Stable dark matter from Pauli blocking in the degenerate fermion background with Quantum Field Theory

arXiv: 2407.08229

Junghoon Joh

With Wonsub Cho, Ki-Young Choi and Osamu Seto

7th CUBES Workshop, 2025-04-26

Motivation & Idea

- Dark matter's lifetime should be longer than the age of the Universe.
- If dark matter has interactions, some conditions (e.g., Z2 parity, U(1) symmetry) or constraints (e.g., minimal coupling, small mass) are needed.
- Main idea: If dark matter couples only to fermions, Pauli blocking can make dark matter live longer without other conditions.
- We calculate the effect of Pauli blocking using:
  - Boltzmann equations
  - Quantum Field Theory (QFT)

Motivation & Idea

- Dark matter's lifetime should be longer than the age of the Universe.
- If dark matter has interactions, some conditions (e.g., Z2 parity, U(1) symmetry) or constraints (e.g., minimal coupling, small mass) are needed.
- Main idea: If dark matter couples only to fermions, Pauli blocking can make dark matter live longer without other conditions.

  ➡ O.E.Bjaelde, S.Das, Phys.Rev.D 82 (2010), 043504
- We calculate the effect of Pauli blocking using:
  - Boltzmann equations ➡ B.Batell, W.Yin, Phys.Rev.D 110 (2024) 7, 075038
  - Quantum Field Theory (QFT) ➡ T.Moroi, W.Yin, JHEP 03 (2021), 296

\+ Fermion production in preheating

➡ P.B.Greene, L.Kofman, Phys.Lett.B 448 (1999)

➡ P.B.Greene, L.Kofman, Phys.Rev.D 62 (2000)

The Toy Model: Yukawa Coupling

- $\mathscr{L} \ni \lambda \bar{\psi} \psi \phi$
- ϕ : scalar dark matter, ψ & $\bar{\psi}$: fermion & anti-fermion
- Assumption: Matter-dominated era
- Range of mass & coupling: $m_\phi \lesssim 0.01\,\mathrm{eV}, \quad \lambda \ll 2 \times 10^{-8} \left(\frac{\mathrm{m}_\phi}{10^{-3}\,\mathrm{eV}} \right)^{3/4}$
- Lifetime w/o Pauli blocking:

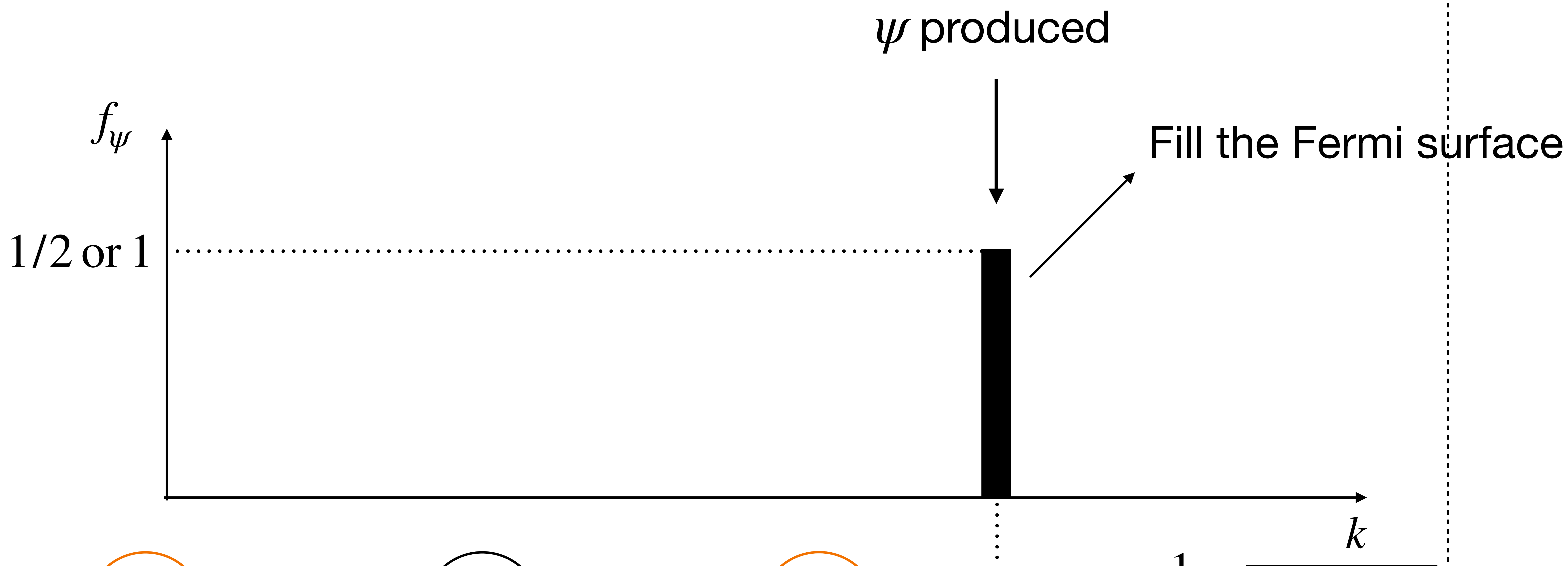
$$\tau_{\phi,0} = \frac{1}{\Gamma_\phi} \simeq 1.6 \times 10^{13}\,\mathrm{sec} \left(\frac{10^{-12}}{\lambda} \right)^2 \left(\frac{10^{-3}\,\mathrm{eV}}{\mathrm{m}_\phi} \right) \left(1 - \frac{4\mathrm{m}_\psi^2}{\mathrm{m}_\phi^2} \right)^{-3/2}$$

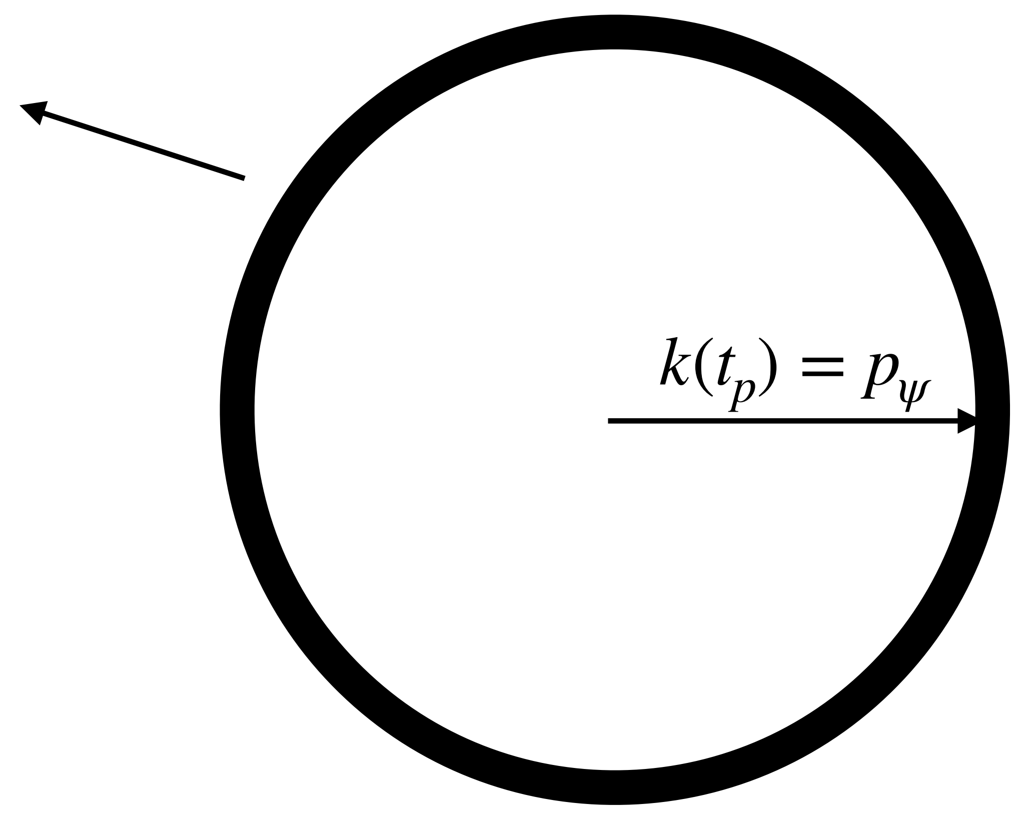
- Lifetime w/ Pauli blocking: $\tau_\phi \simeq t_i \left(\frac{\rho_\phi(t_i)}{m_\phi} \right)^{1/2} \left(\frac{p_\psi^3}{4\pi^2} \right)^{-1/2}$ $\rightarrow$ Result

Pauli-Blocking Idea

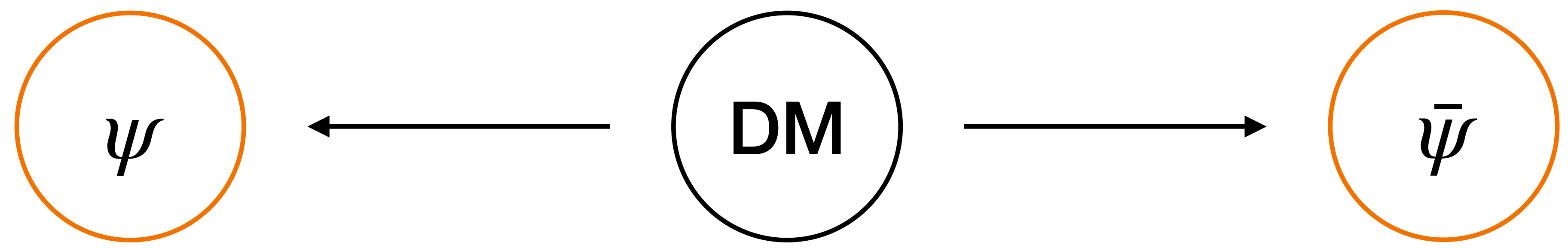
Value of distribution function

Fermi sphere in momentum space





$$k(t_p) = p_\psi \equiv \frac{1}{2}\sqrt{m_\phi^2 - 4m_\psi^2}$$



Momentum: p_ψ Mass: m_ψ

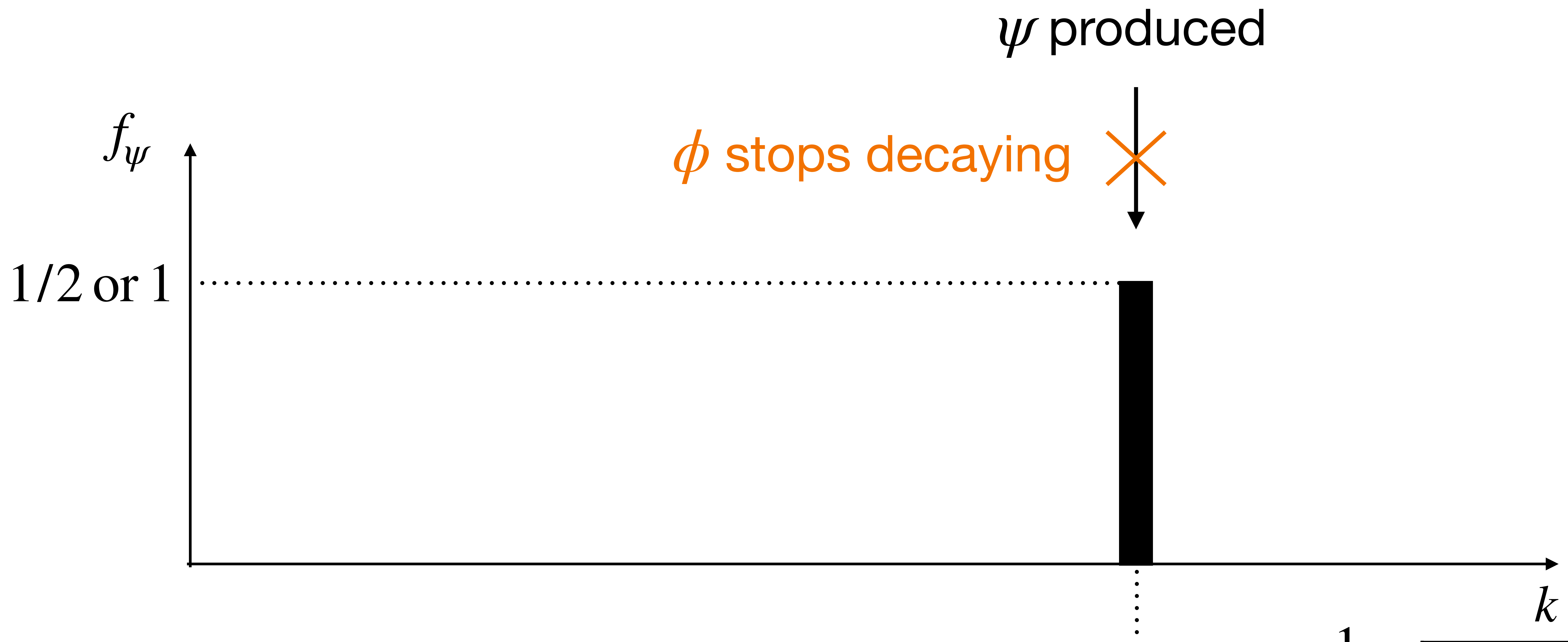
Momentum: 0 Mass: m_ϕ

Momentum: p_ψ Mass: m_ψ

$m_\phi > 2m_\psi$

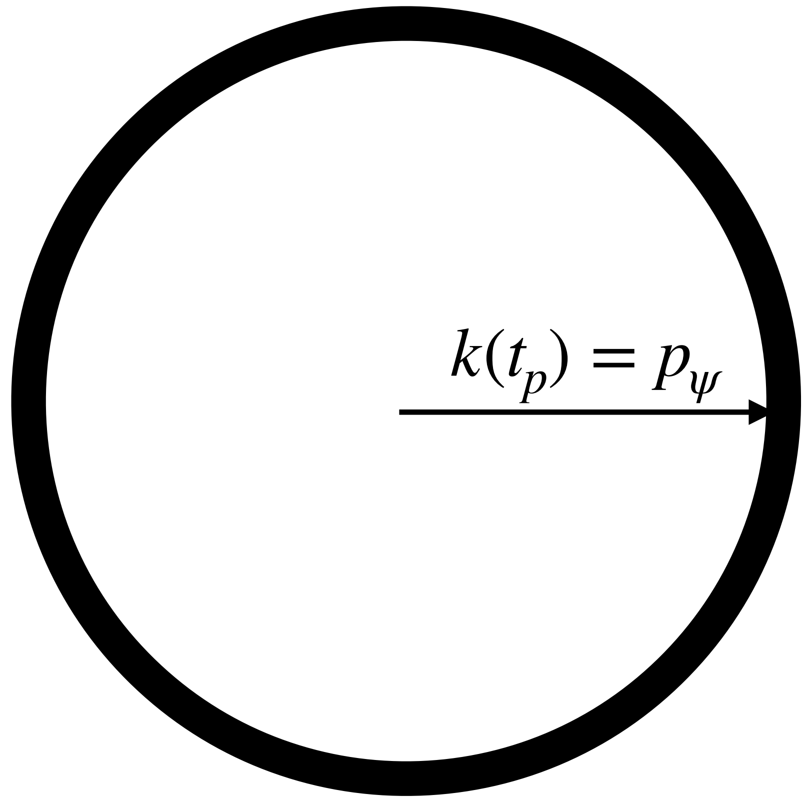
Pauli-Blocking Idea

Value of distribution function



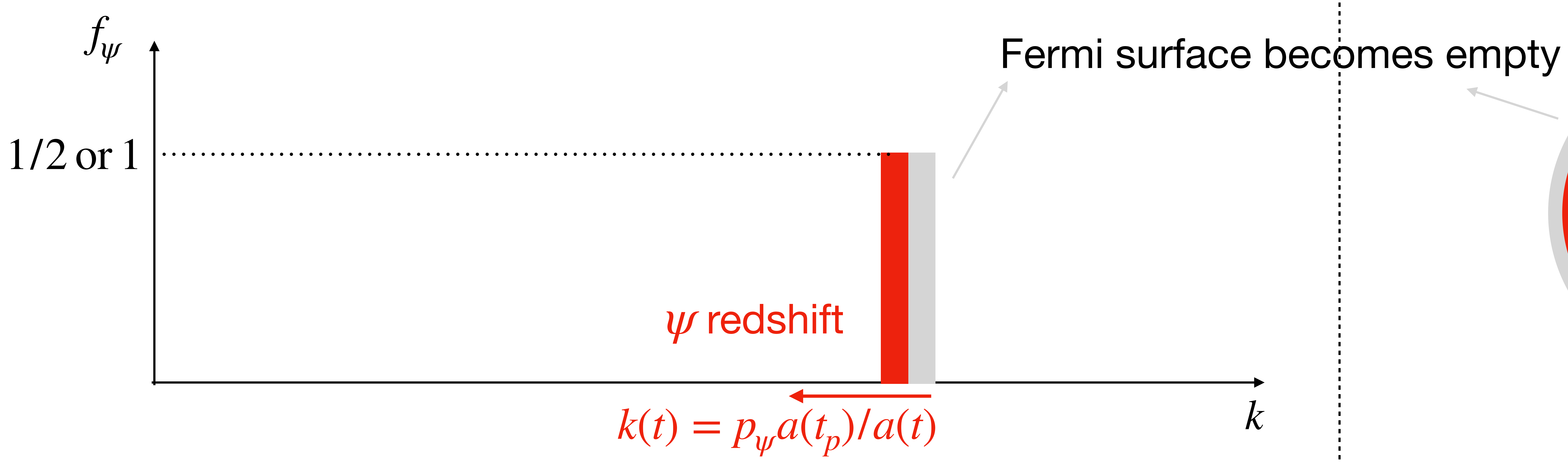
$$k(t_p) = p_\psi \equiv \frac{1}{2}\sqrt{m_\phi^2 - 4m_\psi^2}$$

