V. From quantum to classical perturbation

1. Classical interpretation for free theory
   a. Large-scale solutions for mode functions
   b. Commutator for canonical variables on large-scale
2. Effects of non-linear interactions
   a. System and environment in inflation
   b. Reduced density matrix and Lindblad equation
   c. Decoherence of cosmological perturbation

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a. Large-scale solutions for mode functions

- What is meant by "classical mechanics"?
  1. Classical mechanics is all about what we can see around
     - Thus, classical mechanics naturally incorporate a large number of particles (although the description is given by a limited number)
     - Indeed, during inflation the number of perturbation particles become very very large even if we have started from vacuum:

$$|0\rangle \;\to\; \sum_{n=0}^{\infty} |n, \boldsymbol{k}; n, -\boldsymbol{k}\rangle$$

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     - But the distinction between "classical" & "quantum" is not very clear: Is 1 particle quantum? 10 particles quantum?
  2. Classical mechanics is the regime where $[x(t), p(t)] \to 0$
     - Commutation relation between canonical variables is the key!

- Once we separately consider the conjugate variables $v_{\boldsymbol{k}}$ and $\pi_{\boldsymbol{k}}$, we can define their mode functions:

$$u_{\boldsymbol{k}} = f_k(\tau)a_{\boldsymbol{k}}(\tau_0) + f_k^*(\tau)a^\dagger_{-\boldsymbol{k}}(\tau_0)$$

$$\pi_{\boldsymbol{k}} = -ig_k(\tau)a_{\boldsymbol{k}}(\tau_0) + ig_k^*(\tau)a^\dagger_{-\boldsymbol{k}}(\tau_0)$$

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- These mode functions f_k and g_k are given in terms of the Bogoliubov coefficients α_k and β_k:

$$f_k = \frac{1}{\sqrt{2k}}(\alpha_k + \beta_k^*)$$
$$g_k = \sqrt{\frac{k}{2}}(\alpha_k - \beta_k^*)$$

- Note that the constraint $|\alpha_k|^2 - |\beta_k|^2 = 1$ can be written as

$$|\alpha_k|^2 - |\beta_k|^2 = 1 = f_k^* g_k + f_k g_k^* = 2\Big[\Re f_k \Re g_k + \Im f_k \Im g_k\Big]$$

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- The equations for f_k and g_k follow from those for $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$ (or equivalently those for α_k and β_k):

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- We are mostly interested in large-scale limit $k^2 \ll a''/a \sim (aH)^2$
  1. During inflation, all observable scales go outside the horizon
  2. On super-horizon scales, the cosmological perturbations of our interest (both scalar and tensor) are frozen to maintain constant amplitude

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$$\frac{(y_k')'}{y_k'} = -2\frac{a'}{a} \qquad \therefore \;\; y_k' = \frac{c_2}{a^2}\,, \qquad \therefore \;\; y_k = c_1 + c_2 \int \frac{d\tau}{a^2}$$

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$$f_k = \boxed{c_1 a} + \boxed{c_2 a \int \frac{d\tau}{a^2}}$$

growing

$$g_k = i\frac{c_2}{a}$$

$\propto a^{-2}$ ($\because a \propto \tau^{-1}$), decaying

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- Remember the hyperbolic parametrization of the Bogoliubov coeff:

$$\alpha_k = \frac{1}{2}\left(\sqrt{2k} f_k + \sqrt{\frac{2}{k}} g_k\right) = e^{-i\theta_k} \cosh r_k$$

$$\beta_k = \frac{1}{2}\left(\sqrt{2k} f_k^* - \sqrt{\frac{2}{k}} g_k^*\right) = e^{i(\theta_k + 2\phi_k)} \sinh r_k$$

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- Multiplying these 2 and taking its complex conjugate give respectively:

$$e^{2i\phi_k} \cosh r_k \sinh r_k = \frac{1}{4}\left(\sqrt{2k} f_k + \sqrt{\frac{2}{k}} g_k\right)\left(\sqrt{2k} f_k^* - \sqrt{\frac{2}{k}} g_k^*\right)$$

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- Taking the absolute value of any of these 2 gives:

$$\cosh r_k \sinh r_k = \frac{1}{2}\sqrt{\left(k|f_k|^2 - \frac{1}{k}|g_k|^2\right)^2 + 4\Big[\Im(f_k g_k^*)\Big]^2}$$

$$\therefore \quad \cos(2\phi_k) = \frac{k|f_k|^2 - \frac{1}{k}|g_k|^2}{\sqrt{\left(k|f_k|^2 - \frac{1}{k}|g_k|^2\right)^2 + 4\Big[\Im(f_k g_k^*)\Big]^2}}$$

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- We need $\cos(2\phi_k)$ because we want to solve the equation for the squeezing parameter $r_k' = \cos(2\phi_k)a'/a$, from which we can find the Bogoliubov coefficients α_k and β_k so that the commutation relation $[u_{\boldsymbol{k}}, \pi_{\boldsymbol{q}}]$ on large-scale can be found!

- Now, plugging the large-scale solutions for f_k and g_k gives $\cos(2\phi_k)$ as

$$\cos(2\phi_k) \underset{-k\tau\to 0}{\longrightarrow} 1 - 2k^2\tau^2 + \cdots$$

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- This means on large-scales $\phi_k \to \pi$, so that we can consistently find

$$\cos\phi_k = -\sqrt{\frac{1+\cos(2\phi_k)}{2}} = -1 + \frac{1}{2}(-k\tau)^2 + \cdots$$
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- Thus we can solve $r_k(\tau)$ to find:

$$r'_k = -\frac{1}{\tau} + 2k^2\tau + \cdots$$
$$\therefore\ r_k = -\log(-k\tau) + (-k\tau)^2 + \cdots$$ $\gg 1$ as $-k\tau \to 0$

- In this limit $r_k \gg 1$, f_k and g_k approach $(\theta_k + \phi_k \equiv \delta_k)$

$$f_k \underset{-k\tau\to 0}{\longrightarrow} \frac{e^{-i\delta_k}}{\sqrt{2k}}\left(\frac{\cos\phi_k}{-k\tau} - e^{i\phi_k}k\tau + \cdots\right)$$

$$g_k \underset{-k\tau\to 0}{\longrightarrow} \sqrt{\frac{k}{2}}e^{-i\delta_k}\left(\frac{i\sin\phi_k}{-k\tau} - e^{i\phi_k}k\tau + \cdots\right)$$

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- It is sufficient to proceed further with the above form of f_k and g_k, but we can further expand $\cos\phi_k$ and $\sin\phi_k$ to find $(\delta_k \equiv \theta_k + \phi_k)$

$$f_k = \frac{e^{-i\delta_k}}{\sqrt{2k}}\left(-\frac{1}{-k\tau} + \frac{k\tau}{2} + \cdots\right)$$

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- In this limit $r_k \gg 1$, f_k and g_k approach ($\theta_k + \phi_k \equiv \delta_k$)

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$$g_k = \sqrt{\frac{k}{2}} e^{-i\delta_k} \left[i - (-k\tau) + \cdots \right]$$

