pm \mathbf{1/2}}$	[35, 41]
SU(6)	Sp(6)	✓*	14	$\mathbf{14} = 2 \cdot (\mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}) + 3 \cdot (\mathbf{1}, \mathbf{1})$	[30, 47]
$[\mathrm{SO}(6)]^2$	SO(6)	✓*	15	$\mathbf{15} = (\mathbf{1}, \mathbf{1}) + 2 \cdot (\mathbf{2}, \mathbf{2}) + (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3})$	[36]

[Bellazzini, Csaki, Serra (2014)]

One-loop analysis

Collaborators and I analyze the phase transition dynamics in composite Higgs model.

[KF, Nakai, Sato, Wang (2023)]

$\mathcal{G} \to \mathcal{H}$	PT dynamics	Model	Order
$SO(N) \to SO(N-1)$	E. Brezin et al. (1973) P. H. Ginsparg (1980)	$N=5$ K. Agashe et al. (2005) $N=9$ E. Beltuzzo et al. (2013)	2'nd 2'nd
$SO(9) \to \mathrm{SO}(5) \times \mathrm{SO}(4)$	This work	S. Chang (2013)	1'st
$SU(2N)\,(U(2N)) \to Sp(2N)$	J. Wirstam (2000)	$N=2$ J. Barnald et al. (2014) $N=3$ E. Katz et al. (2005) … and many works	anomaly 1'st
$SU(N)\,(U(N)) \to SO(N)$	F. Basile et al. (2005)	$N=5$ N. Arkani-Hamed et al. (2005)	1'st

Summary: 1’st lecture

- Order of phase transition can be determined by computing the Landau free energy.

- A thermal field theory is a field theory on $\mathbf{R}^3 \times \mathbf{S}^1$.

- At high-temperature, $\mathbf{S}^1$ direction can be integrated out and obtain three-dimensional effective theory.

- A non-perturbative fluctuation should be taken into account to determine the order of a thermal phase transition.