Fermi sphere in momentum space

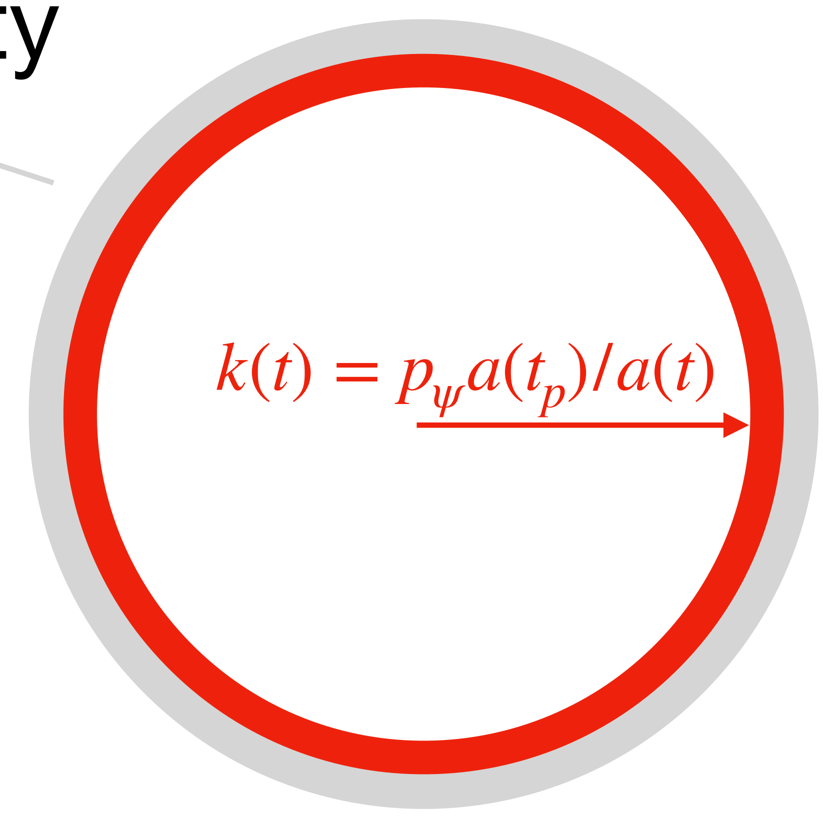


Pauli-Blocking Idea

Value of distribution function

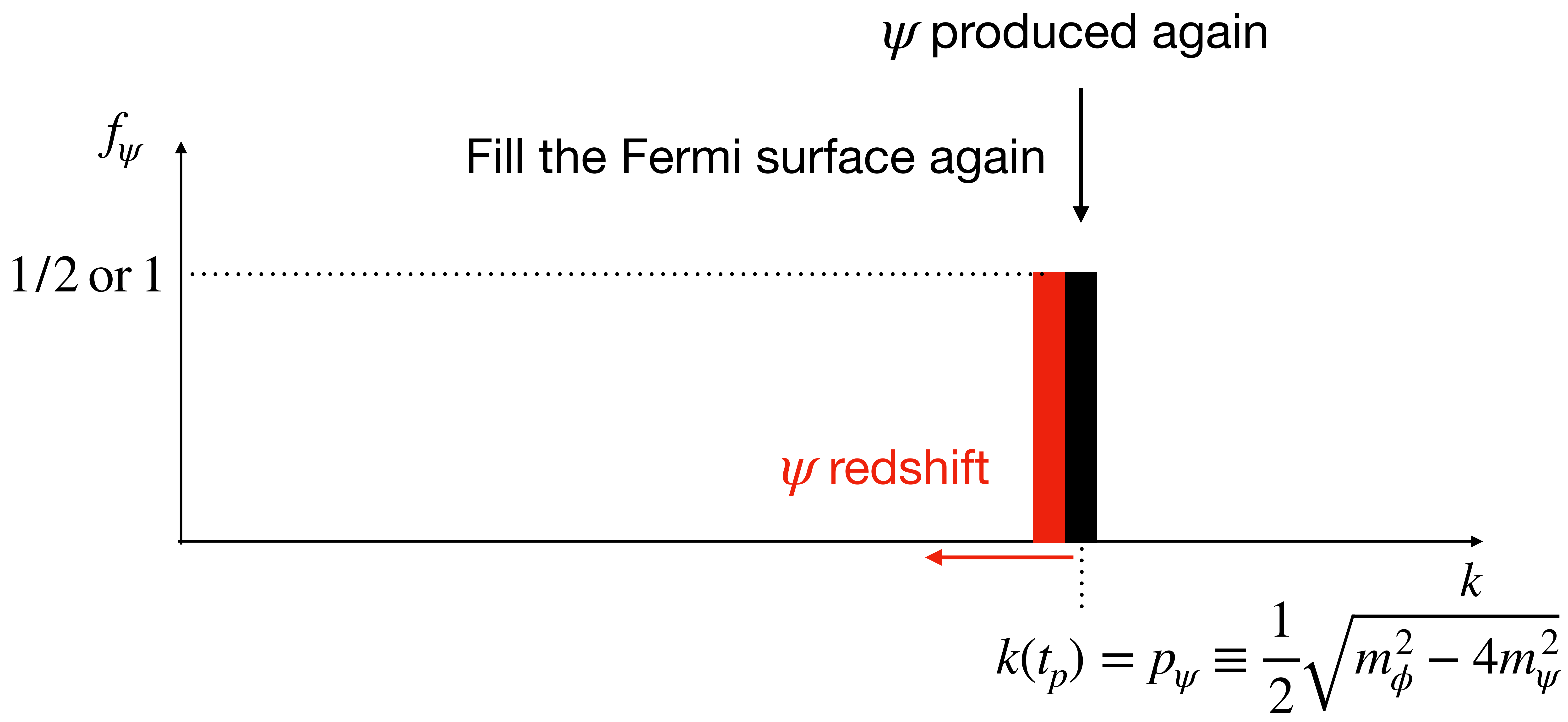


Fermi sphere in momentum space

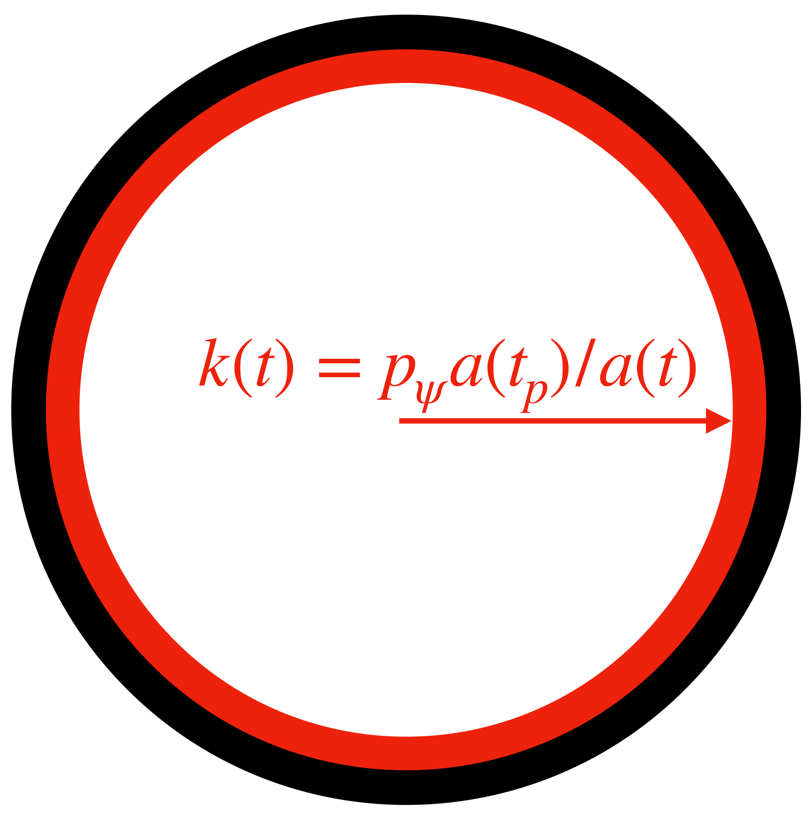


Pauli-Blocking Idea

Value of distribution function



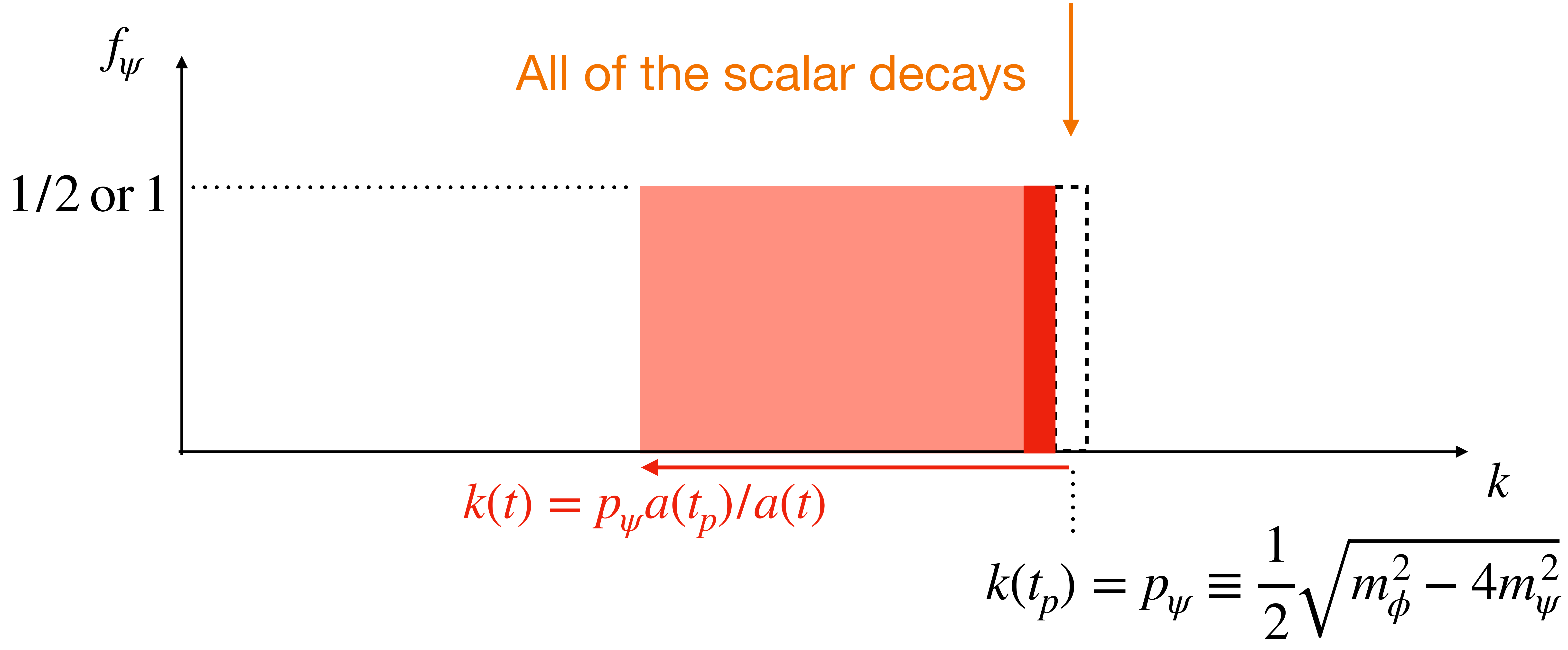
Fermi sphere in momentum space



Pauli-Blocking Idea

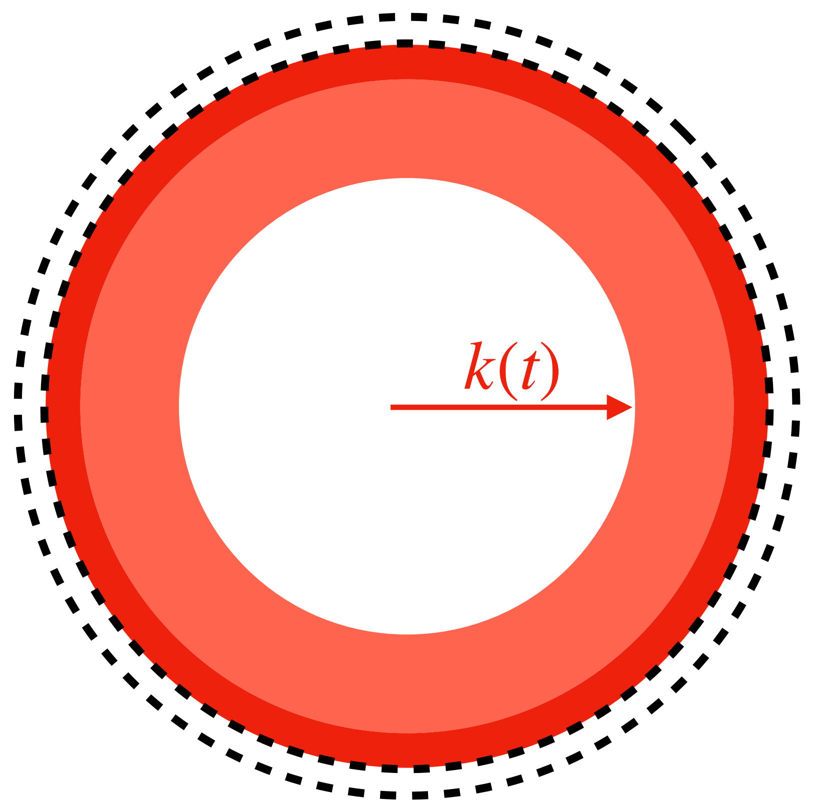
Value of distribution function

$n_\phi(t) \lesssim$ Volume of Fermi sphere



Lifetime with Pauli blocking τ_ϕ

Fermi sphere in momentum space



Boltzmann Equation

- Momentum distribution function:
  - $$\frac{\partial f_\phi(\vec{p},t)}{\partial t} - H\vec{k}\cdot\vec{\nabla}_{\vec{p}} f_\phi(\vec{p},t) = f_\phi^{\rm coll}(\vec{p},t)$$
  - $$\frac{\partial f_\psi(\vec{k},t)}{\partial t} - H\vec{k}\cdot\vec{\nabla}_{\vec{k}} f_\psi(\vec{k},t) = f_\psi^{\rm coll}(\vec{k},t)$$
- Number density:
  - $$\dot{n}_\phi(t) + 3H(t)n_\phi(t) = -\Gamma_\phi n_\phi(t)\left(1 - 2f_\psi(p_\psi, t)\right)$$
  - $$\dot{n}_\psi(t) + 3H(t)n_\psi(t) = \Gamma_\phi n_\phi(t)\left(1 - 2f_\psi(p_\psi, t)\right)$$

Boltzmann Equation

- Momentum distribution function:
  - $$\frac{\partial f_\phi(\vec{p},t)}{\partial t} - H\vec{k}\cdot\overrightarrow{\nabla}_{\vec{p}} f_\phi(\vec{p},t) = f_\phi^{\rm coll}(\vec{p},t)$$
  - $$\frac{\partial f_\psi(\vec{k},t)}{\partial t} - H\vec{k}\cdot\overrightarrow{\nabla}_{\vec{k}} f_\psi(\vec{k},t) = f_\psi^{\rm coll}(\vec{k},t)$$

$$f_\phi^{\rm coll}(\overrightarrow{\mathbf{p}},t) \propto \left[\underbrace{f_\psi(\overrightarrow{\mathbf{k}},t)f_{\bar{\psi}}(\overrightarrow{\mathbf{k}}',t)(1+f_\phi(\overrightarrow{\mathbf{p}},t))}_{\text{back-reaction term}} - \underbrace{f_\phi(\overrightarrow{\mathbf{p}},t)(1-f_\psi(\overrightarrow{\mathbf{k}},t))(1-f_{\bar{\psi}}(\overrightarrow{\mathbf{k}}',t))}_{\text{reaction term}}\right]$$

- Number density:
  - $$\dot{n}_\phi(t) + 3H(t)n_\phi(t) = -\Gamma_\phi n_\phi(t)\left(1 - 2f_\psi(p_\psi,t)\right)$$
  - $$\dot{n}_\psi(t) + 3H(t)n_\psi(t) = \Gamma_\phi n_\phi(t)\left(1 - 2f_\psi(p_\psi,t)\right)$$

Boltzmann Equation

- Momentum distribution function:
  - $$\frac{\partial f_\phi(\vec{p},t)}{\partial t} - H\vec{k}\cdot\vec{\nabla}_{\vec{p}} f_\phi(\vec{p},t) = f_\phi^{\mathrm{coll}}(\vec{p},t)$$
  - $$\frac{\partial f_\psi(\vec{k},t)}{\partial t} - H\vec{k}\cdot\vec{\nabla}_{\vec{k}} f_\psi(\vec{k},t) = f_\psi^{\mathrm{coll}}(\vec{k},t)$$

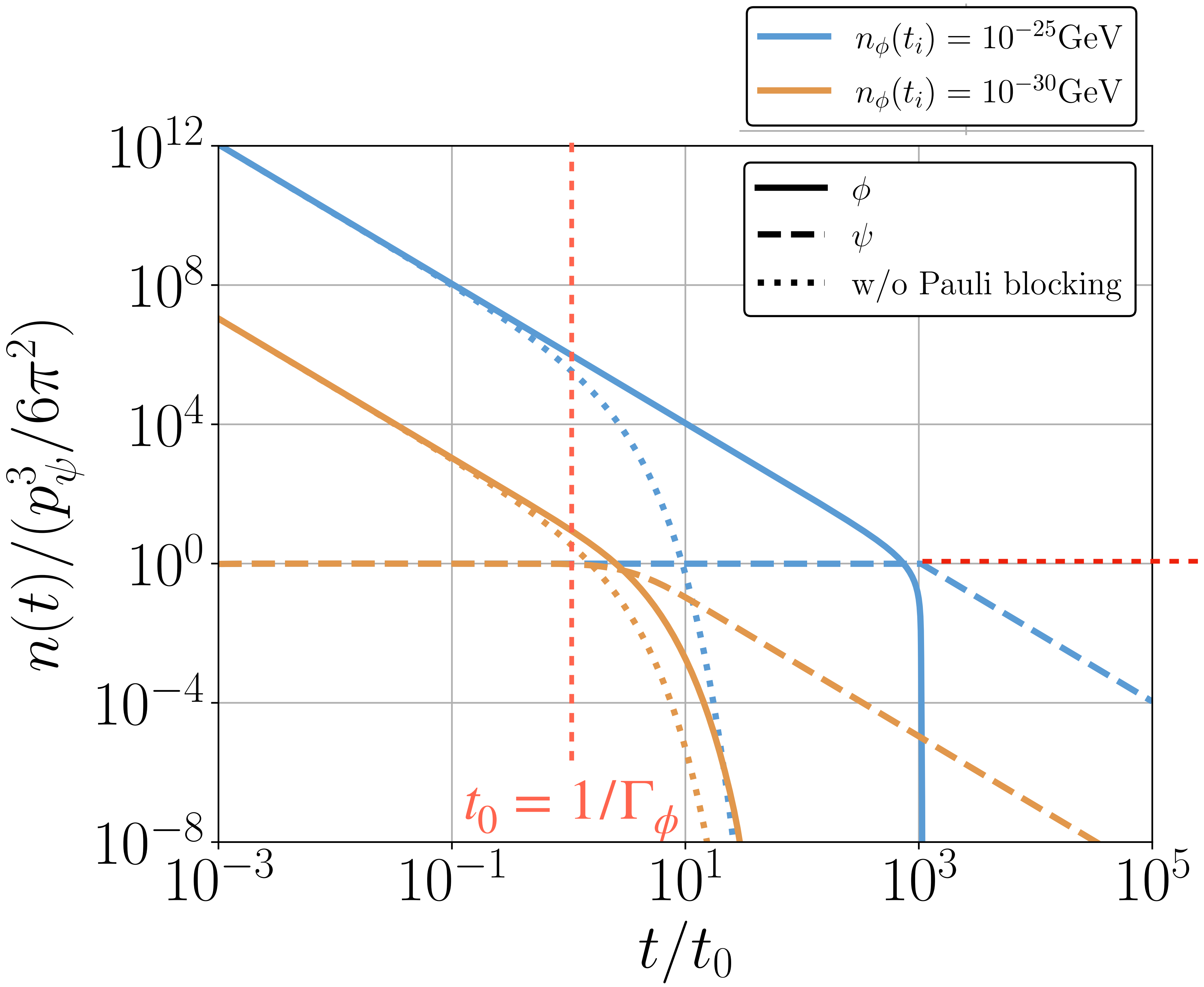
$$f_\phi^{\mathrm{coll}}(\overrightarrow{\mathbf{p}},t) \propto \left[f_\psi(\overrightarrow{\mathbf{k}},t) f_{\bar{\psi}}(\overrightarrow{\mathbf{k}}',t)(1+f_\phi(\overrightarrow{\mathbf{p}},t)) - f_\phi(\overrightarrow{\mathbf{p}},t)(1-f_\psi(\overrightarrow{\mathbf{k}},t))(1-f_{\bar{\psi}}(\overrightarrow{\mathbf{k}}',t))\right]$$

- Number density:
  - $$\dot{n}_\phi(t) + 3H(t)n_\phi(t) = -\Gamma_\phi n_\phi(t)\Big(1-2f_\psi(p_\psi,t)\Big)$$
  - $$\dot{n}_\psi(t) + 3H(t)n_\psi(t) = \Gamma_\phi n_\phi(t)\Big(1-2f_\psi(p_\psi,t)\Big)$$