- Thus, upon fixing $\delta_k = \pi/2$ we find f_k is consistent with full expression (only c_1-term contributes) and g_k is not (but in fact it is, as i comes from c_1-term)

- Indeed, from the exact de Sitter solutions we find

$$r_k = \sinh^{-1}\left[\frac{1}{2(-k\tau)}\right] \underset{-k\tau\to 0}{\longrightarrow} -\log(-k\tau) + (-k\tau)^2 + \cdots$$

$$\theta_k = k\tau - \tan^{-1}\left[\frac{1}{2(-k\tau)}\right] \underset{-k\tau\to 0}{\longrightarrow} -\frac{\pi}{2} + (-k\tau) + \cdots$$

$$\phi_k = \frac{3}{4}\pi + \frac{1}{2}\tan^{-1}\left[\frac{1}{2(-k\tau)}\right] \underset{-k\tau\to 0}{\longrightarrow} \pi - (-k\tau) + \cdots$$

- Thus, the results from exact de Sitter solutions and from large-scale approximation are consistent!

- Keeping up to leading-order in large-scale approximation gives:

$$f_k^* g_k + f_k g_k^* \underset{-k\tau \to 0}{\longrightarrow} 0$$

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- Dynamical evolution of cosmological perturbations **guarantees** that on very large scales, viz. as inflation proceeds, the commutator between canonical variables vanishes
- Quantum cosmological perturbations on super-horizon scales become **classical** in the sense that the commutator vanishes

- If, however, keeping the next-to-leading contributions in the large-scale approximation gives:

$$
\begin{aligned}
f_k^* g_k + f_k g_k^* \underset{-k\tau\to 0}{\longrightarrow} \frac{1}{2}\Bigg[&\left(\frac{\cos\phi_k}{-k\tau} - e^{-i\phi_k}k\tau + \cdots\right)\left(\frac{i\sin\phi_k}{-k\tau} - e^{i\phi_k}k\tau + \cdots\right) \\
&+ \left(\frac{\cos\phi_k}{-k\tau} - e^{-i\phi_k}k\tau + \cdots\right)\left(\frac{-i\sin\phi_k}{-k\tau} - e^{-i\phi_k}k\tau + \cdots\right)\Bigg] \\
= \frac{1}{2}\Big[\cos\phi_k\big(e^{i\phi_k} + e^{-i\phi_k}\big) - i\sin\phi_k\big(e^{i\phi_k} - e^{-i\phi_k}\big) + \mathcal{O}(k\tau)\Big] \\
= 1 + \mathcal{O}(k\tau)
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- We recover the **non-vanishing** commutator
- In that sense, large-scale approximation resembles WKB approximation in QM: Quantumness (viz. non-vanishing commutator) is revealed only if we go to sub-leading order in large-scale approximation!

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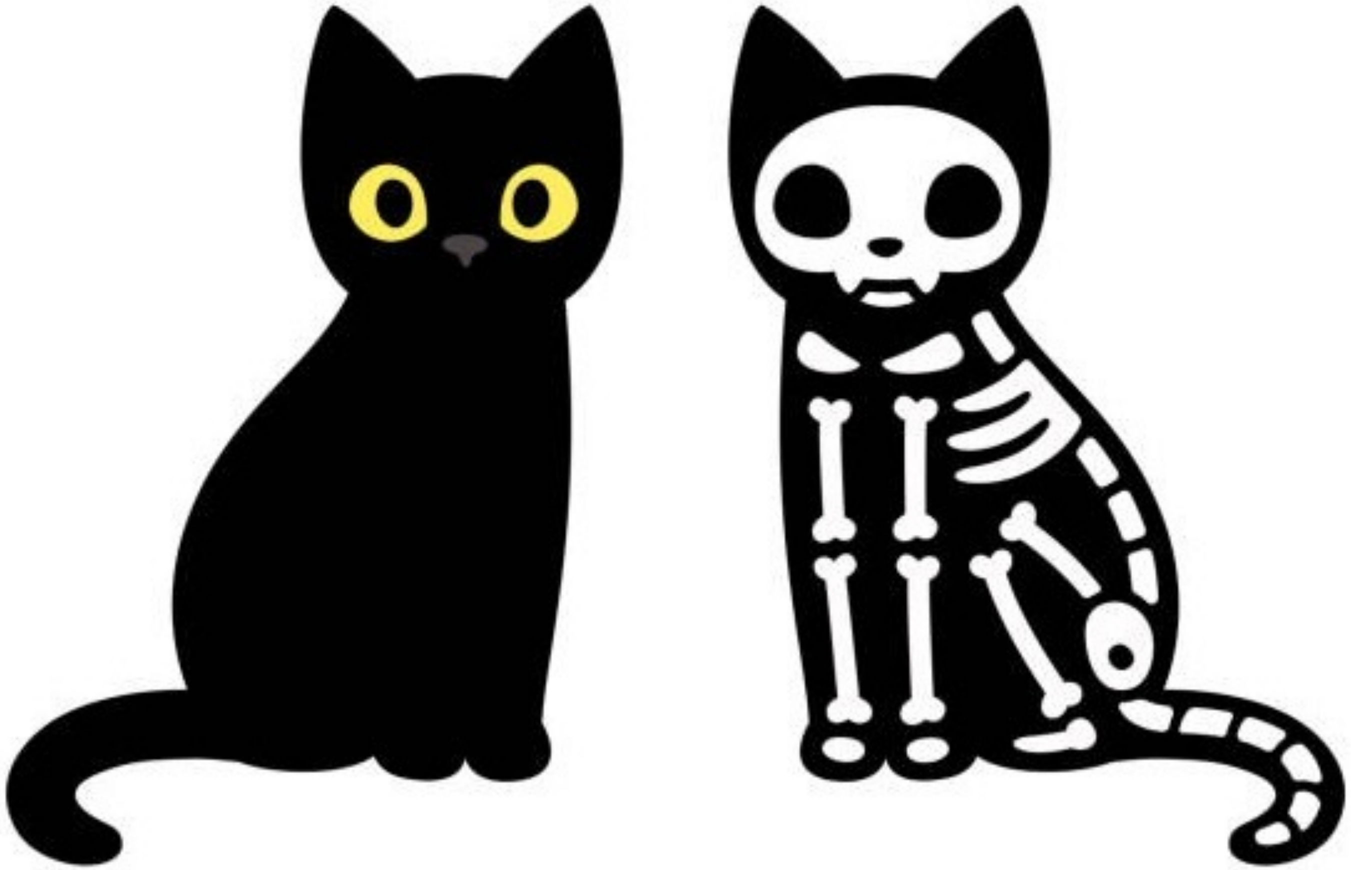
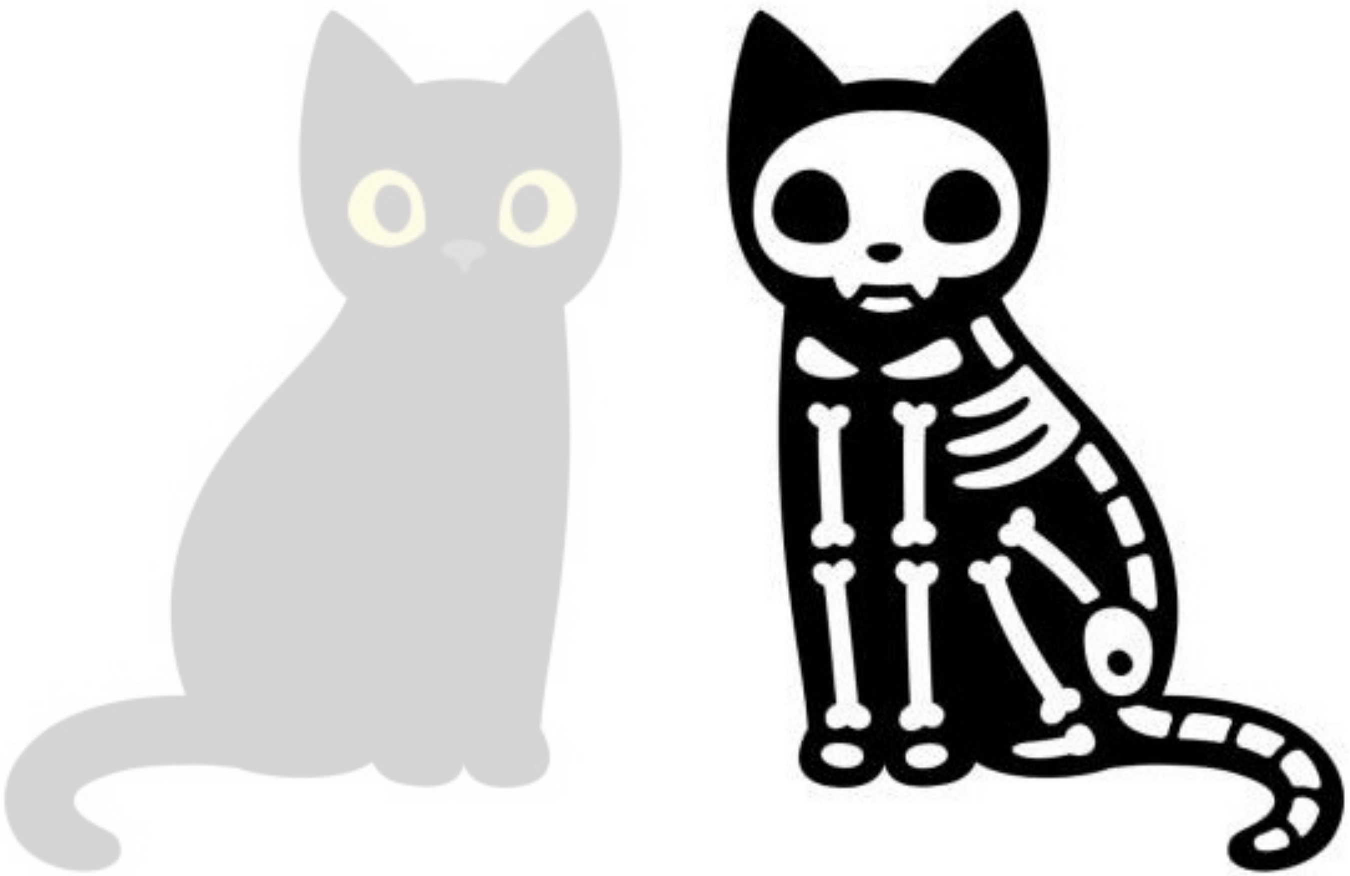
- In QM, if (say) 2 states are possible, so is their linear combination
- If a QM cat has 2 states $|\mathrm{dead}\rangle$ and $|\mathrm{alive}\rangle$, the state $|\mathrm{dead}\rangle + |\mathrm{alive}\rangle$ is possible
- A QM cat can be dead **AND** alive