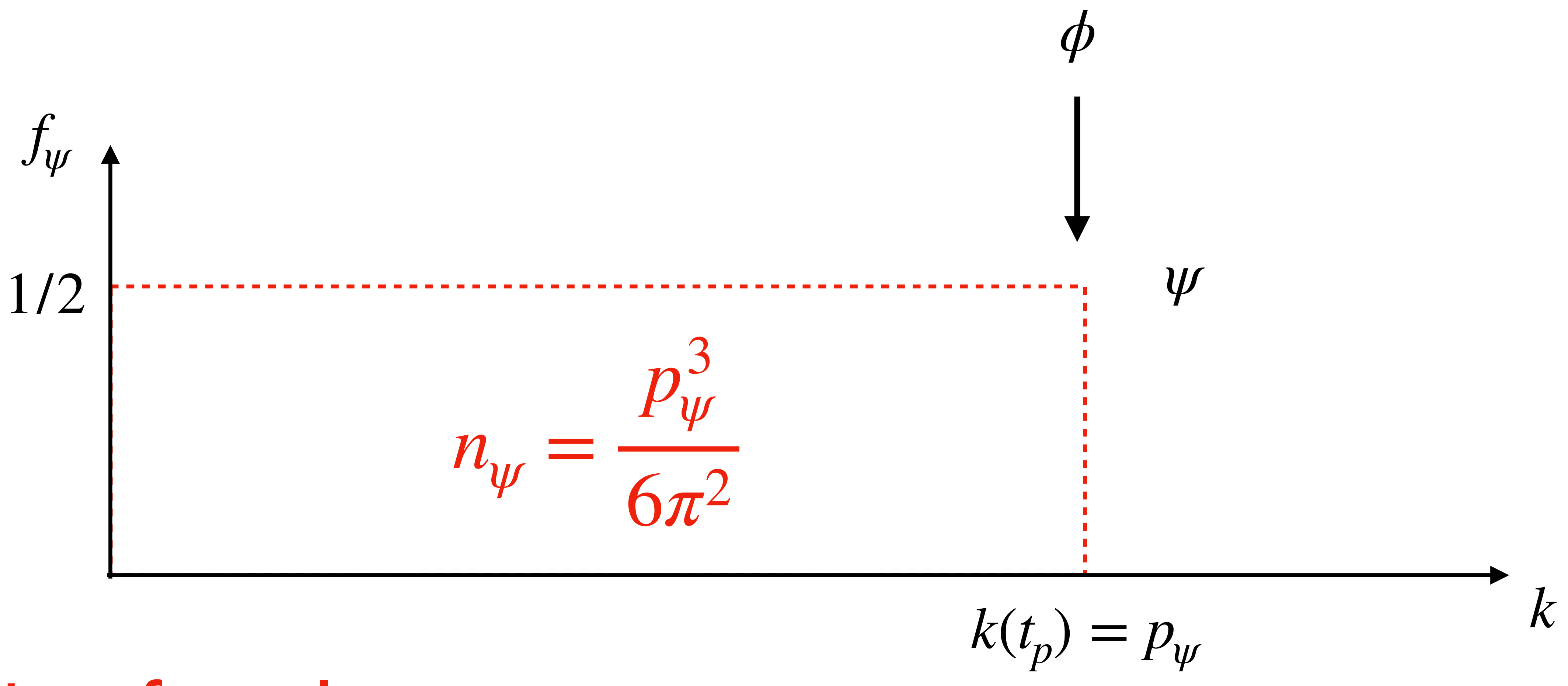
Dark matter stops decaying when $f_\psi(p_\psi,t) = 1/2$

Result of Boltzmann Equation

W.Cho, K.Y.Choi, J.Joh, O.Seto. e-Print: 2407.08229 [hep-ph]



$$n_\phi(t) \simeq \frac{p_\psi^3}{6\pi^2}$$



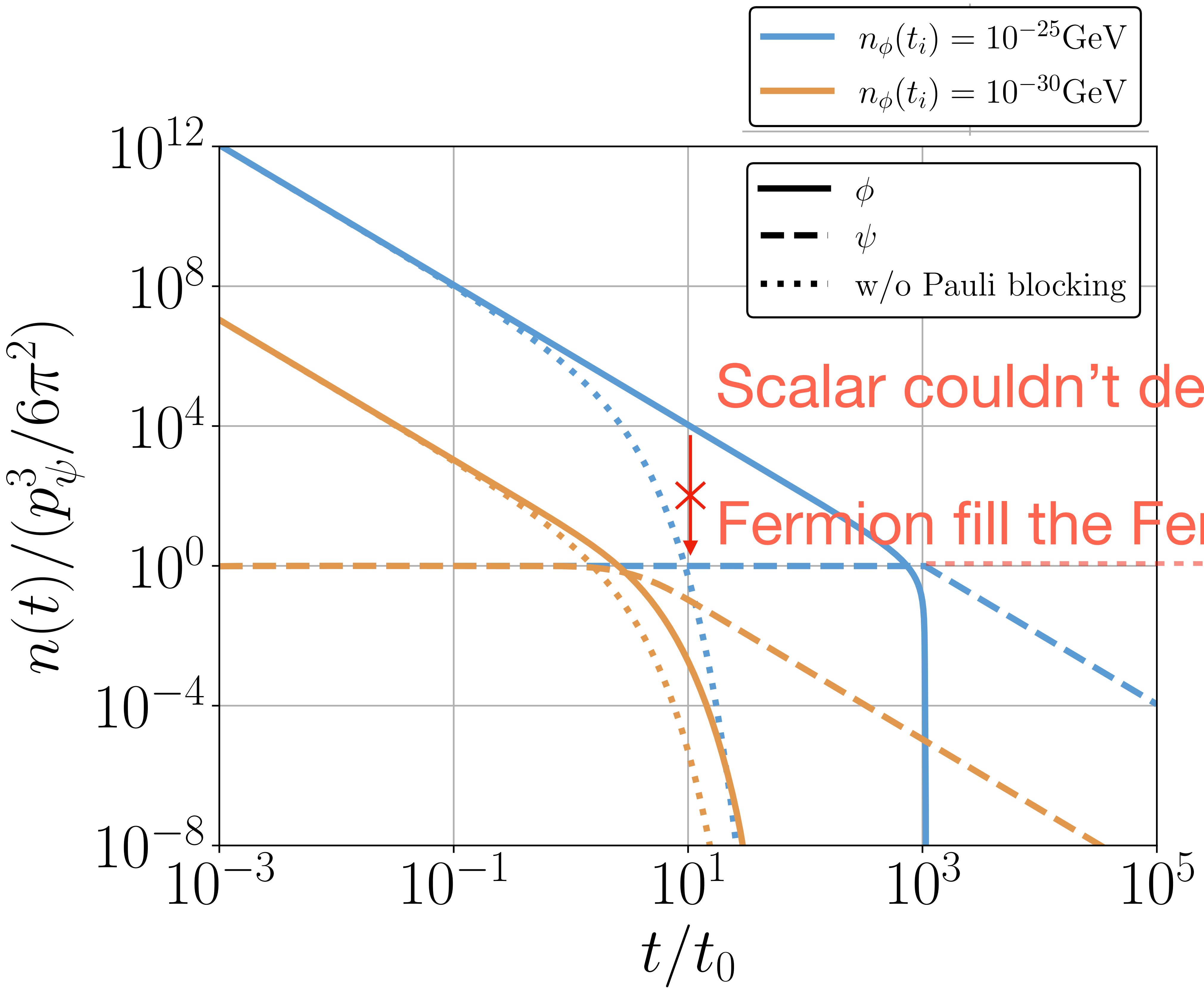
Number density of scalar =

Volume of Fermi sphere in momentum space

$$\left(\dot n_\phi(t) + \dot n_\psi(t)\right) + 3H(t)\left(n_\phi(t) + n_\psi(t)\right) = 0$$

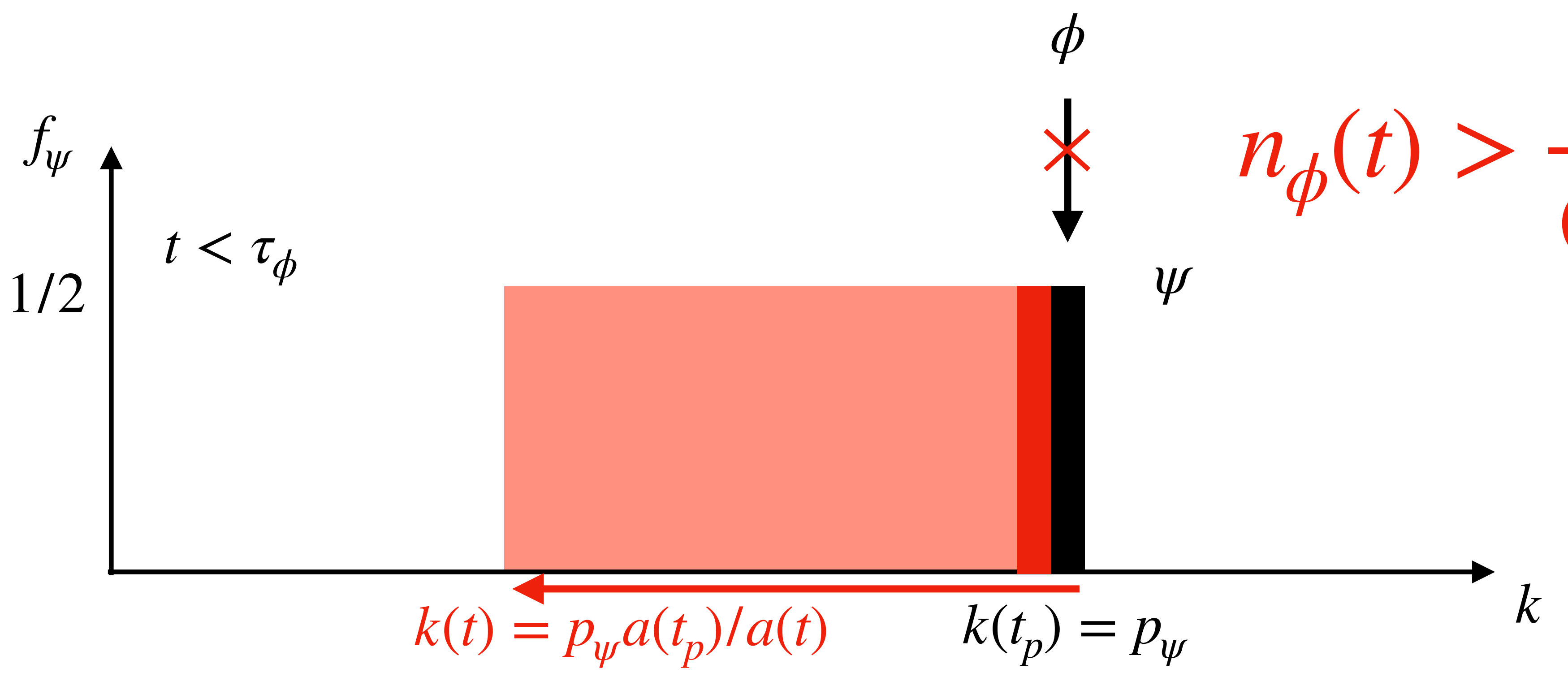
Result of Boltzmann Equation

W.Cho, K.Y.Choi, J.Joh, O.Seto. e-Print: 2407.08229 [hep-ph]



Scalar couldn't decay more

Fermion fill the Fermi surface



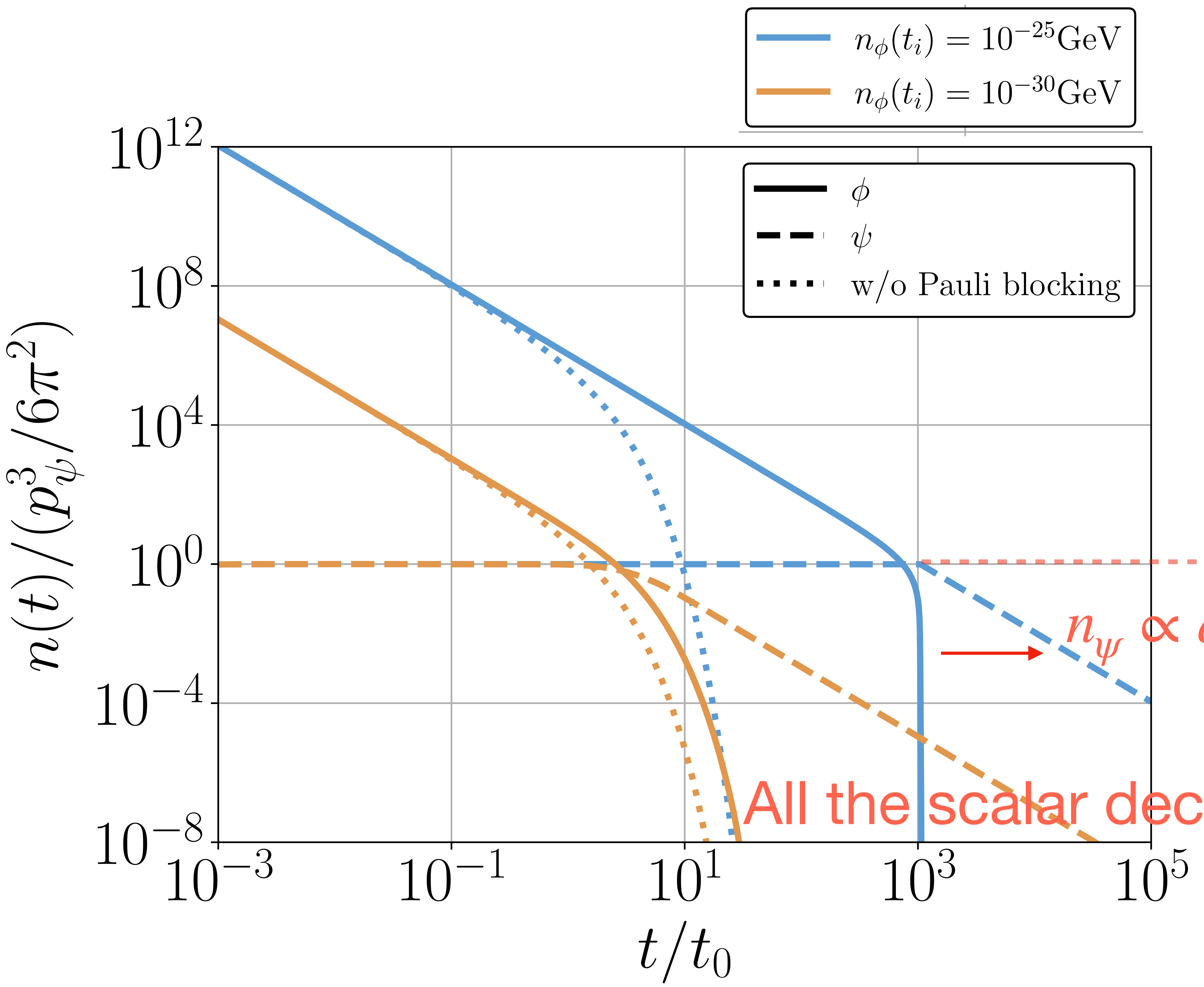
$$n_\phi(t) > \frac{p_\psi^3}{6\pi^2}$$

$$n_\phi(t) \simeq \frac{p_\psi^3}{6\pi^2}$$

$$\left(\dot{n}_\phi(t) + \dot{n}_\psi(t)\right) + 3H(t)\left(n_\phi(t) + n_\psi(t)\right) = 0$$

Result of Boltzmann Equation

W.Cho, K.Y.Choi, J.Joh, O.Seto. e-Print: 2407.08229 [hep-ph]



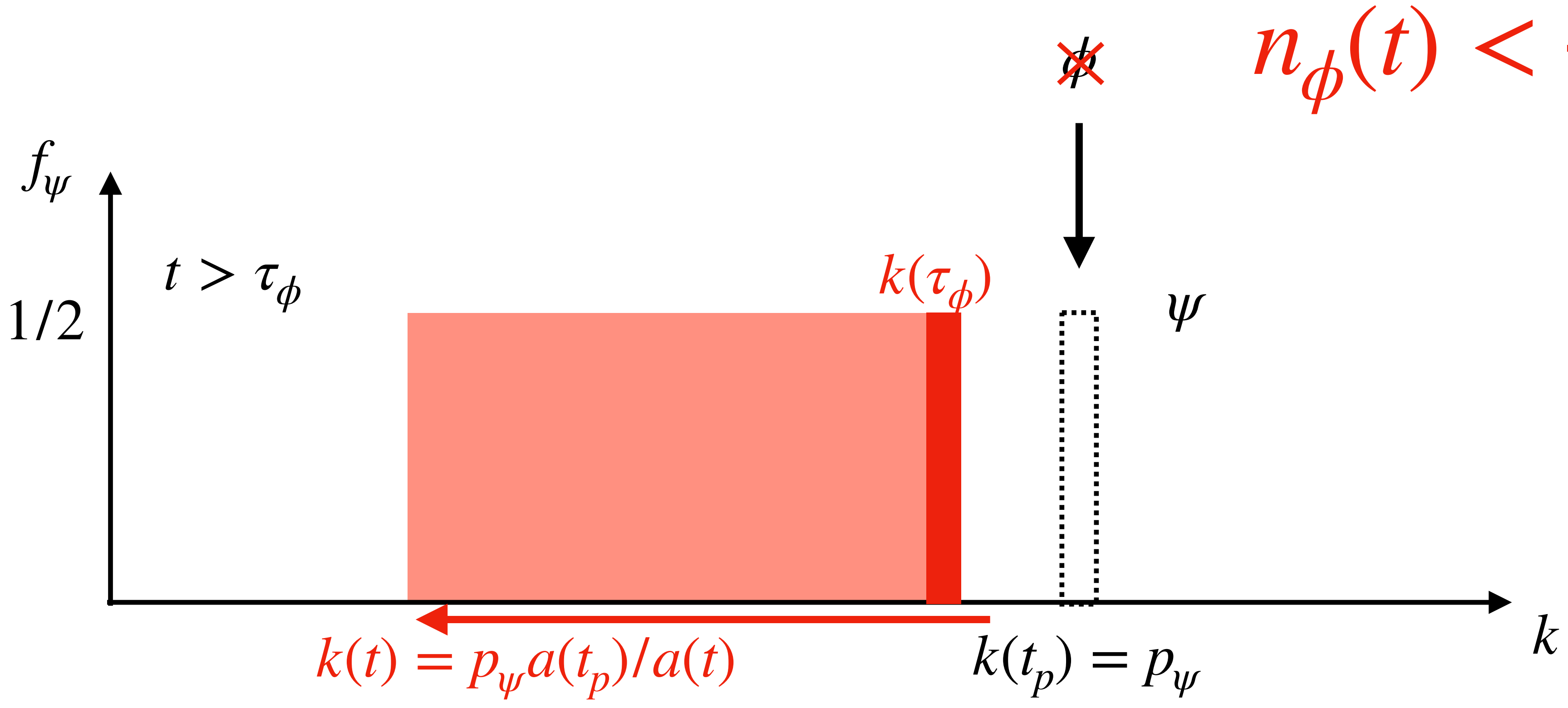
$n_\psi \propto a^{-3}(t)$

All the scalar decays to fermion

$n_\phi(t) > \dfrac{p_\psi^3}{6\pi^2}$

$n_\phi(t) \simeq \dfrac{p_\psi^3}{6\pi^2}$

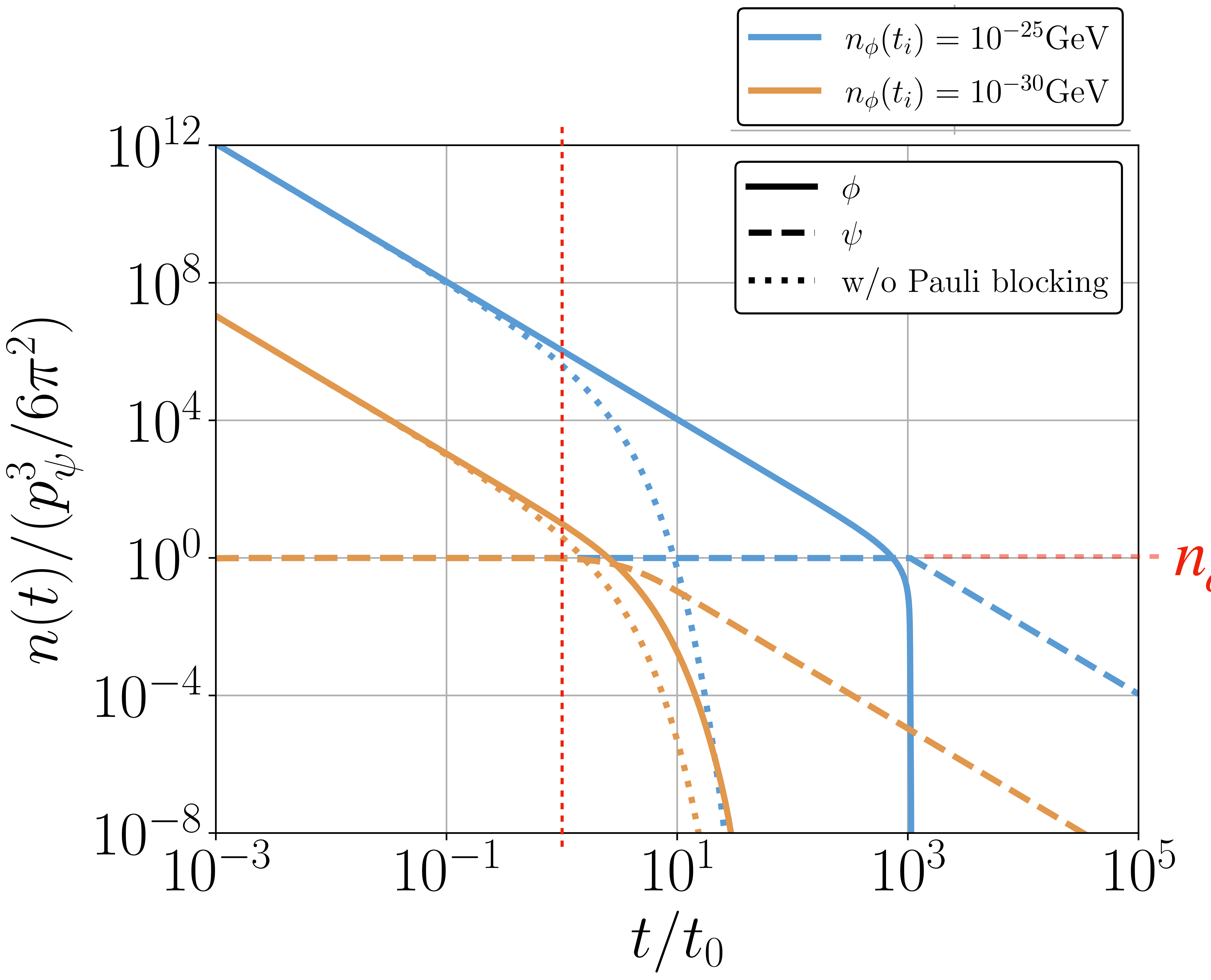
$n_\phi(t) < \dfrac{p_\psi^3}{6\pi^2}$



$$\Big(\dot{n}_\phi(t) + \dot{n}_\psi(t)\Big) + 3H(t)\Big(n_\phi(t) + n_\psi(t)\Big) = 0$$

Result of Boltzmann Equation

W.Cho, K.Y.Choi, J.Joh, O.Seto. e-Print: 2407.08229 [hep-ph]

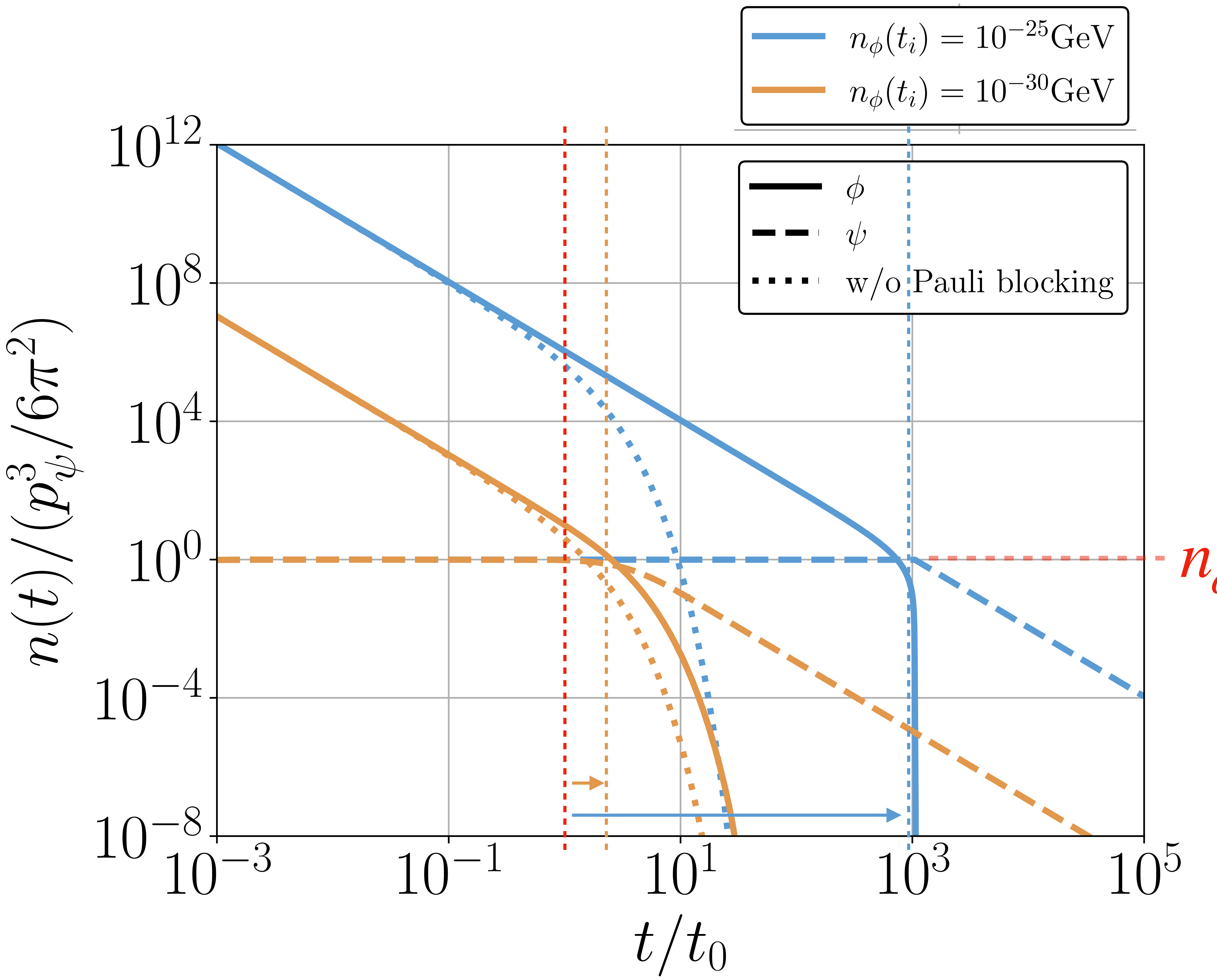


$$n_\phi(\tau_\phi) \simeq \frac{p_\psi^3}{6\pi^2}$$

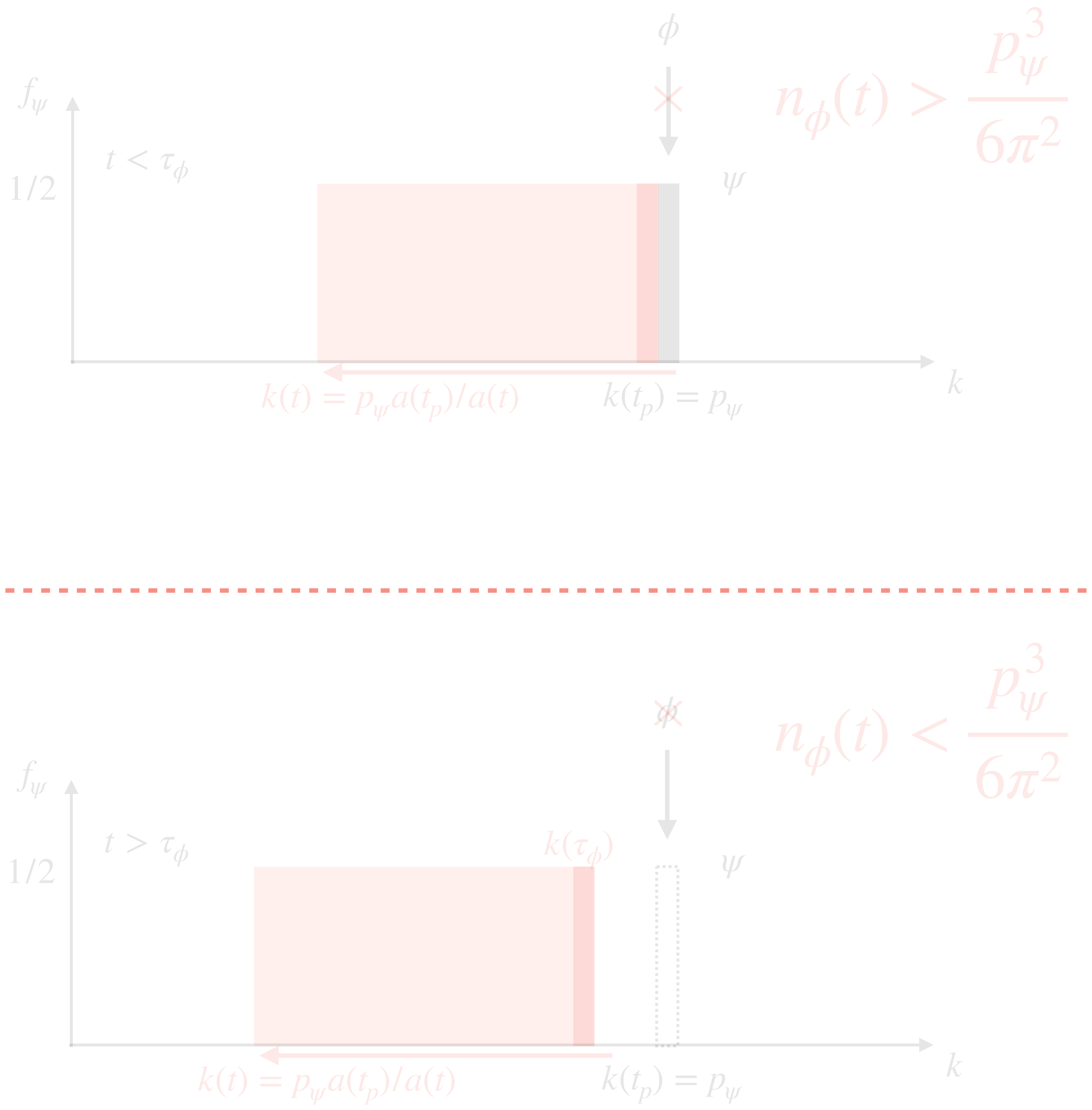
$$\tau_{\phi,0} = 1/\Gamma_\phi$$

Result of Boltzmann Equation

W.Cho, K.Y.Choi, J.Joh, O.Seto. e-Print: 2407.08229 [hep-ph]



$$n_\phi(\tau_\phi) \simeq \frac{p_\psi^3}{6\pi^2}$$



$$\tau_{\phi,0} = 1/\Gamma_\phi \rightarrow \quad \tau_\phi \simeq t_i \left(\frac{\rho_\phi(t_i)}{m_\phi}\right)^{1/2} \left(\frac{p_\psi^3}{4\pi^2}\right)^{-1/2}$$

Dark matter becomes much more sustainable

Quantum Field Theory

- Distribution function:

$$f_{\psi,\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle a^{\dagger r}_{\vec{k}} a^r_{\vec{k}} \right\rangle \quad \text{:fermion}$$

$$f_{\bar{\psi},\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle b^{\dagger r}_{\vec{k}} b^r_{\vec{k}} \right\rangle \quad \text{:anti-fermion}$$

- Fermion related function:

$$g_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r\sum_s \underbrace{\left\langle b^s_{-\vec{k}} a^r_{\vec{k}} \right\rangle}_{\to\text{annihilation}} \bar{v}^s_{-\vec{k}} u^r_{\vec{k}}$$

$$g^*_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r\sum_s \underbrace{\left\langle a^{r\dagger}_{\vec{k}} b^{s\dagger}_{-\vec{k}} \right\rangle}_{\to\text{production}} \bar{u}^r_{\vec{k}} v^s_{-\vec{k}}$$

Quantum Field Theory

- Distribution function:

$$f_{\psi,\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle a^{\dagger r}_{\vec{k}} a^r_{\vec{k}} \right\rangle \quad \text{:fermion}$$

$$f_{\bar{\psi},\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle b^{\dagger r}_{\vec{k}} b^r_{\vec{k}} \right\rangle \quad \text{:anti-fermion}$$

- Fermion related function:

$$g_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r \sum_s \underbrace{\left\langle b^s_{-\vec{k}} a^r_{\vec{k}} \right\rangle}_{\to\text{annihilation}} \bar{v}^s_{-\vec{k}} u^r_{\vec{k}}$$

$$g^*_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r \sum_s \underbrace{\left\langle a^{r\dagger}_{\vec{k}} b^{s\dagger}_{-\vec{k}} \right\rangle}_{\to\text{production}} \bar{u}^r_{\vec{k}} v^s_{-\vec{k}}$$

- Differential equations: $\frac{d}{dt}\langle \mathcal{O} \rangle = -i\left\langle \left[\mathcal{O}, H_{\text{int}}\right] \right\rangle + \langle \dot{\mathcal{O}} \rangle$

Quantum Field Theory

- Distribution function:

$$f_{\psi,\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle a^{\dagger r}_{\vec{k}} a^r_{\vec{k}} \right\rangle$$:fermion

$$f_{\bar{\psi},\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle b^{\dagger r}_{\vec{k}} b^r_{\vec{k}} \right\rangle$$:anti-fermion

- Fermion related function:

$$g_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r\sum_s \underline{\left\langle b^s_{-\vec{k}} a^r_{\vec{k}} \right\rangle} \bar{v}^s_{-\vec{k}} u^r_{\vec{k}}$$

→annihilation

$$g^*_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r\sum_s \underline{\left\langle a^{r\dagger}_{\vec{k}} b^{s\dagger}_{-\vec{k}} \right\rangle} \bar{u}^r_{\vec{k}} v^s_{-\vec{k}}$$

→ production

- Differential equations:

$$\frac{d}{dt}\langle \mathcal{O} \rangle = -i\left\langle \left[\mathcal{O}, H_{\rm int}\right] \right\rangle + \langle \dot{\mathcal{O}} \rangle$$

$$\phi(x,t) \to \phi(t) = \bar{\phi}\cos(m_\phi t)$$

*assumption: oscillation, homogeneous & classical / decay of scalar is negligible

Quantum Field Theory

- Distribution function:

$$f_{\psi,\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle a^{\dagger r}_{\vec{k}} a^r_{\vec{k}} \right\rangle$$:fermion

$$f_{\bar{\psi},\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle b^{\dagger r}_{\vec{k}} b^r_{\vec{k}} \right\rangle$$:anti-fermion

- Fermion related function:

$$g_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r \sum_s \underline{\left\langle b^s_{-\vec{k}} a^r_{\vec{k}} \right\rangle} \bar{v}^s_{-\vec{k}} u^r_{\vec{k}}$$

→annihilation

$$g^*_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r \sum_s \underline{\left\langle a^{r\dagger}_{\vec{k}} b^{s\dagger}_{-\vec{k}} \right\rangle} \bar{u}^r_{\vec{k}} v^s_{-\vec{k}}$$

→ production

- Differential equations: $\frac{d}{dt}\langle \mathcal{O} \rangle = -i\left\langle \left[\mathcal{O}, H_{\text{int}}\right] \right\rangle + \langle \dot{\mathcal{O}} \rangle$

$$\phi(x,t) \to \phi(t) = \bar{\phi}\cos(m_\phi t)$$

*assumption: oscillation, homogeneous & classical / decay of scalar is negligible

$$\dot{f}_{\psi,\vec{k}} = i\lambda\bar{\phi}\,\underline{\left(g_{\vec{k}} - g^*_{\vec{k}}\right)}\cos m_\phi t$$

→ |production - annihilation|

$$\dot{f}_{\bar{\psi},\vec{k}} = i\lambda\bar{\phi}\,\underline{\left(g_{-\vec{k}} - g^*_{-\vec{k}}\right)}\cos m_\phi t$$

In boson case, $1 + f_{\chi,\vec{k}} + f_{\chi,-\vec{k}}$

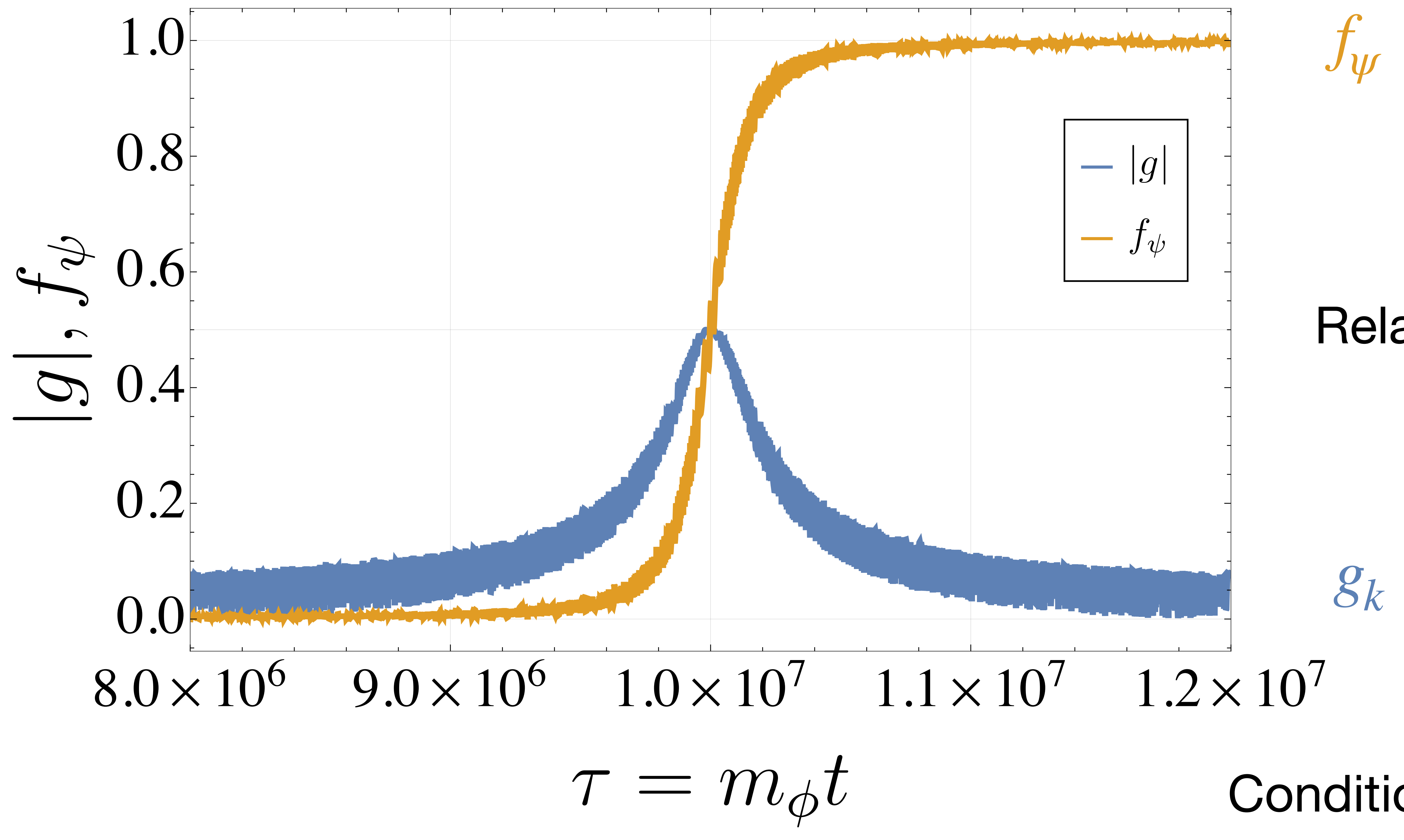
Ref. T.Moroi, W.Yin, JHEP 03 (2021), 296

*Pauli blocking term

$$\dot{g}_{\vec{k}} = -i\left[2\omega_k g_{\vec{k}}(t) + \lambda\bar{\phi}\left\{\frac{k^2}{\omega_k^2}\left(1 - f_{\psi,\vec{k}} - f_{\bar{\psi},-\vec{k}}\right) + \frac{2m_\psi}{\omega_k} g_{\vec{k}}\right\}\cos m_\phi t\right]$$

Result of Quantum Field Theory

Modified from W.Cho, K.Y.Choi, J.Joh, O.Seto. e-Print: 2407.08229 [hep-ph]