- In CM, however, we have not seen a cat that is dead **AND** alive
- A cat is **either** dead…



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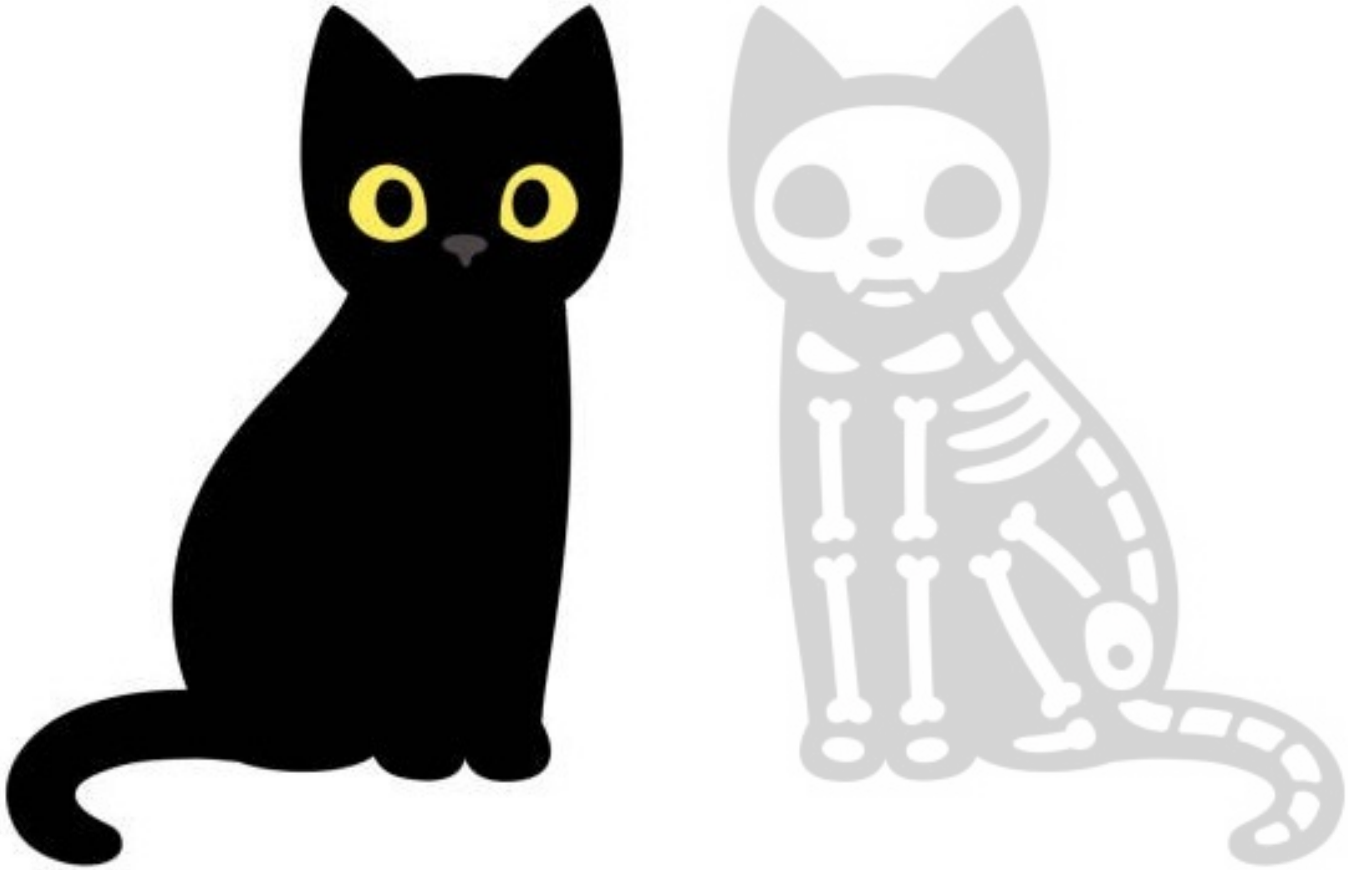