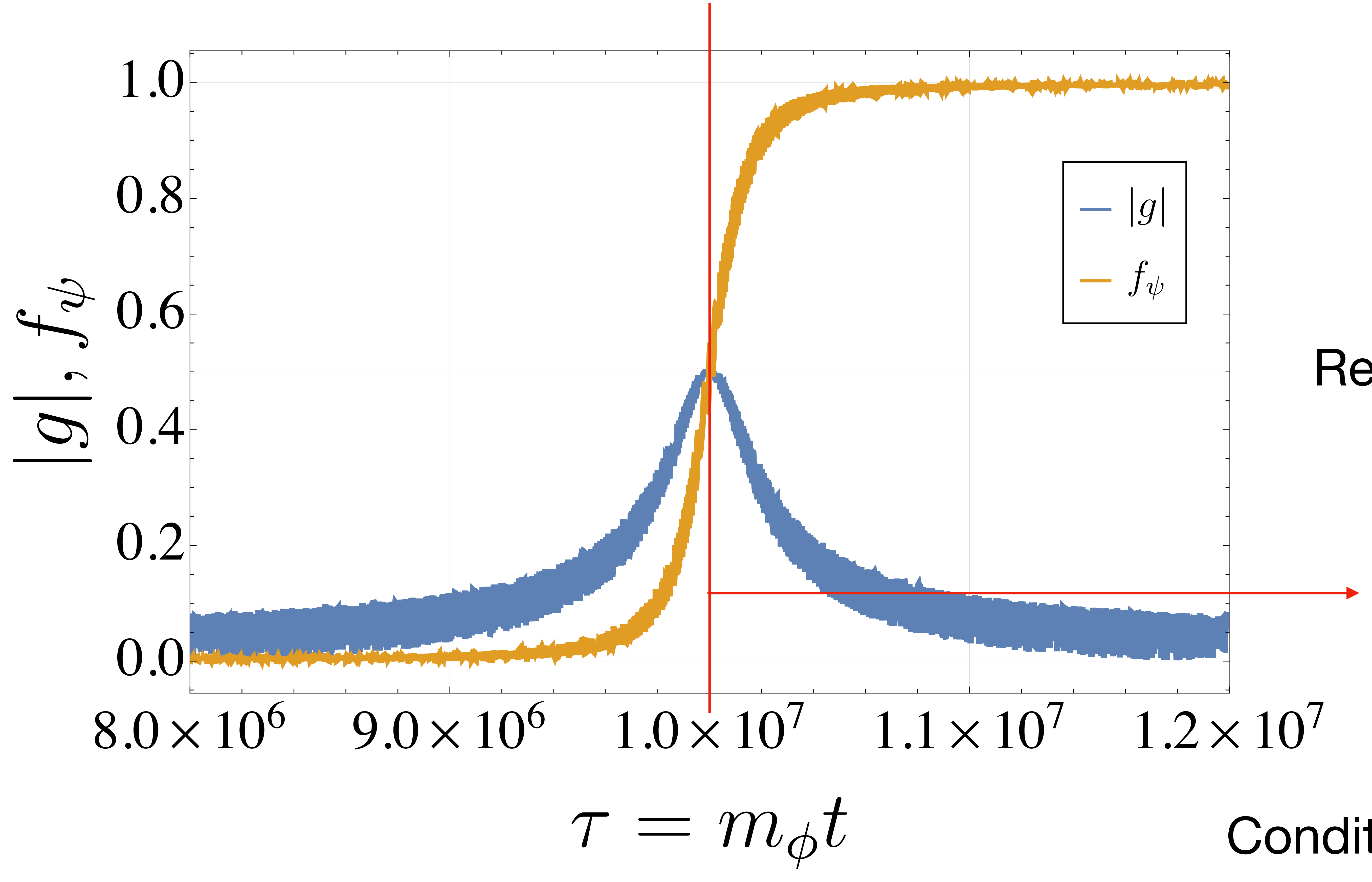


Relations: $f_{\psi,k}(1 - f_{\psi,k}) = g_k g_k^* = |g_k|^2$

Conditions: $q \equiv \lambda\bar{\phi}/m_\phi = 0.01, \quad \tau_0 = 10^2 q^{-2}$

Result of Quantum Field Theory

Modified from W.Cho, K.Y.Choi, J.Joh, O.Seto. e-Print: 2407.08229 [hep-ph]



Relations: $f_{\psi,k}(1-f_{\psi,k}) = g_k g_k^* = |g_k|^2$

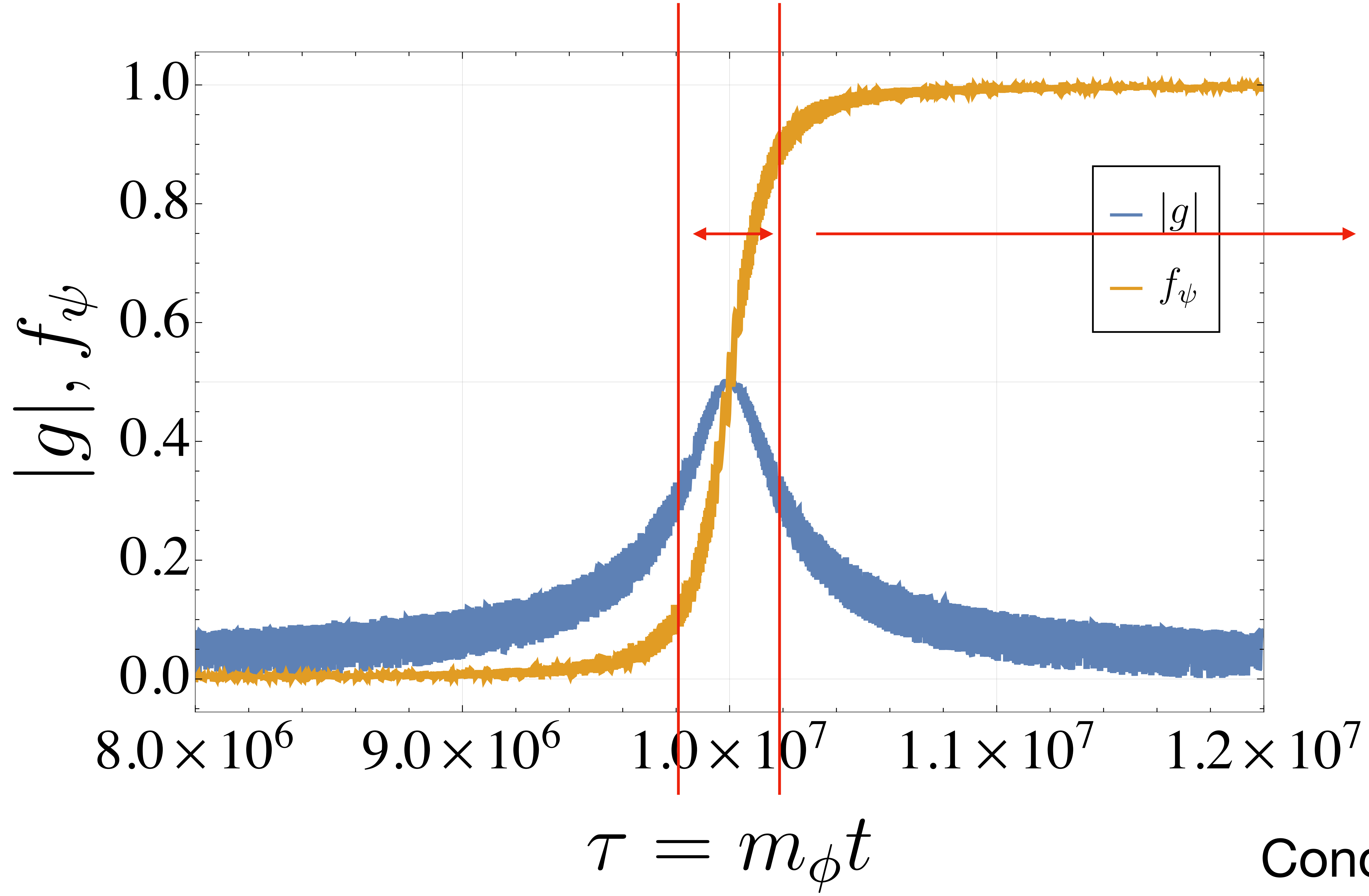
Fixed comoving momentum

$k(\tau_0) = m_\phi/2$

Conditions: $q \equiv \lambda\bar{\phi}/m_\phi = 0.01, \quad \tau_0 = 10^2 q^{-2}$

Result of Quantum Field Theory

Modified from W.Cho, K.Y.Choi, J.Joh, O.Seto. e-Print: 2407.08229 [hep-ph]



$1 - \frac{2}{3}q \lesssim \frac{\tau}{\tau_0} \lesssim 1 + \frac{2}{3}q$ = narrow band

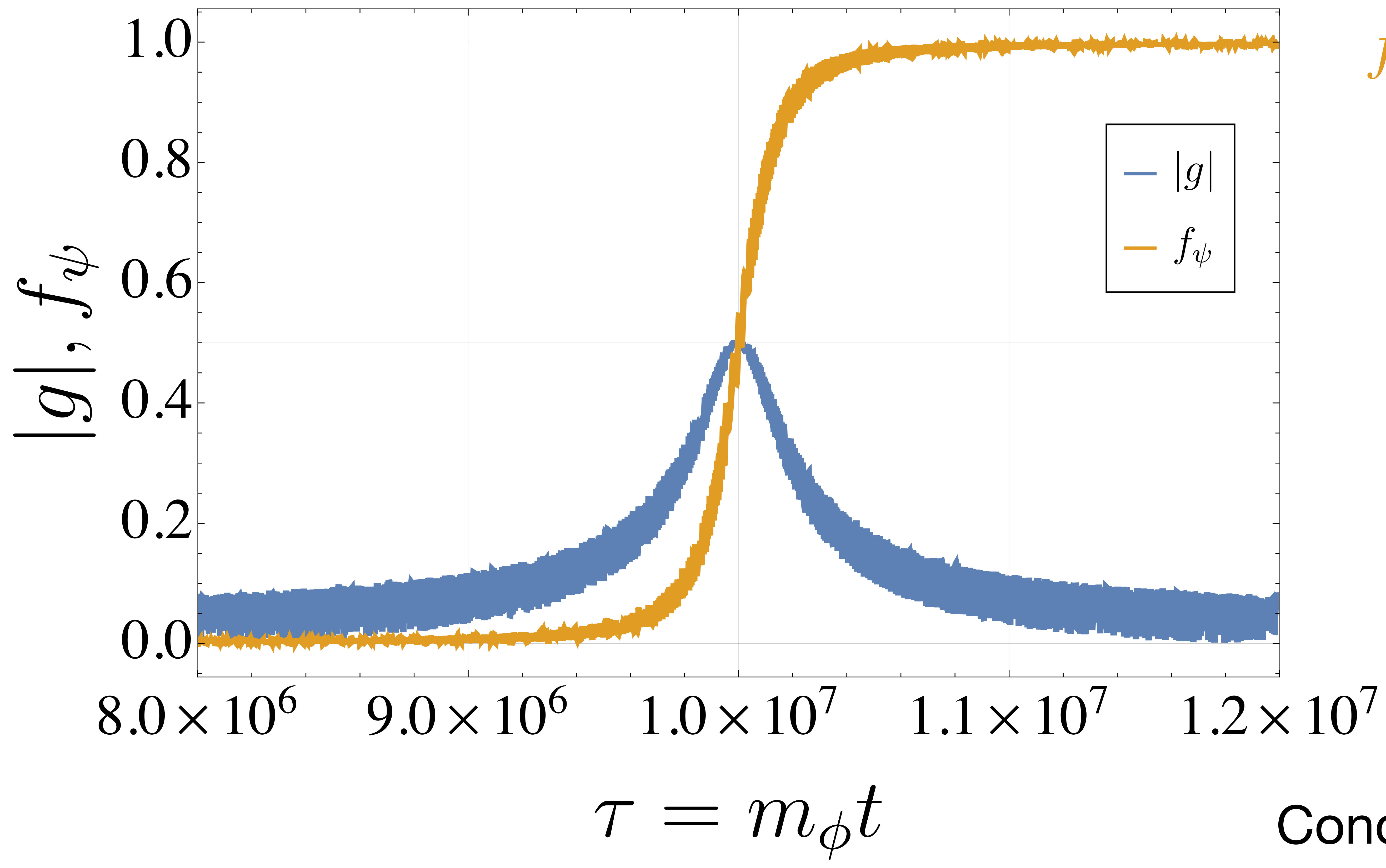
where $q = \dfrac{\lambda\bar{\phi}}{m_\phi}$

$k(\tau_0) = m_\phi/2$

Conditions: $q \equiv \lambda\bar{\phi}/m_\phi = 0.01, \quad \tau_0 = 10^2 q^{-2}$

Result of Quantum Field Theory

Modified from W.Cho, K.Y.Choi, J.Joh, O.Seto. e-Print: 2407.08229 [hep-ph]



$f_\psi \to 1 \Rightarrow$ Pauli blocking is working

$1-\frac{2}{3}q \lesssim \frac{\tau}{\tau_0} \lesssim 1+\frac{2}{3}q$ = narrow band

where $q = \dfrac{\lambda\bar{\phi}}{m_\phi}$

$k(\tau_0) = m_\phi/2$

Conditions: $q \equiv \lambda\bar{\phi}/m_\phi = 0.01, \quad \tau_0 = 10^2 q^{-2}$

Example_100% Dark Matter

- Dark matter (condensed scalar): $\phi(t) = \bar{\phi}\left(\frac{a(t)}{a(t_0)}\right)^{-3/2}\cos(m_\phi t)$
- Energy density of dark matter: $\rho_\phi(t_0) = \frac{1}{2}m_\phi^2\bar{\phi}^2 \simeq 3.7 \times 10^{-47}\,\mathrm{GeV}^4$
- Lifetime w/o Pauli blocking: $\tau_{\phi,0} \simeq 1.6 \times 10^{13}\,\mathrm{sec}\left(\frac{10^{-12}}{\lambda}\right)^2\left(\frac{10^{-3}\,\mathrm{eV}}{\mathrm{m}_\phi}\right)$
- Lifetime w/ Pauli blocking: $\tau_\phi \simeq 132 \times t_0\left(\frac{10^{-3}\,\mathrm{eV}}{m_\phi}\right)^2$

$t_0 \simeq 4.3 \times 10^{17}\,\mathrm{sec}$: age of the Universe

Conclusion

- When the scalar field can decay only to the fermions, the Pauli blocking can delay the decay of the DM
$\rightarrow$ the lifetime getting longer than the age of the Universe.

- The results from using Boltzmann equations and QFT shows the Pauli blocking is working.

- In the example, as the scalar is 100% dark matter, the lifetime becomes 100 times the age of the Universe where $m_\phi \simeq 10^{-3}\,\mathrm{eV}$, and independent of the coupling λ

Appendix

Isotropic Fermion

- Scalar DM is condensate when the mass is small
- Lots of dark matter decays into fermion almost at the same time
- Since the direction of the fermion is determined randomly, the fermion becomes isotropic $\Rightarrow f_\psi(\vec{k}, t) = f_\psi(k, t)$

Boltzmann Equation Collision Term

$$f_\phi^{\mathrm{coll}}(\overrightarrow{\mathbf{p}}, t) = \frac{1}{2E_{\overrightarrow{\mathbf{p}}}} \int \frac{d^3\overrightarrow{\mathbf{k}'}}{(2\pi)^3 2E_{\overrightarrow{\mathbf{k}'}}} \frac{d^3\overrightarrow{\mathbf{k}}}{(2\pi)^3 2E_{\overrightarrow{\mathbf{k}}}} (2\pi)^4 \delta^4(p - k' - k)\,|\,\mathscr{M}\,|^2$$

$$\times \underline{\left[f_\psi(\overrightarrow{\mathbf{k}}, t) f_{\bar\psi}(\overrightarrow{\mathbf{k}}', t)(1 + f_\phi(\overrightarrow{\mathbf{p}}, t)) - f_\phi(\overrightarrow{\mathbf{p}}, t)(1 - f_\psi(\overrightarrow{\mathbf{k}}, t))(1 - f_{\bar\psi}(\overrightarrow{\mathbf{k}'}, t))\right]}$$

$$\to n_\phi(1 - 2f_\psi)$$

Boltzmann Equation without Expansion

- Momentum distribution function:
  - $$\frac{\partial f_\phi(\vec{p}, t)}{\partial t} = f_\phi^{\mathrm{coll}}(\vec{p}, t)$$
  - $$\frac{\partial f_\psi(\vec{k}, t)}{\partial t} = f_\psi^{\mathrm{coll}}(\vec{k}, t)$$
- Number density:
  - $$\dot{n}_\phi(t) = -\Gamma_\phi n_\phi(t)\left(1 - 2f_\psi(p_\psi, t)\right)$$

Dark matter stops decaying when $f_\psi(p_\psi, t) = 1/2$

  - $$\dot{n}_\psi(t) = \Gamma_\phi n_\phi(t)\left(1 - 2f_\psi(p_\psi, t)\right)$$

$$\Rightarrow n_\phi(t) + n_\psi(t) = n_\phi(t) + \frac{p_\psi^3}{\pi^2} f_\psi(p_\psi, t) = C_0$$: Total number density

Boltzmann Equation without Expansion

- Solution:

$$\frac{1}{n_\phi(t)} = \frac{B}{A} + \left(\frac{1}{n_\phi(0)} - \frac{B}{A}\right) e^{A(t-t_0)}$$

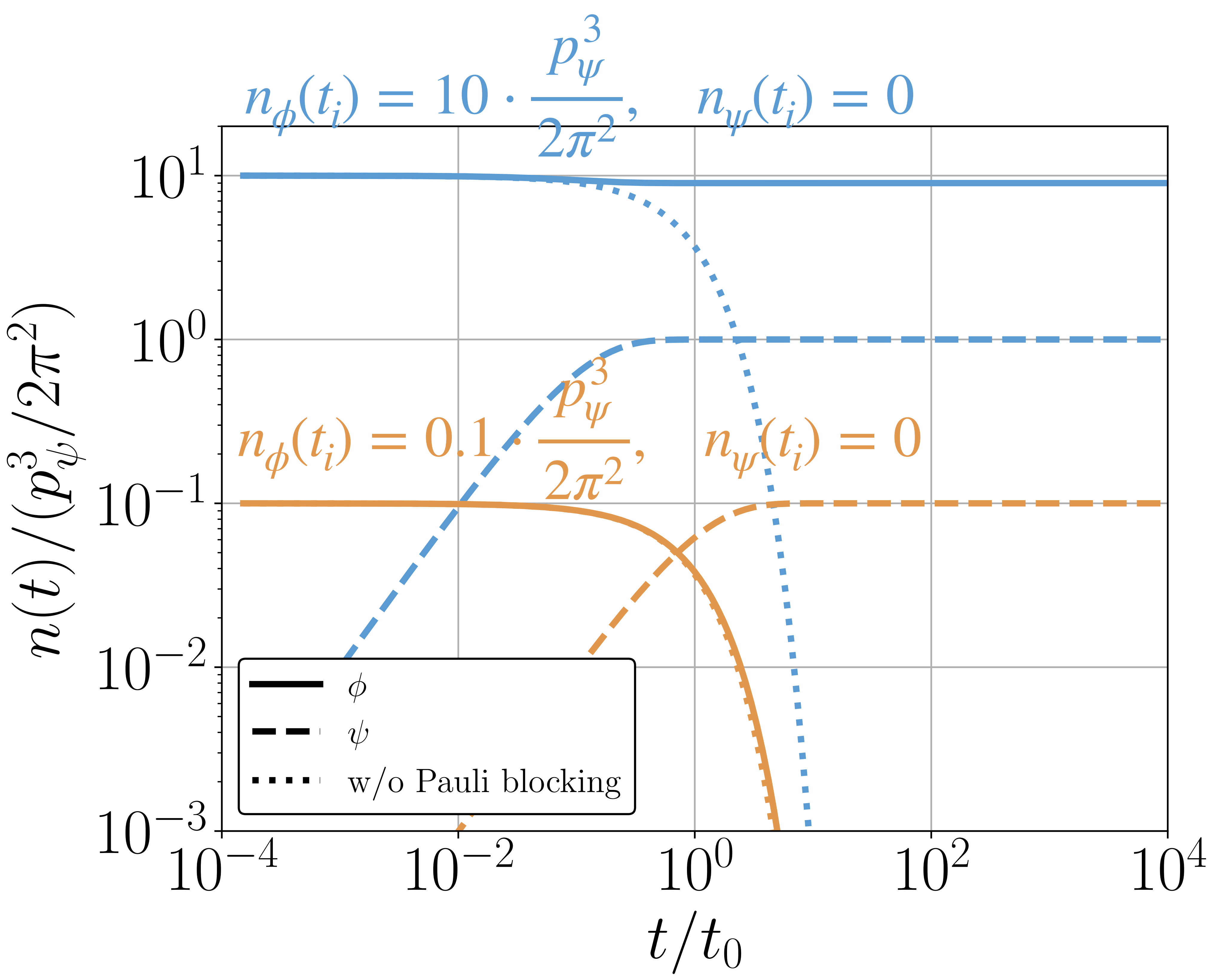
$$= \left(C_0 - \frac{p_\psi^3}{4\pi^2}\right)^{-1} + \left[\frac{1}{n_\phi(t_0)} - \left(C_0 - \frac{p_\psi^3}{4\pi^2}\right)^{-1}\right] \mathrm{Exp}\left(\Gamma_\phi(1 - C_0\frac{4\pi^2}{p_\psi^3})(t-t_0)\right)$$