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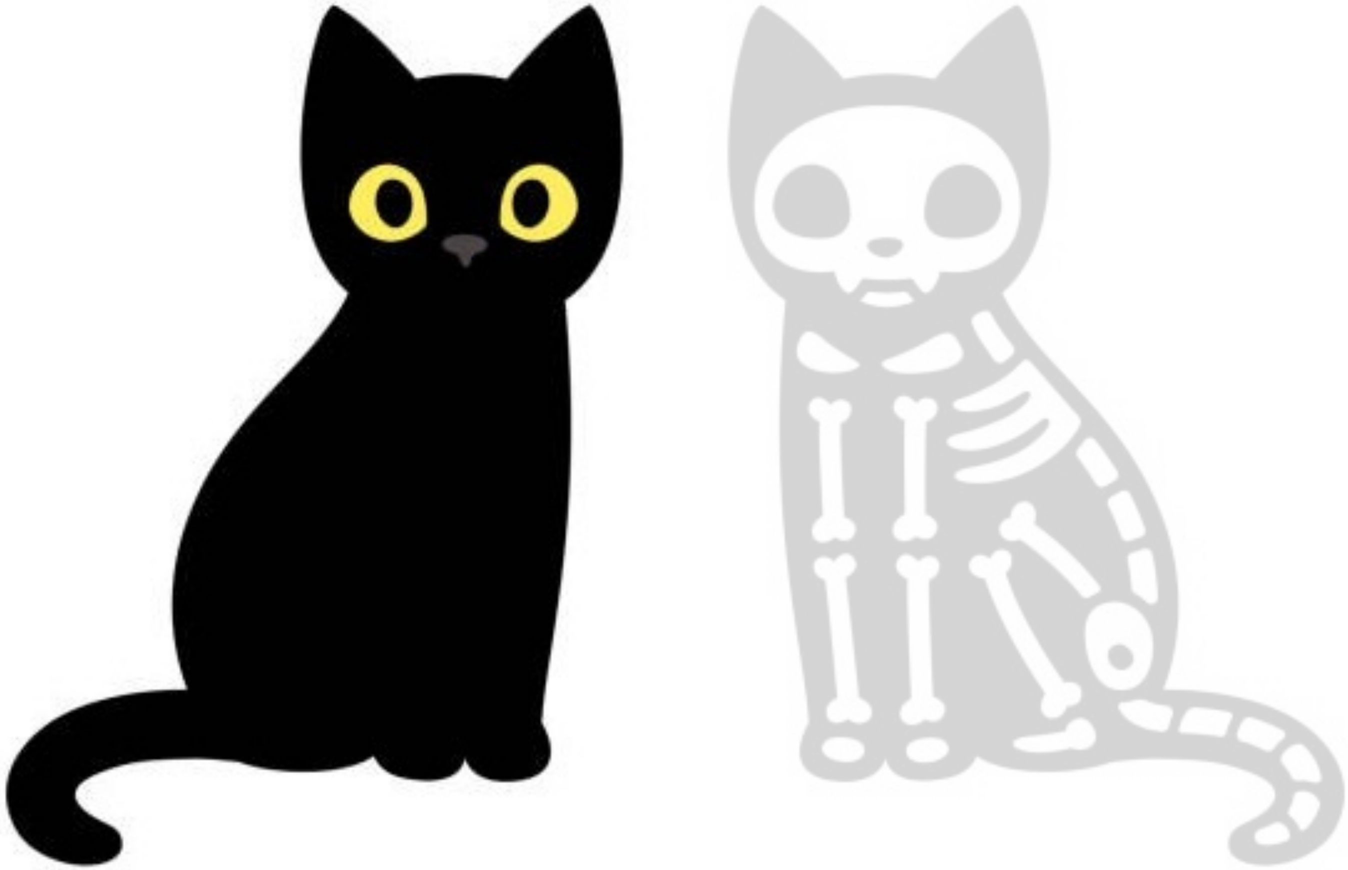
- Superposition of possible QM physical states is not maintained on CM scales



- Most recent superposition is made for molecules of mass 25,000 Da (1 Da $\approx 1.66\times10^{-27}$ kg) consisting of up to 2,000 atoms (2019)

nature physics

LETTERS

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Quantum superposition of molecules beyond 25 kDa

Yaakov Y. Fein[1], Philipp Geyer[1], Patrick Zwick[2], Filip Kiałka[1], Sebastian Pedalino[1], Marcel Mayor[2,3,4], Stefan Gerlich[1] and Markus Arndt[1*]

- In the experiments the molecules remained in a superposition for more than 7 ms

- One possible explanation why a larger quantum system does not maintain superposition for a long time is because the constituent parts of the quantum system interacts with outer, external “environment”, and it is very difficult to control those environmental interactions
- The loss of quantum superposition due to the environmental interaction is called “decoherence”

- One possible explanation why a larger quantum system does not maintain superposition for a long time is because the constituent parts of the quantum system interacts with outer, external “environment”, and it is very difficult to control those environmental interactions
- The loss of quantum superposition due to the environmental interaction is called “decoherence”
- Now we apply this idea to inflationary universe
- The system – part of the world of our interest – is the perturbations with their wavelength longer than the Hubble radius, viz. super-horizon modes, as they are observationally relevant
- The environment – the rest of the world other than the system – is those with shorter wavelength than the Hubble radius, i.e. sub-horizon modes
- How can the system and environment interact?

- The interaction between system and environment is impossible at quadratic order, since a mode with comoving momentum $\boldsymbol{k}$ can interact only with the mode with $-\boldsymbol{k}$ (momentum conservation)
- Thus, for the $\boldsymbol{k}$-mode to interact with another with momentum different from $\boldsymbol{k}$, we need cubic or higher order interactions – non-linear interactions!
- Then (for e.g. cubic order) the momentum conservation demands the sum of momenta of the participating modes vanish, $\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = \boldsymbol{0}$, so the (say) $\boldsymbol{k}_1$-mode can interact with 2 modes with different momenta

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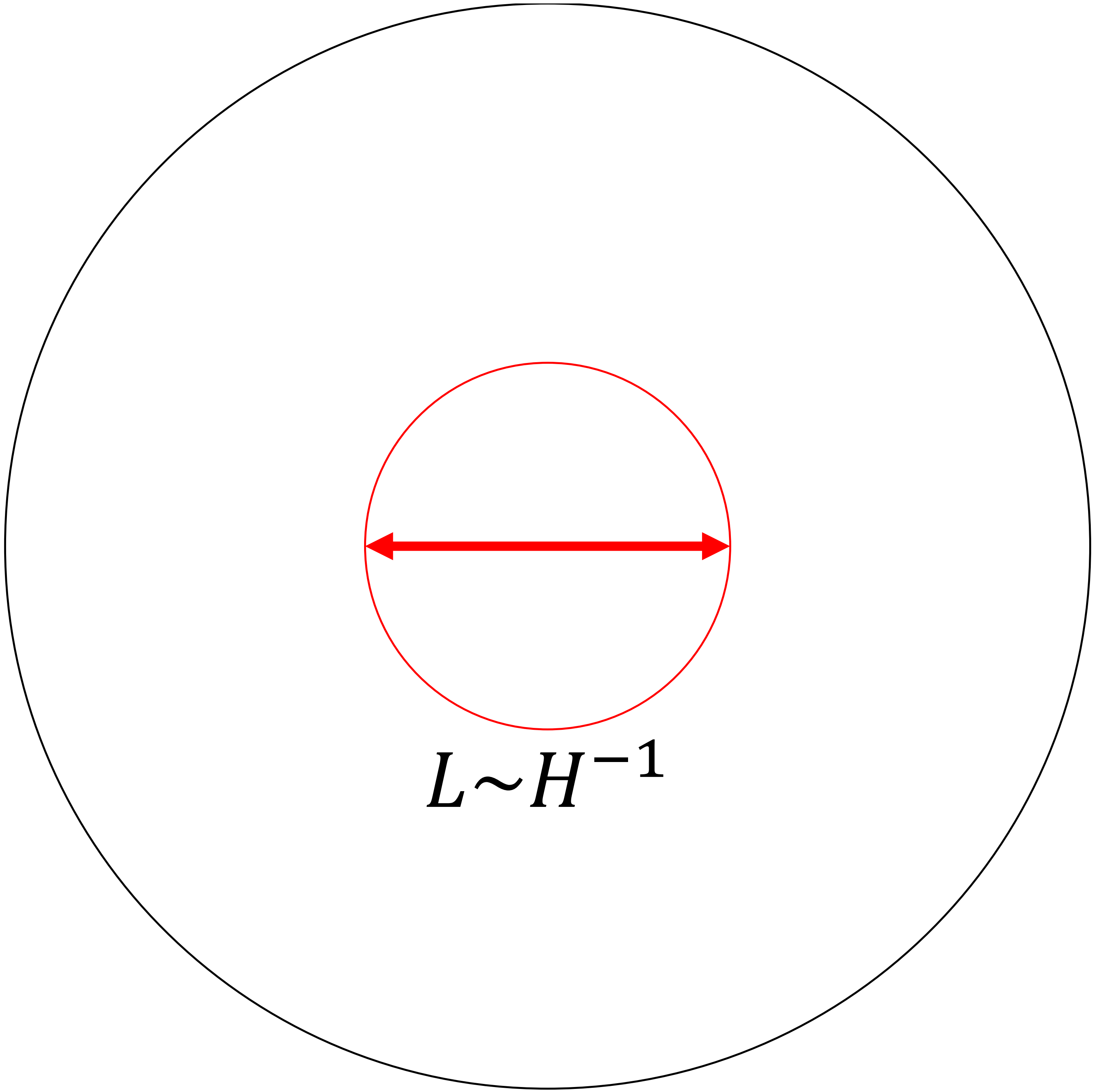
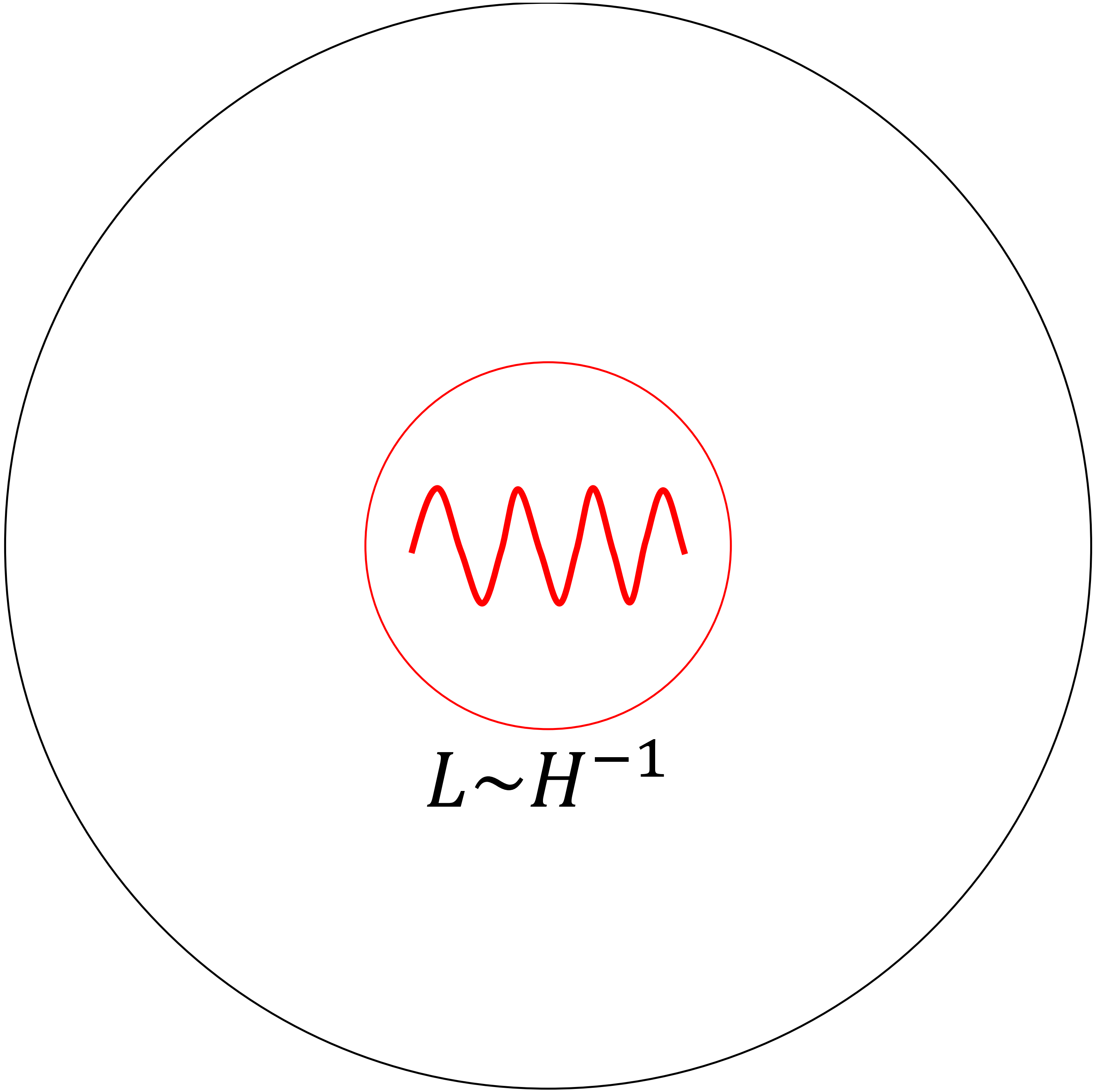
- Then (for e.g. cubic order) the momentum conservation demands the sum of momenta of the participating modes vanish, $\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = \boldsymbol{0}$, so the (say) $\boldsymbol{k}_1$-mode can interact with 2 modes with different momenta
- However, the typical time scale for a super-horizon mode is much longer than that for a sub-horizon mode, so a super-horizon mode cannot echo back to every input a sub-horizon mode supplies via non-linear interaction
- That is, a super-horizon can only respond to the "smeared-out" interactions of the coupled sub-horizon mode(s), which corresponds to integrate out the environment