- $C_0 - \frac{p_\psi^3}{4\pi^2} = n_\phi(t) + \frac{p_\psi^3}{2\pi^2}\left(f_\psi(p_\psi, t) - \frac{1}{2}\right) < 0 \qquad \Rightarrow n_\phi(t) \to 0, \quad f_\psi \to \frac{2\pi^2}{p_\psi^3} C_0$
- $C_0 - \frac{p_\psi^3}{4\pi^2} = n_\phi(t) + \frac{p_\psi^3}{2\pi^2}\left(f_\psi(p_\psi, t) - \frac{1}{2}\right) > 0$

$$\Rightarrow n_\phi(t) \to n_\phi(t_0) - \frac{p_\psi^3}{2\pi^2}\left(\frac{1}{2} - f_\psi(p_\psi, t_0)\right), \quad f_\psi \to \frac{1}{2}$$

Boltzmann Equation without Expansion

Analytical results



$$n_\phi(t) + n_\psi(t) = n_\phi(t) + \frac{p_\psi^3}{\pi^2} f_\psi(p_\psi, t) = C_0$$

$$n_\phi(t) + n_\psi(t) > \frac{p_\psi^3}{2\pi^2}$$

$$\Rightarrow n_\phi(t) \to n_\phi(t_0) - \frac{p_\psi^3}{\pi^2}\left(\frac{1}{2} - f_\psi(p_\psi, t_0)\right),$$

$$n_\psi(t) \to \frac{p_\psi^3}{2\pi^2}, \quad f_\psi \to \frac{1}{2}$$

$$n_\phi(t) + n_\psi(t) < \frac{p_\psi^3}{2\pi^2}$$

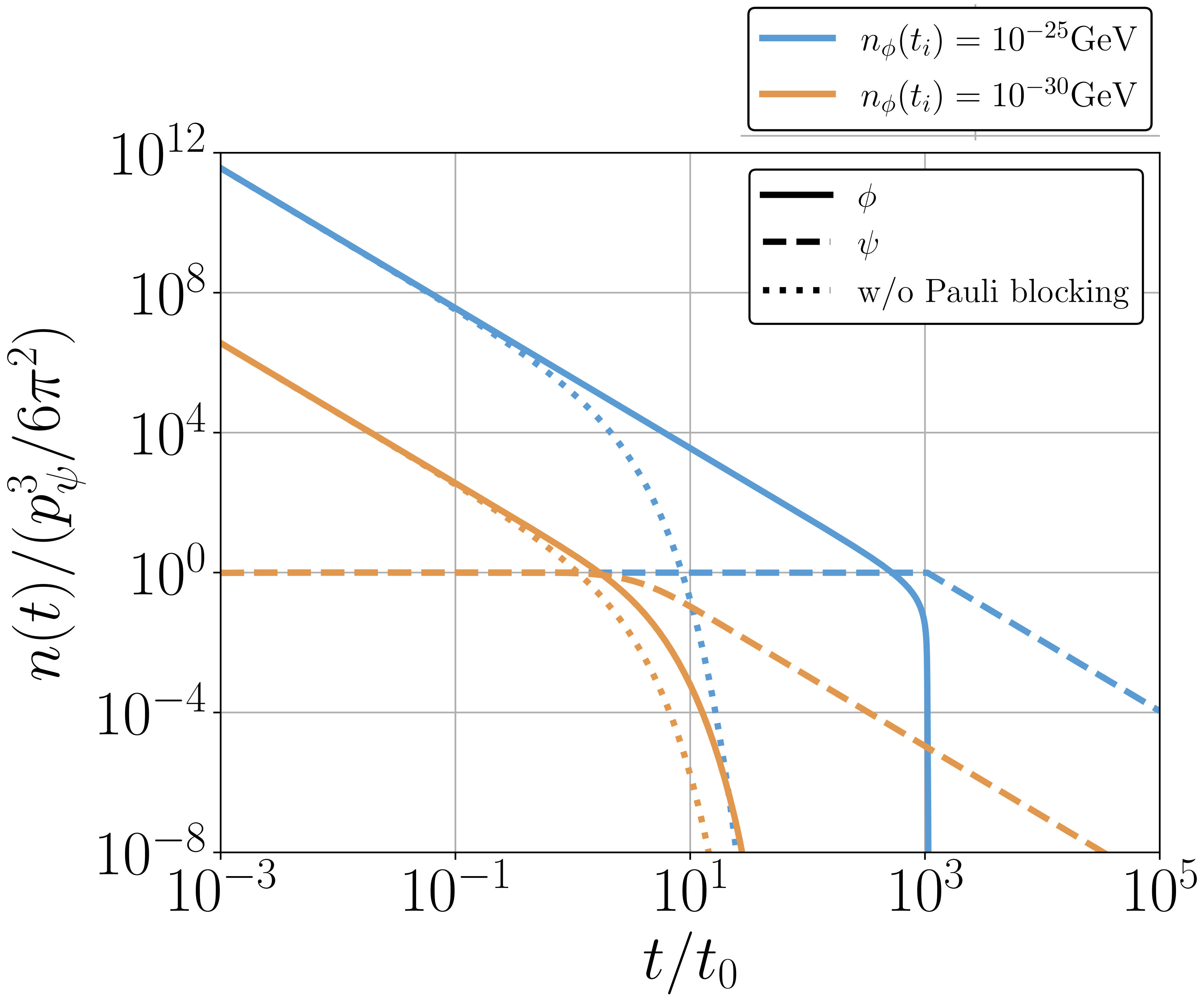
$$\Rightarrow n_\phi(t) \to 0$$

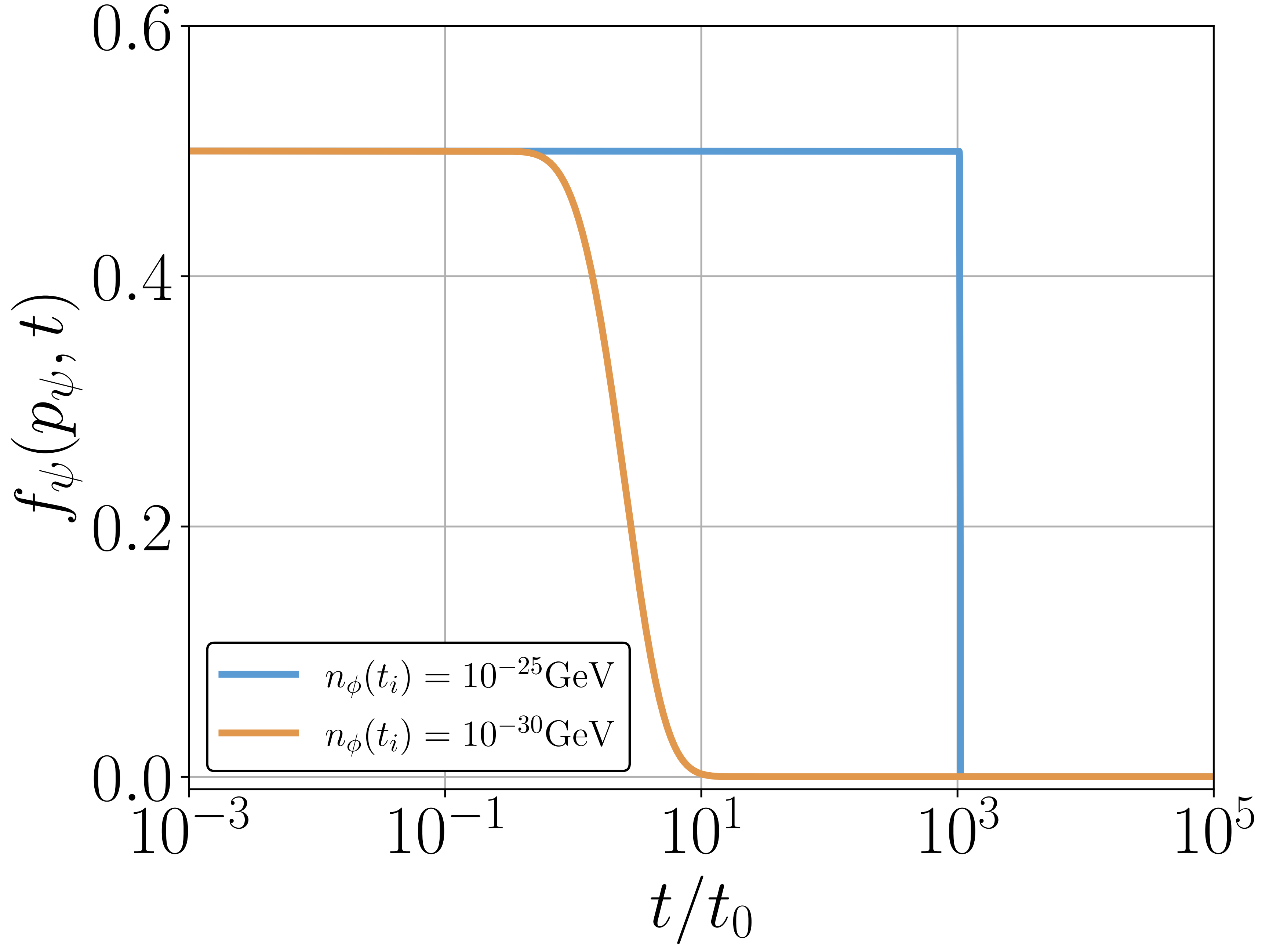
$$n_\psi(t) \to n_\phi(t_i), \quad f_\psi \to \frac{2\pi^2}{p_\psi^3} n_\phi(t_i)$$

Boltzmann Equation with Expansion

$$\dot{n}_\phi(t) + 3H(t)n_\phi(t) = -\Gamma_\phi n_\phi(t)\left(1 - 2f_\psi(p_\psi, t)\right)$$

$$\dot{n}_\psi(t) + 3H(t)n_\psi(t) = \Gamma_\phi n_\phi(t)\left(1 - 2f_\psi(p_\psi, t)\right)$$





Quantum Field Theory

- Hamiltonian:

$$H_{\text{int}}(t) = \lambda \int d^3x \phi(x)\overline{\psi}(x)\psi(x)$$

$$H_{\text{free}} = \int \frac{d^3p}{(2\pi)^3} \sum_s \omega_{\vec{\mathbf{p}}}(a^{s\dagger}_{\vec{\mathbf{p}}} a^s_{\vec{\mathbf{p}}} + b^{s\dagger}_{\vec{\mathbf{p}}} b^s_{\vec{\mathbf{p}}})$$

- Distribution function:

$$f_{\psi,\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle a^{\dagger r}_{\vec{k}} a^r_{\vec{k}}\right\rangle$$

$$f_{\bar{\psi},\vec{k}}(t) \equiv \frac{1}{2}\sum_r \frac{1}{V}\left\langle b^{\dagger r}_{\vec{k}} b^r_{\vec{k}}\right\rangle$$

- Particles/Fields:

*assumption: oscillation, homogeneous & classical

$$\phi(t) = \bar{\phi}\cos(m_\phi t)$$

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a^s_{\vec{p}}(t)u^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}} + b^{s\dagger}_{\vec{p}}(t)v^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}\right)$$

$$\bar{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a^{s\dagger}_{\vec{p}}(t)\bar{u}^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + b^s_{\vec{p}}(t)\bar{v}^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right)$$

- Fermion related function:

$$g_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r\sum_s \underline{\left\langle b^s_{-\vec{k}} a^r_{\vec{k}}\right\rangle}\, \bar{v}^s_{-\vec{k}} u^r_{\vec{k}}$$

→annihilation

$$g^*_{\vec{k}}(t) \equiv \frac{1}{4V\omega_k}\sum_r\sum_s \underline{\left\langle a^{r\dagger}_{\vec{k}} b^{s\dagger}_{-\vec{k}}\right\rangle}\, \bar{u}^r_{\vec{k}} v^s_{-\vec{k}}$$

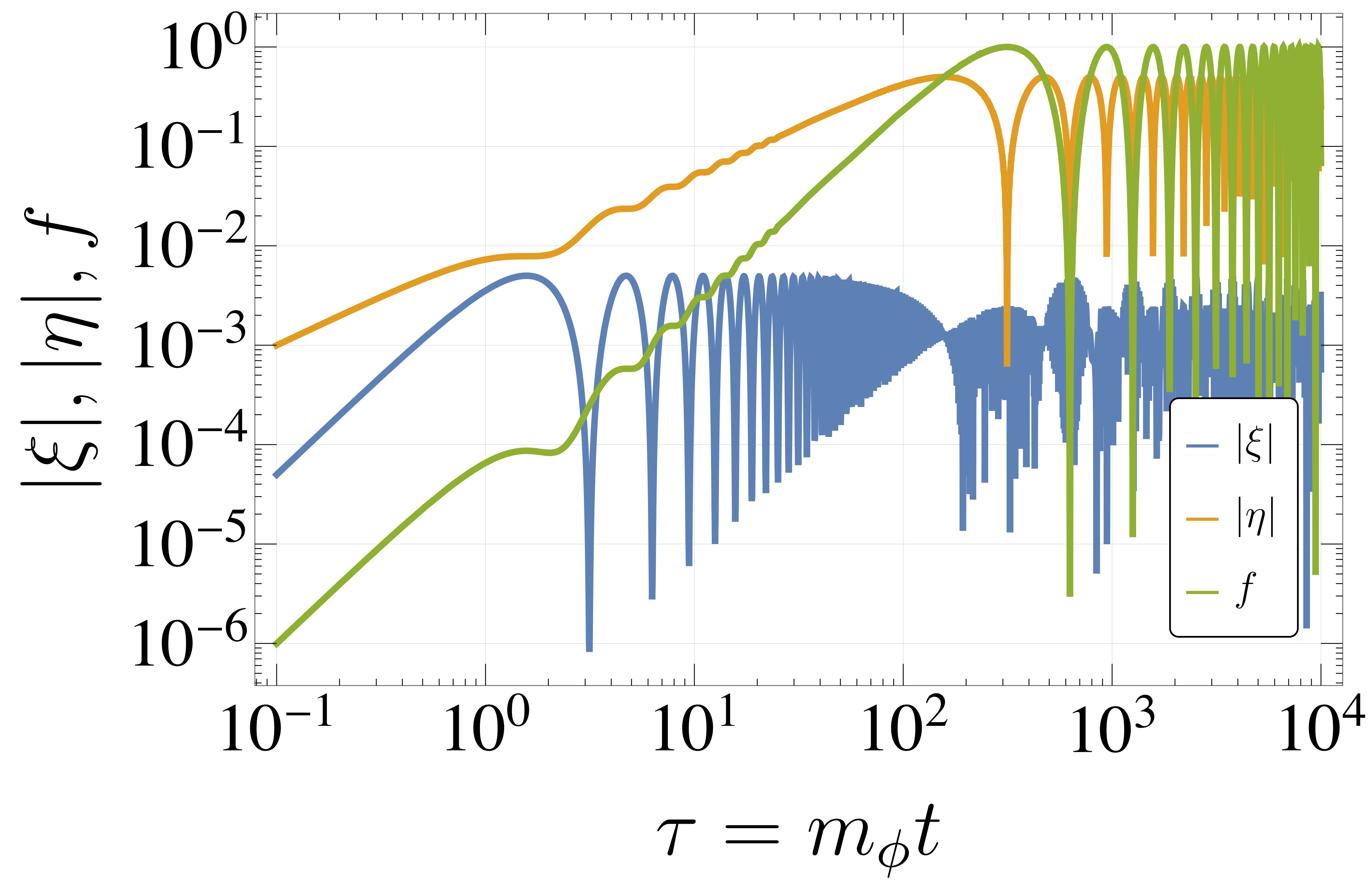
→ production

Quantum Field Theory without Expansion

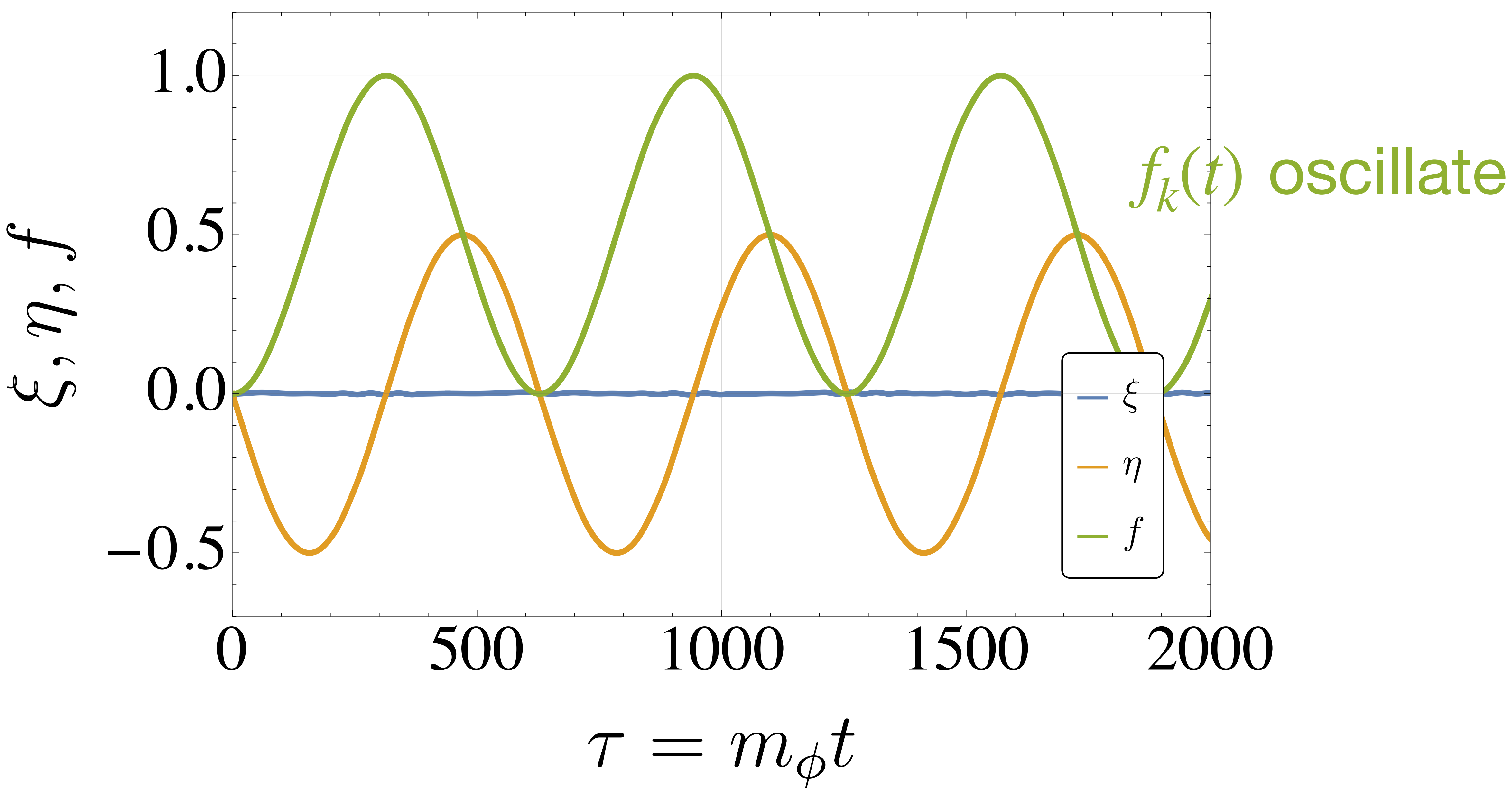
- Differential equations
  - $\phi(x,t) \to \phi(t) = \bar{\phi}\cos(m_\phi t)$
  - $\dot{f}_{\psi,\vec{k}} = i\lambda\bar{\phi}\left(g_{\vec{k}} - g^*_{\vec{k}}\right)\cos m_\phi t$
  - $\dot{f}_{\bar{\psi},\vec{k}} = i\lambda\bar{\phi}\left(g_{-\vec{k}} - g^*_{-\vec{k}}\right)\cos m_\phi t$
  - $\dot{g}_{\vec{k}} = -i\left[2\omega_k g_{\vec{k}}(t) + \lambda\bar{\phi}\left\{\frac{k^2}{\omega_k^2}\left(1 - f_{\psi,\vec{k}} - f_{\bar{\psi},-\vec{k}}\right) + \frac{2m_\psi}{\omega_k} g_{\vec{k}}\right\}\cos m_\phi t\right]$
- Differential equations
  - $\frac{df_k}{d\tau} = q\left[2\xi_k \sin\left(2\tilde{\omega}_k\tau\right) - 2\eta_k\cos\left(2\tilde{\omega}_k\tau\right)\right]\cos\tau$
  - $\frac{d\xi_k}{d\tau} = q\left[\left(1 - 2f_k\right)\sin(2\tilde{\omega}_k\tau)\right]\cos\tau$
  - $\frac{d\eta_k}{d\tau} = -q\left[\left(1 - 2f_k\right)\cos\left(2\tilde{\omega}_k\tau\right)\right]\cos\tau$
  - where $g_k(t) = e^{-2i\omega_k t}[\xi_k(t) + i\eta_k(t)],\ \tau = m_\phi t, \quad q = \lambda\bar{\phi}/m_\phi, \quad \text{and} \quad \tilde{\omega}_k = \omega_k/m_\phi$