$L \sim H^{-1}$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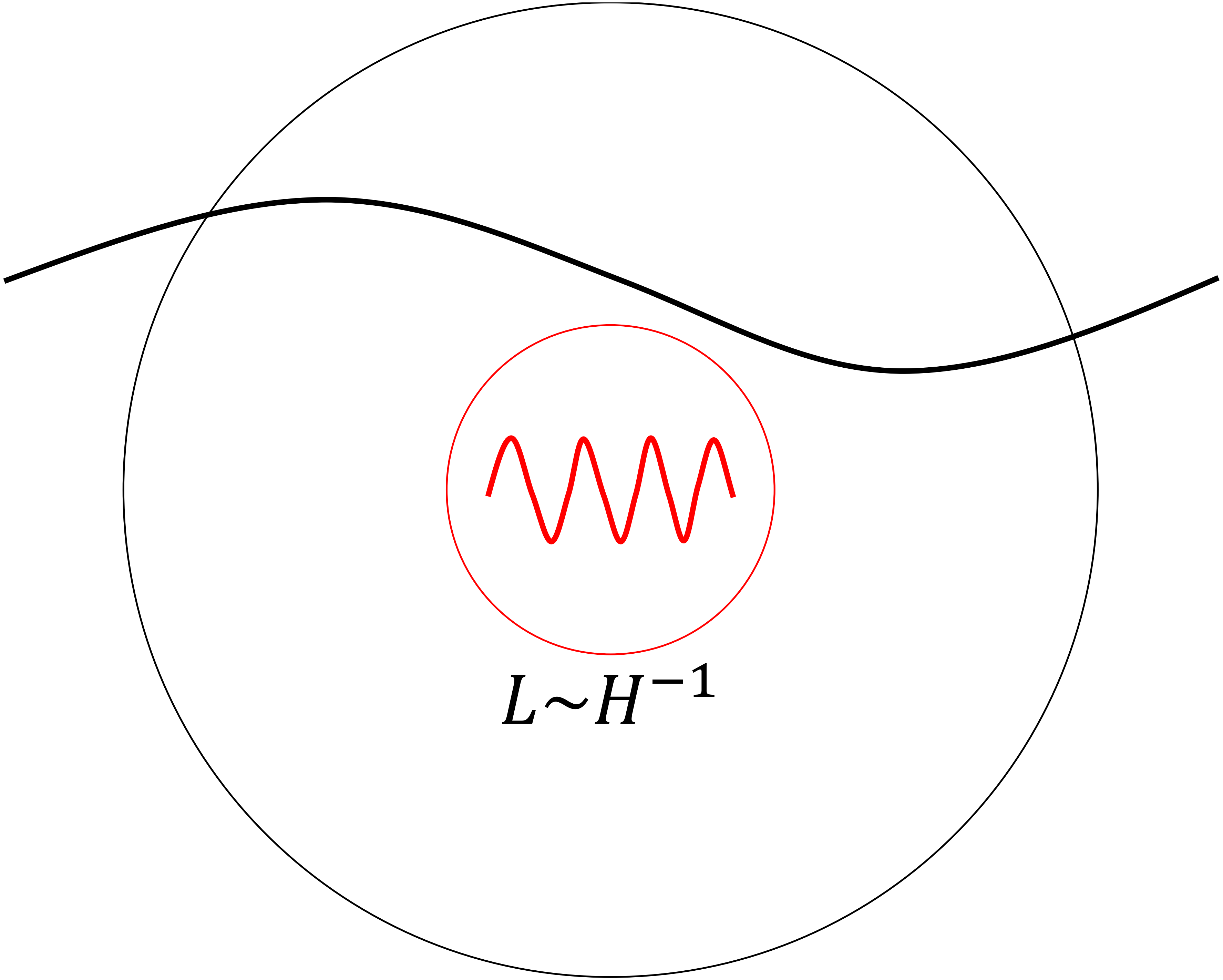
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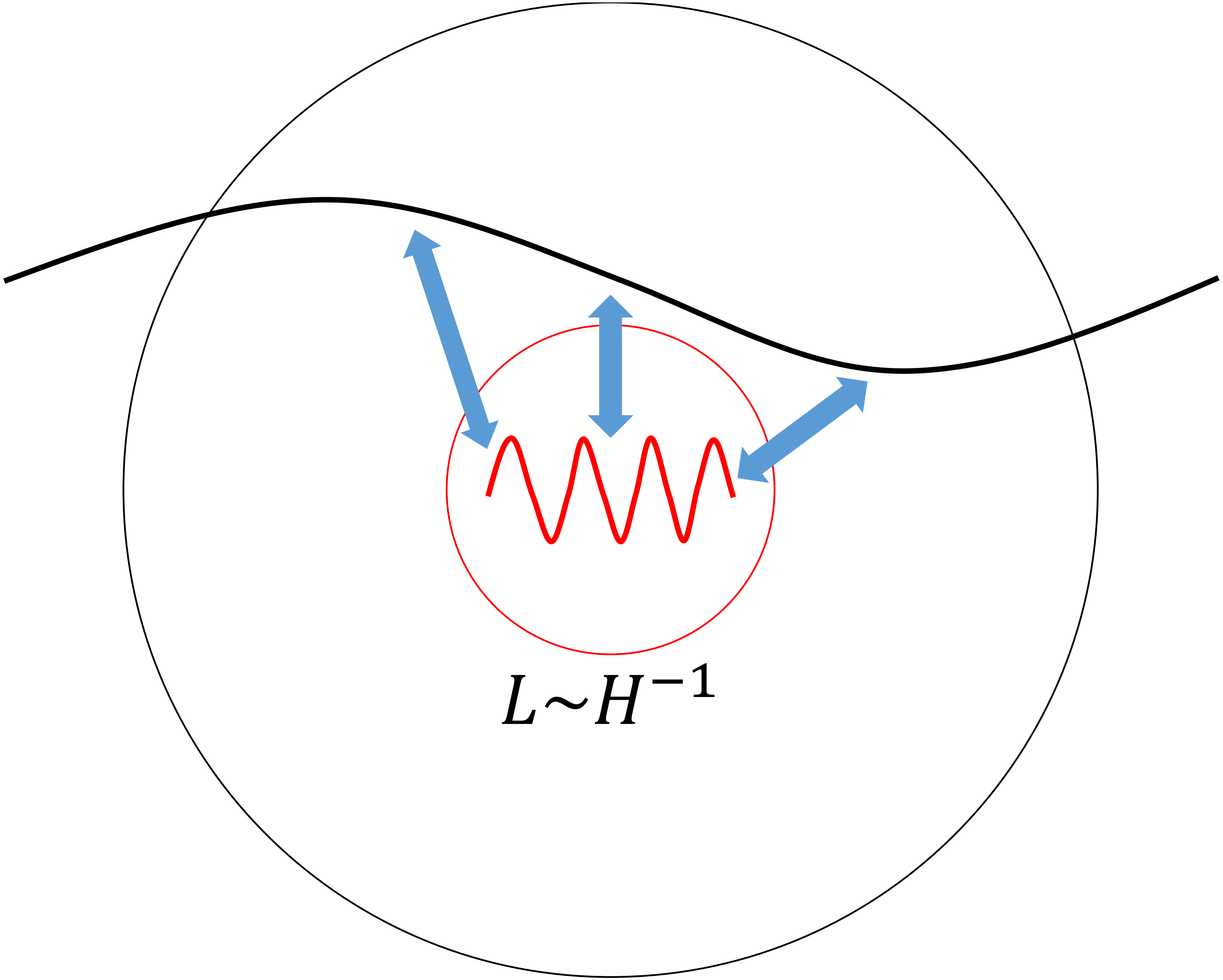