Quantum Field Theory without Expansion

e-Print: 2407.08229 [hep-ph]



e-Print: 2407.08229 [hep-ph]



- Approximate solutions: *assumption: $m_\psi = 0$
  *approximation: $\omega_k \sim m_\phi/2 \Rightarrow \xi_k(\tau) \simeq 0$

$$f_k(\tau) \simeq \sin^2\left[\frac{1}{2}q\left(\tau + \cos\tau\sin\tau\right)\right] \sim \sin^2(q\tau/2)$$ → average value is 1/2

$$\eta_k(\tau) \simeq -\frac{1}{2}\sin\left[q(\tau + \cos\tau\sin\tau)\right] \sim -\frac{1}{2}\sin(q\tau)$$

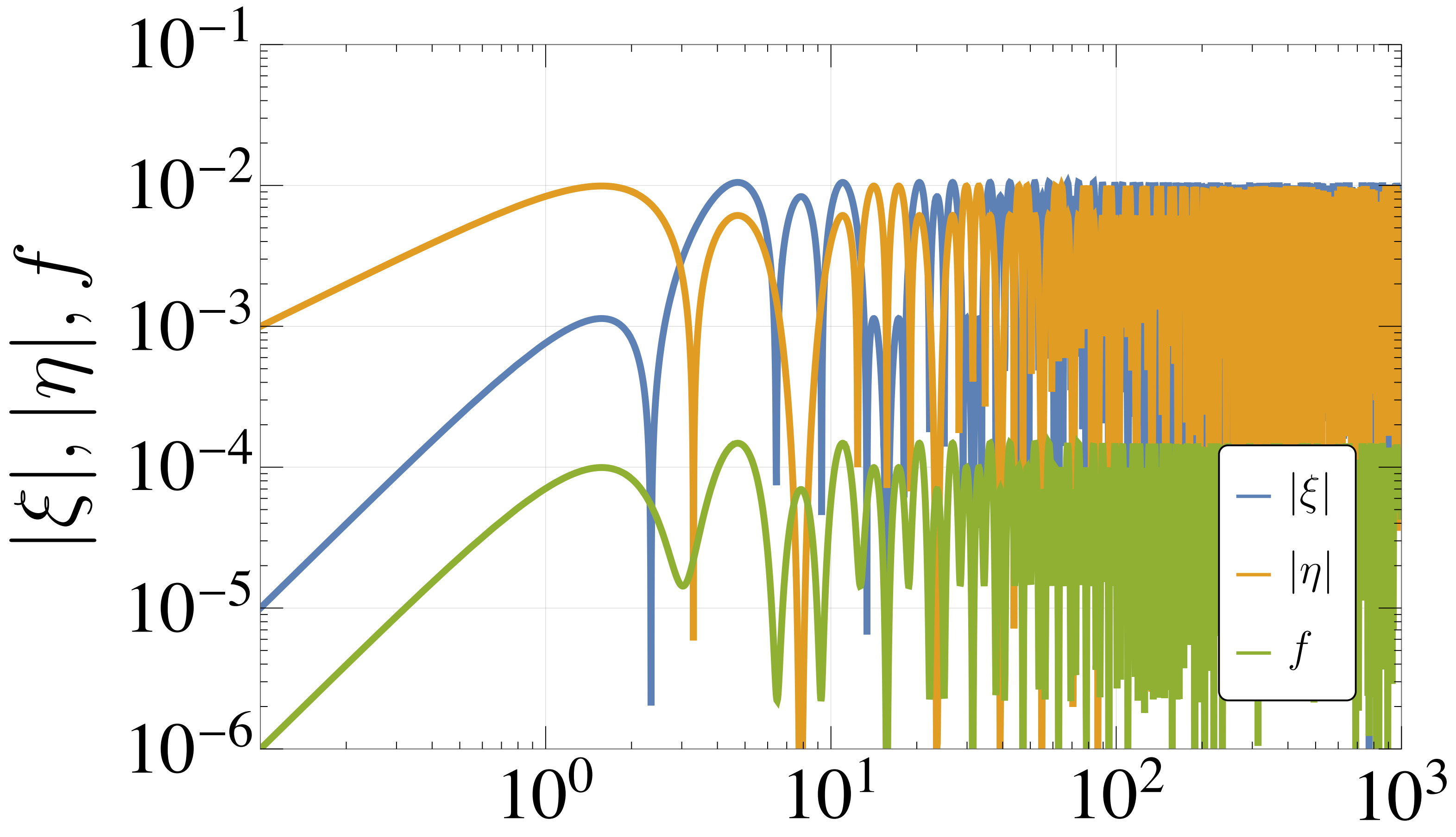
$$g_k(t) = e^{-2i\omega_k t}[\xi_k(t) + i\eta_k(t)]$$

$$\tau = m_\phi t, \quad q = \lambda\bar{\phi}/m_\phi, \quad \text{and} \quad \tilde{\omega}_k = \omega_k/m_\phi$$

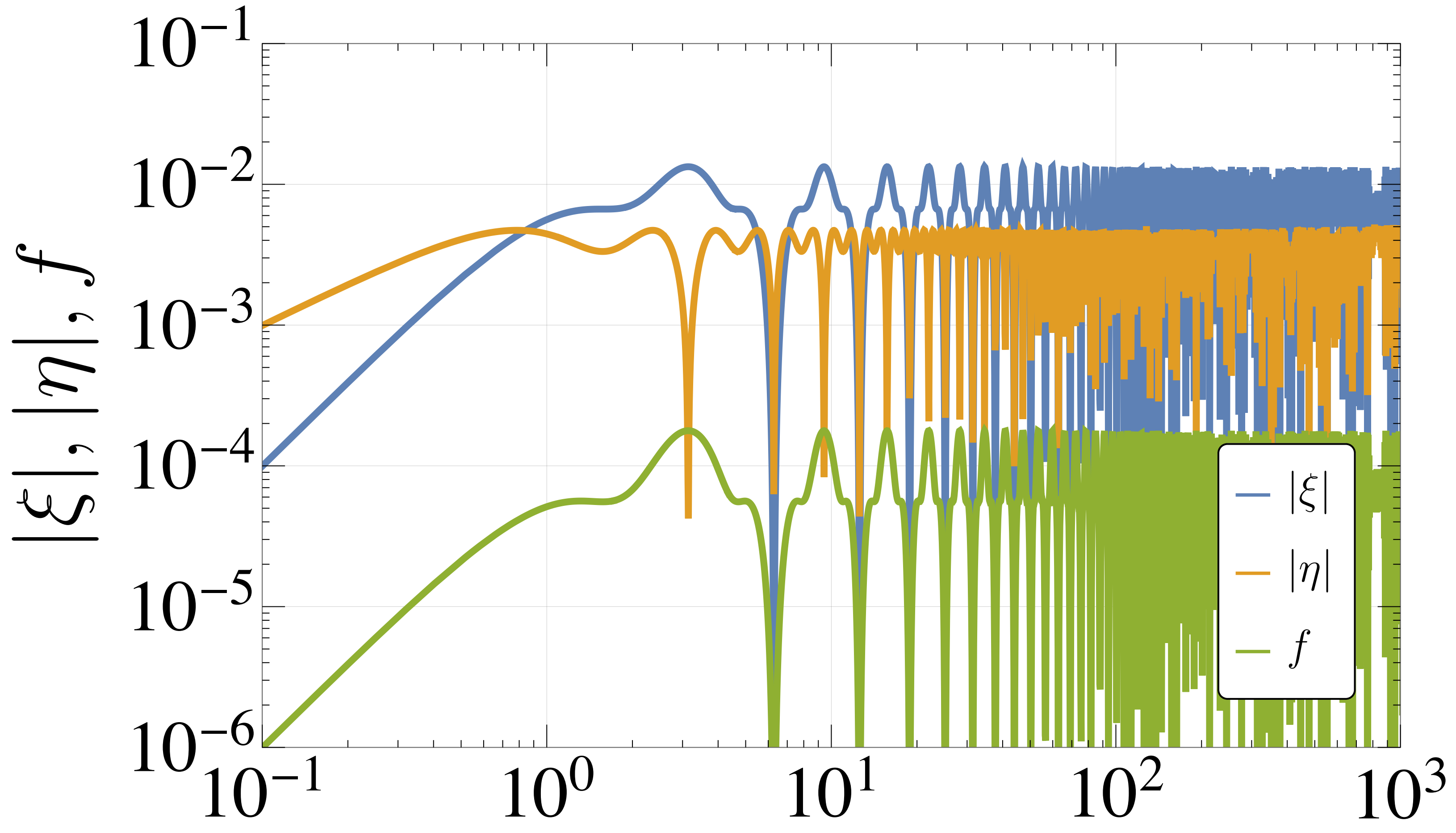
Quantum Field Theory without Expansion

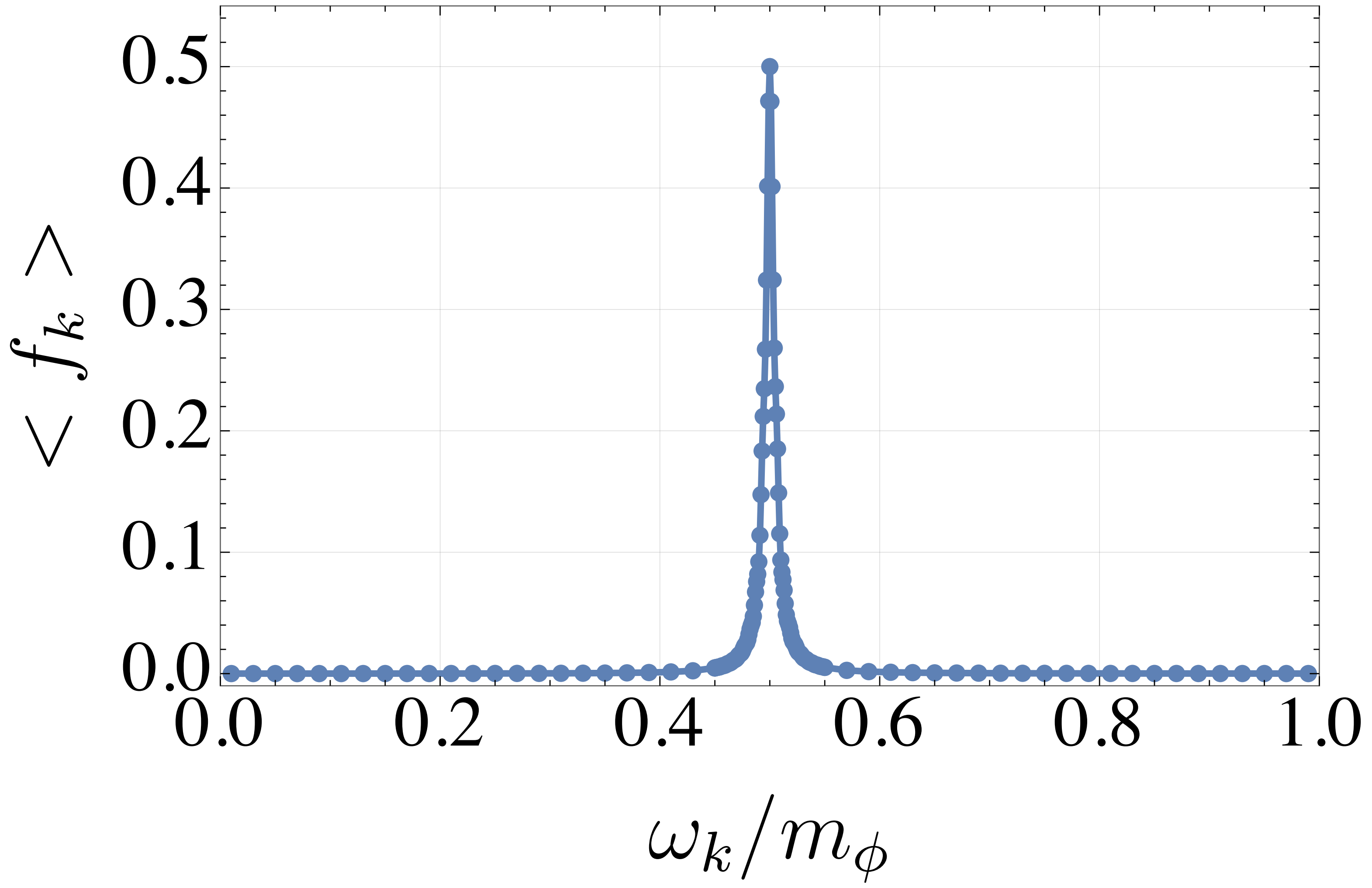
When ω_k is far from $m_\phi/2$

$\omega = 0.1 m_\phi$



$\omega = m_\phi$





$$f_k(\tau) \simeq \frac{q^2\left(1 + 12\tilde{\omega}_k^2 + (-1 + 4\tilde{\omega}_k^2)\cos(2\tau) - 8\tilde{\omega}_k(2\tilde{\omega}_k\cos(\tau)\cos(2\tilde{\omega}_k\tau) + \sin(\tau)\sin(2\tilde{\omega}_k\tau))\right)}{2(1 - 4\tilde{\omega}_k^2)^2}$$

$$\xi_k(\tau) \simeq \frac{-q\left(-2\tilde{\omega}_k + 2\tilde{\omega}_k\cos(\tau)\cos(2\tilde{\omega}_k\tau) + \sin(\tau)\sin(2\tilde{\omega}_k\tau)\right)}{-1 + 4\tilde{\omega}_k^2}$$

$$\eta_k(\tau) \simeq \frac{q\left(\cos(2\tilde{\omega}_k\tau)\sin(\tau) - 2\tilde{\omega}_k\cos(\tau)\sin(2\tilde{\omega}_k\tau)\right)}{-1 + 4\tilde{\omega}_k^2}$$

Quantum Field Theory with Expansion

- Differential equations:
  - $\dot{f}_k = kH\dfrac{\partial f_k}{\partial k} + i\lambda\phi(t)\left(g_k - g_k^*\right)$
  - $\dot{g}_k = kH\dfrac{\partial g_k}{\partial k} - i\left[2\Omega_k(t)g_k(t) + \lambda\phi(t)\left\{\dfrac{K(t)^2}{\Omega(t)^2}\left(1 - 2f_k\right) + \dfrac{2m_\psi}{\Omega(t)}g_k\right\}\right]$

$$\Omega(\tau) = \left(\frac{\tau_0}{\tau}\right)^{2/3}\omega_k$$

$$\Theta(\tau) = \frac{3\omega_k}{m_\phi}\tau_0\left(\frac{\tau}{\tau_0}\right)^{1/3}$$

$$K(t) = \frac{a_0}{a(t)}\hat{k} = \frac{m_\phi}{2}$$

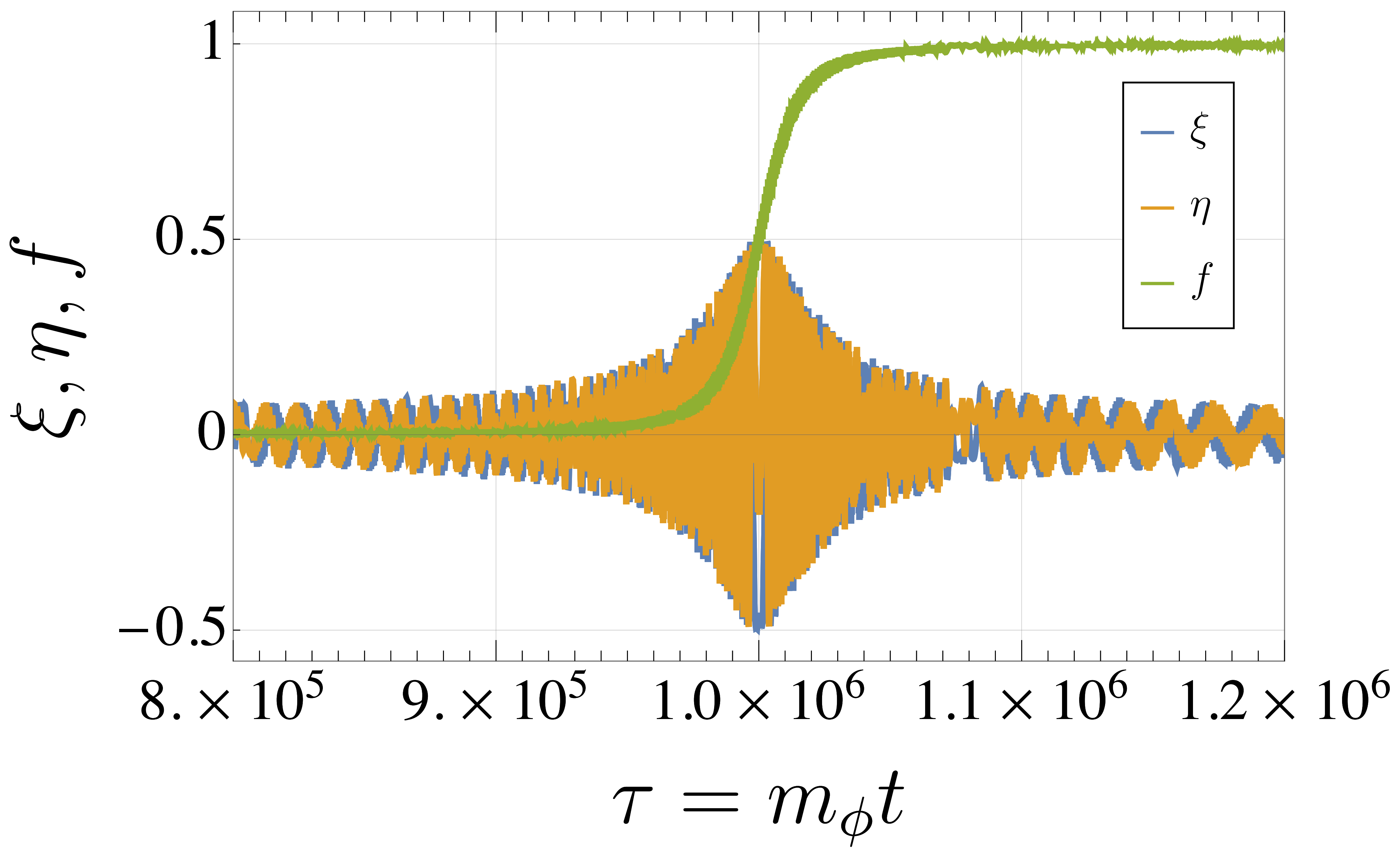
- Differential equations:
  - $\dfrac{df_*(\tau)}{d\tau} = q\left(\dfrac{\tau}{\tau_0}\right)^{-1}\left(2\xi_*(\tau)\sin 2\Theta(\tau) - 2\eta_*(\tau)\cos 2\Theta(\tau)\right)\cos\tau$
  - $\dfrac{d\xi_*(\tau)}{d\tau} = q\left(\dfrac{\tau}{\tau_0}\right)^{-1}\left[\left(1 - 2f_*(\tau)\right)\sin 2\Theta(\tau)\right]\cos\tau$
  - $\dfrac{d\eta_*(\tau)}{d\tau} = -q\left(\dfrac{\tau}{\tau_0}\right)^{-1}\left[\left(1 - 2f_*(\tau)\right)\cos 2\Theta(\tau)\right]\cos\tau$

$$q = \frac{\lambda\bar{\phi}}{m_\phi}, \quad \tilde{K}(t) = \frac{K(t)}{m_\phi}, \quad \tilde{\Omega}(t) = \frac{\Omega(t)}{m_\phi}$$

$$f_k(1 - f_k) = \xi_k^2 + \eta_k^2 = g_k g_k^* \geq 0$$

Quantum Field Theory with Expansion

e-Print: 2407.08229 [hep-ph]



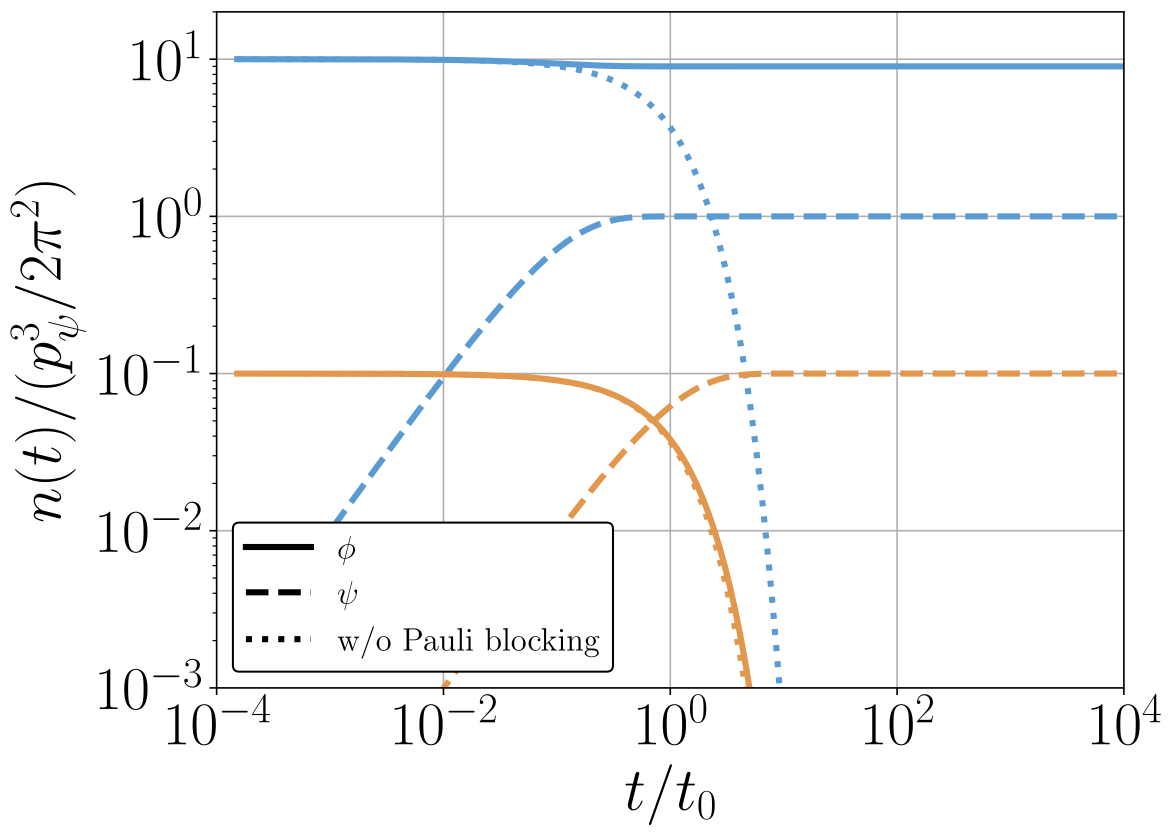
$f_\psi \to 1$ instead of $f_\psi \to 1/2$

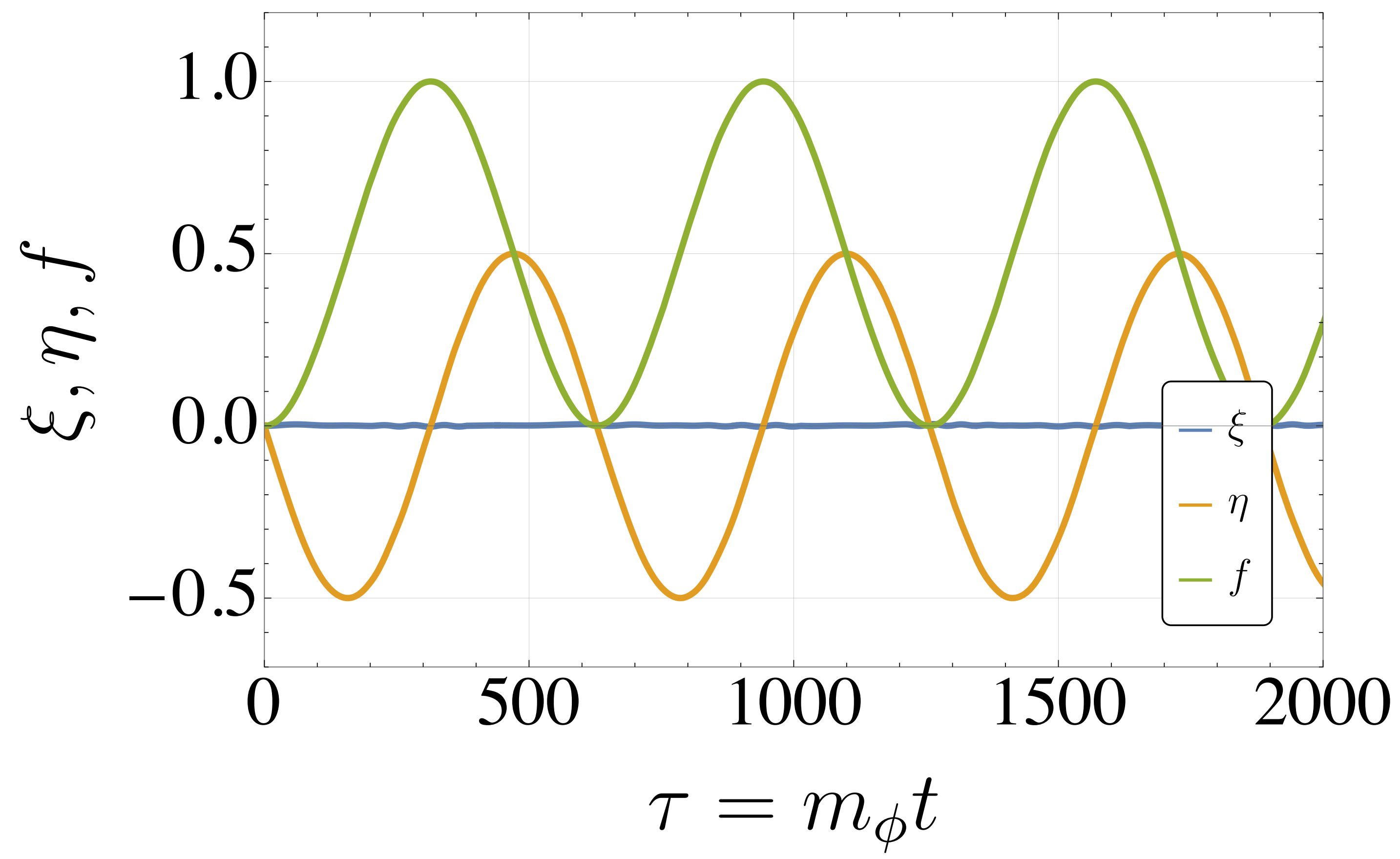
"The difference can be cured if the factor $1 \pm 2f_\psi$ in the Boltzmann equation is modified to $1 \pm f_\psi$"

Boson case: Ref. Takeo Moroi, et al. JHEP 03 (2021), 296

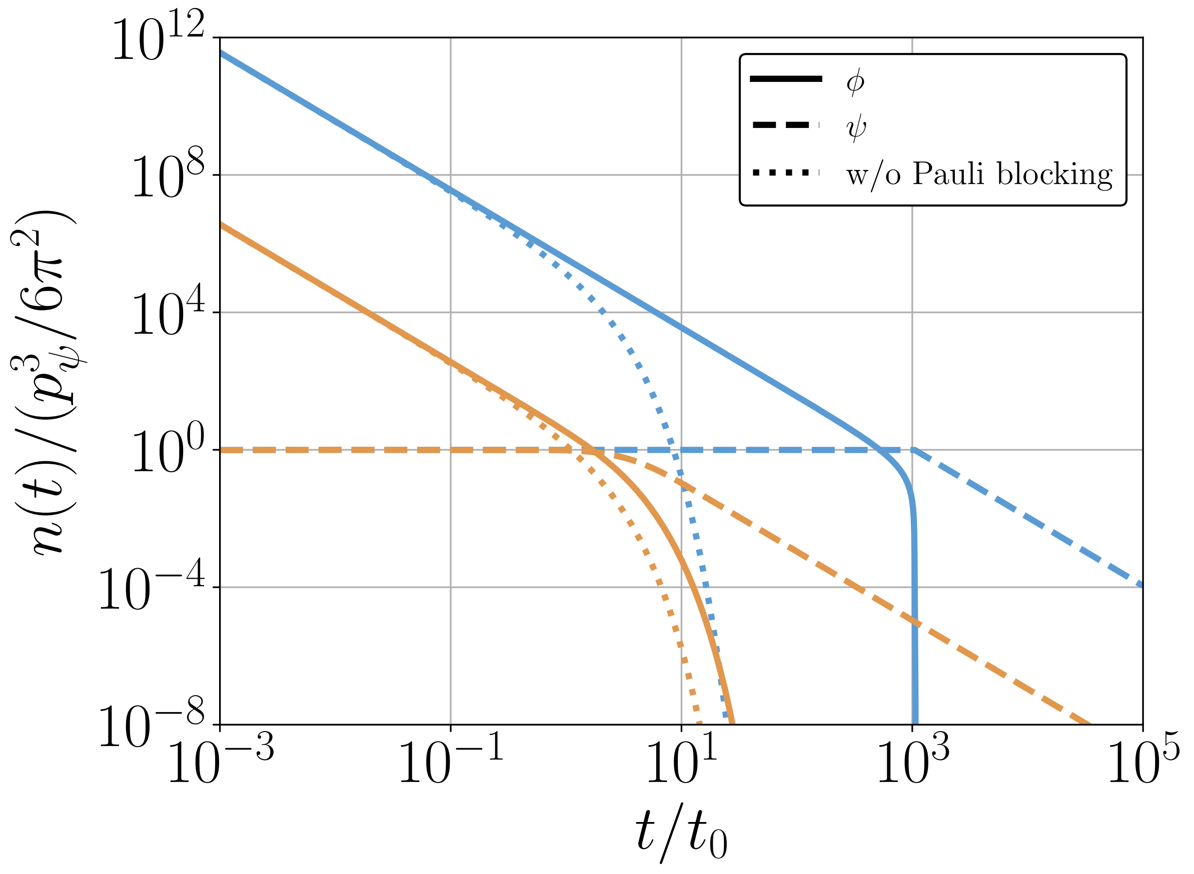
*assumption: $m_\psi = 0$

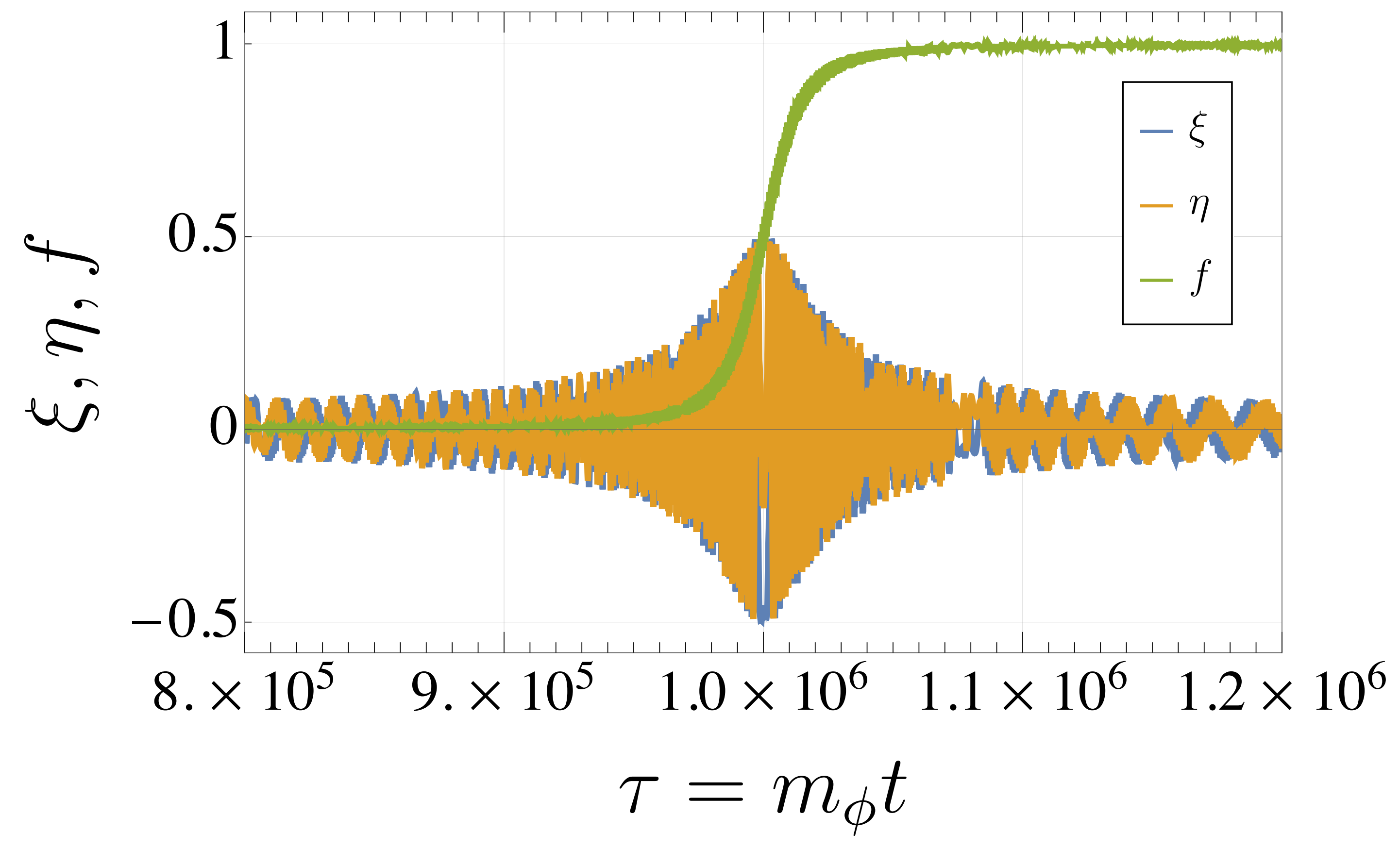
Comparison between Boltzmann & QFT





	Boltzmann	QFT
w/o Expansion	$f_\psi \to 1/2$	Oscillate & $\langle f_\psi \rangle \to 1/2$
w/ Expansion	$f_\psi \to 1/2$	$f_\psi \to 1$





Example_Type-II seesaw mechanism

- Lagrangian of Yukawa interaction:

$$\mathscr{L}_{\text{Yukawa}} \supset -\frac{1}{\sqrt{2}} Y_\Delta^{ij} \overline{l_L^{iC}} \cdot \Delta l_L^j + h.c.$$

- Left-handed neutrino mass matrix:

$$\left(\mathscr{M}_\nu\right)_{ij} = Y_\Delta^{ij} v_\Delta$$

- The scalar potential:

$$V = V_1 + V_2$$

$$\begin{aligned} V_1 = & -\hat{\mu}_1{}^2 \left|\Phi\right|^2 + \frac{1}{2}\lambda_1 \left|\Phi\right|^4 \\ & + \hat{\mu}_3{}^2 \operatorname{Tr}\left(\Delta^\dagger \Delta\right) + \frac{1}{2}\Lambda_1 \left(\operatorname{Tr}\left(\Delta^\dagger \Delta\right)\right)^2 + \frac{1}{2}\Lambda_2 \left(\left(\operatorname{Tr}\left(\Delta^\dagger \Delta\right)\right)^2 - \operatorname{Tr}\left[\left(\Delta^\dagger \Delta\right)^2\right]\right) \\ & + \Lambda_4 \left|\Phi\right|^2 \operatorname{Tr}\left(\Delta^\dagger \Delta\right) + \Lambda_5 \Phi^\dagger \left[\Delta^\dagger, \Delta\right] \Phi - \frac{\Lambda_6}{\sqrt{2}} \left(\Phi^T \cdot \Delta \Phi + \text{ H.c. }\right) \end{aligned}$$

$$V_2 = \frac{1}{2}\mu_S^2 S^2 + \frac{1}{4}\lambda_S S^4 + \frac{1}{2}\lambda_{S1} S^2 \left|\Phi\right|^2 + \frac{1}{2}\lambda_{S\Delta} S^2 \operatorname{Tr}\left(\Delta^\dagger \Delta\right) - \frac{\lambda_6}{\sqrt{2}} S \left(\Phi^T \cdot \Delta \Phi + \text{ H.c. }\right)$$

l_L : left-handed lepton doublet

Δ : triplet Higgs field

Φ : SM doublet Higgs

S : Real singlet scalar (dark matter candidate)

$$\tau_\phi \simeq 132 \times t_0 \left(\frac{10^{-3}\,\mathrm{eV}}{m_\phi} \right)^2 (\mathrm{MD}), \quad \tau_\phi \simeq 3.8 \times t_0 \left(\log \left(\frac{m_\phi}{10^{-3}\,\mathrm{eV}} \right)^4 \right) (\mathrm{DE})$$

Example_Type-II seesaw mechanism

- Neutral component: S, Φ and $\Delta \to s, \phi^0$ and δ^0

- The unitary matrix diagonalizing the mass-squared matrix:

$$U = \begin{pmatrix} U_{s1} & U_{s2} & U_{s3} \\ U_{\phi^0 1} & U_{\phi^0 2} & U_{\phi^0 3} \\ U_{\delta^0 1} & U_{\delta^0 2} & U_{\delta^0 3} \end{pmatrix}$$

- Decay rate:

$$\begin{aligned} \Gamma\left(\phi_1 \to \nu\nu\right) &\sim \frac{1}{4\pi}\left|Y_\Delta U_{\delta^0 1}\right|^2 m_{\phi_1} \\ &\sim \frac{m_{\phi_1}}{4\pi}\frac{m_\nu^2}{v_\Delta^2}\left|U_{\delta^0 1}\right|^2 \\ &\simeq 8 \times 10^{-43}\,\mathrm{GeV}\left(\frac{\mathrm{m}_{\phi_1}}{10^{-3}\,\mathrm{eV}}\right)\left(\frac{\mathrm{m}_\nu}{10^{-6}\,\mathrm{eV}}\right)^2\left(\frac{1\,\mathrm{keV}}{\nu_\Delta}\right)^2\left(\frac{\left|\mathrm{U}_{\delta^0 1}\right|^2}{10^{-11}}\right) \end{aligned}$$

where $U_{\delta^0 1} \sim \dfrac{M_{\delta^0}^2}{M_{\delta^0\delta^0}^2} \simeq \dfrac{\lambda_6 v^2}{\hat{\mu}_3^2} \simeq 10^{-5}\left(\dfrac{\lambda_6}{0.1}\right)\left(\dfrac{10^4\,\mathrm{GeV}}{\hat{\mu}_3}\right)^2$

$\Rightarrow \Gamma^{-1} > \mathcal{O}(10)t_0$