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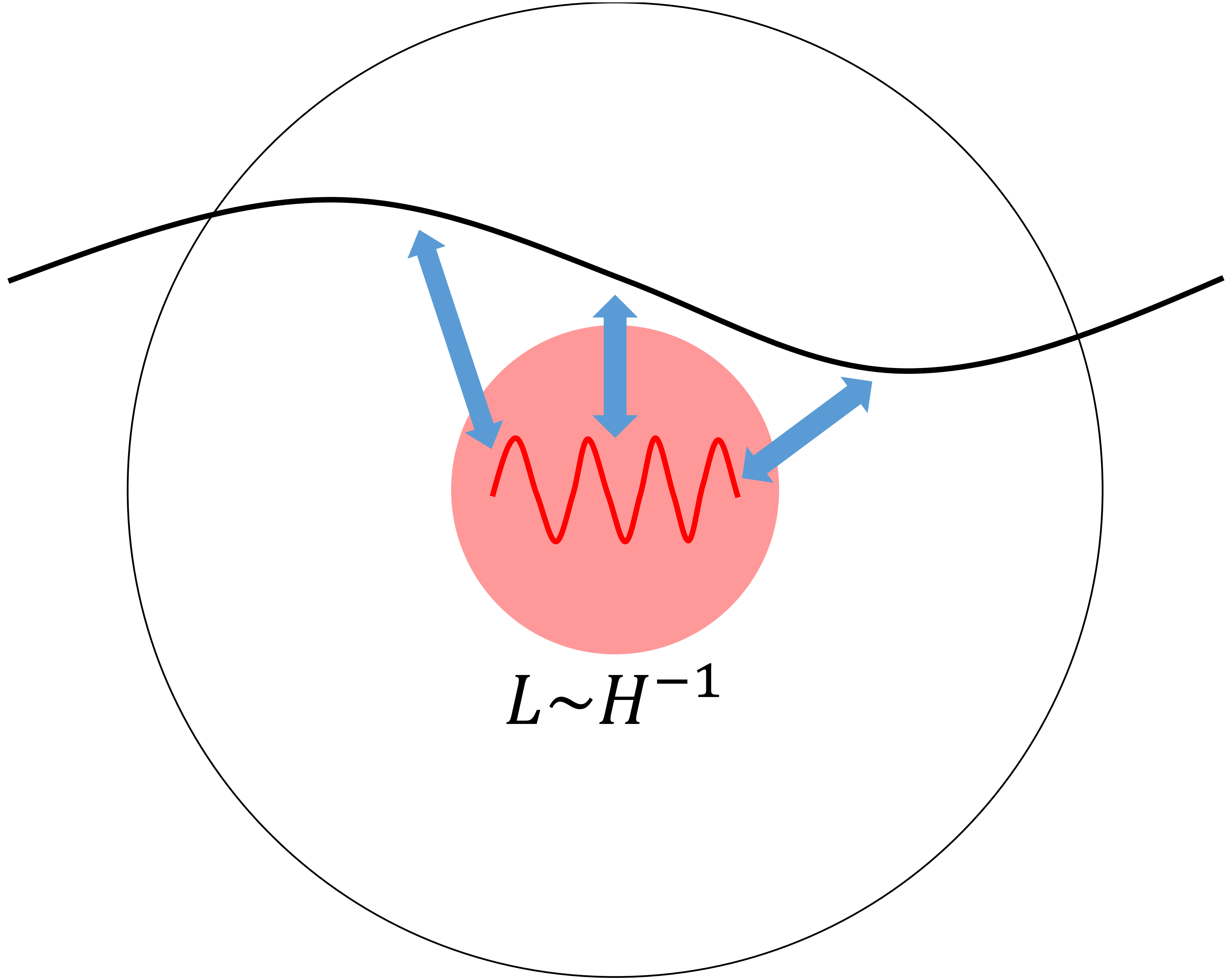
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b. Reduced density matrix and Lindblad equation

- The world can be divided into the fast environment $|\mathcal{E}\rangle$ and (relatively) slow system $|\mathcal{S}\rangle$, and the state consists of them:

$$|\Psi\rangle = |\mathcal{S}\rangle \otimes |\mathcal{E}\rangle$$

- The evolution as an ensemble average of the state can be described in terms of the density matrix ρ defined by:

$$\rho \equiv |\Psi\rangle\langle\Psi|$$

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$$\rho \equiv |\Psi\rangle\langle\Psi|$$

- The evolution of the density matrix is determined by the von Neumann eq:

$$\frac{d\rho}{d\tau} = -i\left[H, \rho\right]$$

- Note the von Neumann eq has the opposite sign to the Heisenberg eq

- We work in the Schroedinger picture
- Since in general the Hamiltonian is t-dep and thus not commuting at different times, the density matrix in the Schroedinger picture (denoted by S) is:

$$
\begin{aligned}
\rho_S(\tau) = \rho_{\text{free},S}(\tau) - iTe^{-i\int_{\tau_0}^{\tau} H_0(\tau')d\tau'} \int_{\tau_0}^{\tau} d\tau_1 \Big[H_{\text{int},I}(\tau_1), \rho_0\Big] \overline{T} e^{i\int_{\tau_0}^{\tau} H_0(\tau')d\tau'} \\
+ (-i)^2 e^{-i\int_{\tau_0}^{\tau} H_0(\tau')d\tau'} \int_{\tau_0}^{\tau} d\tau_2 \int_{\tau_0}^{\tau_2} d\tau_1 \Big[H_{\text{int},I}(\tau_2), \Big[H_{\text{int},I}(\tau_1), \rho_0\Big]\Big] \overline{T} e^{i\int_{\tau_0}^{\tau} H_0(\tau')d\tau'} + \cdots \\
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1. $\rho_{\text{free},S}(\tau)$: Free part of density matrix (evolution only by H_0)

$$\rho_{\text{free},S}(\tau) = Te^{-i\int_{\tau_0}^{\tau} H_0(\tau')d\tau'} \rho_S(\tau_0) \overline{T} e^{i\int_{\tau_0}^{\tau} H_0(\tau')d\tau'}$$

2. T ($\bar{T}$): (Anti) Time-ordering operator
3. $H_{\text{int},I}(\tau)$: Interaction Hamiltonian in the interaction picture

- We are eventually interested in the “reduced” density matrix where the effects of the environment are integrated out

$$\rho_{\mathrm{red},S}(\tau) = \sum_i \langle \mathcal{E}_i | \rho_S(\tau) | \mathcal{E}_i \rangle = \mathrm{Tr}_{\mathcal{E}} \rho_S$$

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Environmental (basis) states are fixed at the time of observation (tracing out)

- Thus to obtain the evolution equation for the reduced density matrix, we only need to take the trace of that of the full density matrix

- The von Neumann equation of the density matrix $\rho_S(\tau)$ is given by

$$\begin{aligned}\frac{d\rho_S(\tau)}{d\tau} &= -i\big[H, \rho_S(\tau)\big] \\ &= -i\big[H, \rho_{\mathrm{free},S}(\tau) + \rho_{(1)}(\tau) + \rho_{(2)}(\tau) + \cdots\big] \\ &= -i\big[H, \rho_{\mathrm{free},S}(\tau)\big] - i\big[H, \rho_{(1)}(\tau)\big] - i\big[H, \rho_{(2)}(\tau)\big] + \cdots\end{aligned}$$

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$$= \boxed{-i\big[H_0, \rho_{\mathrm{free},S}(\tau)\big]} \longrightarrow \text{0}^{\text{th}} \text{ order in } H_{\mathrm{int}}$$

$$\boxed{-\, i\big[H_{\mathrm{int},S}, \rho_{\mathrm{free},S}(\tau)\big] - i\big[H_0, \rho_{(1)}(\tau)\big]} \longrightarrow \text{1}^{\text{st}} \text{ order in } H_{\mathrm{int}}$$

$$\boxed{-\, i\big[H_0, \rho_{(2)}(\tau)\big] - i\big[H_{\mathrm{int},S}, \rho_{(1)}(\tau)\big]} + \cdots$$

$\longrightarrow$ 2nd order in H_{int}

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$$\underbrace{\boxed{- i\big[H_0, \rho_{(2)}(\tau)\big] - i\big[H_{\text{int},S}, \rho_{(1)}(\tau)\big]}}_{2^{\text{nd}} \text{ order in } H_{\text{int}}} + \cdots$$

- Now we have to compute each term, and to take trace over $|\mathcal{E}_i\rangle$ to find the equation for the reduced density matrix
- The necessary steps are straight but quite long, so we only present the results

1. 1st term can be written as, after tracing over $|\mathcal{E}_i\rangle$:

$$-i\sum_i \left\langle \mathcal{E}_i \middle| [H_0, \rho_{\mathrm{free},S}(\tau)] \middle| \mathcal{E}_i \right\rangle = -i\left[H_{0,\mathcal{S}}, \rho_{(0)}^{(\mathrm{red})}\right]$$

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2. 2nd term can also be written as:

$$-\,i\sum_i \left\langle \mathcal{E}_i \middle| \left[H_{\text{int},S}, \rho_{\text{free},S}(\tau)\right] \middle| \mathcal{E}_i \right\rangle = -i\left[H^{(\text{eff1})}_{\text{int}}, \rho_{\text{free},\mathcal{S}}\right]$$

$$H^{(\text{eff1})}_{\text{int}} \equiv \left\langle \mathcal{E} \middle| \overline{T} e^{i\int_{\tau_0}^{\tau} H_{0,\mathcal{E}}\, d\tau'} H_{\text{int},S} T e^{-i\int_{\tau_0}^{\tau} H_{0,\mathcal{E}}\, d\tau'} \middle| \mathcal{E} \right\rangle$$

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2. 2nd term can also be written as:

$$-i\sum_i \left\langle \mathcal{E}_i \middle| \left[H_{\text{int},S}, \rho_{\text{free},S}(\tau)\right] \middle| \mathcal{E}_i \right\rangle = -i\left[H_{\text{int}}^{(\text{eff1})}, \rho_{\text{free},\mathcal{S}}\right]$$

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1. Interaction Hamiltonian (in the Schroedinger picture) averaged over time-evolved environment state at τ
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2. 2nd term can also be written as:

$$-\, i\sum_i \left\langle \mathcal{E}_i \middle| \left[H_{\mathrm{int},S}, \rho_{\mathrm{free},S}(\tau)\right] \middle| \mathcal{E}_i \right\rangle = -i\left[H_{\mathrm{int}}^{(\mathrm{eff1})}, \rho_{\mathrm{free},\mathcal{S}}\right]$$

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3. 3rd and 4th terms:

$$-i\sum_i \left\langle \mathcal{E}_i \middle| \left[H_0, \rho_{(1)}(\tau) + \rho_{(2)}(\tau)\right] \middle| \mathcal{E}_i \right\rangle = -i\sum_i \left[H_{0,\mathcal{S}}, \rho_{(1)}^{(\mathrm{red})} + \rho_{(2)}^{(\mathrm{red})}\right]$$

4. 5^{th} term: After careful calculations, we can write

$$-i\sum_i \left\langle \mathcal{E}_i \middle| \left[H_{\text{int},S}, \rho_{(1)}(\tau)\right] \middle| \mathcal{E}_i \right\rangle = -i\Big[\underbrace{\frac{-i}{2}\sum_i \left(L_1^\dagger L_2 - L_2^\dagger L_1\right)}_{\equiv H_{\text{int}}^{(\text{eff2})}}, \rho_{\text{free},\mathcal{S}}\Big]$$

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$\longrightarrow$ 2nd-order effective Hamiltonian

Purely real: Some components of $\rho_S^{(\text{red})}$ decay exponentially $\longleftarrow$

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$$L_1 \equiv \left\langle \mathcal{E}_i \middle| H_{\text{int},S} T e^{-i\int_{\tau_0}^{\tau} H_{0,\mathcal{E}} d\tau'} \middle| \mathcal{E} \right\rangle$$

$$L_2 \equiv \left\langle \mathcal{E}_i \middle| \int_{\tau_0}^{\tau} d\tau_1 H_{\text{int},I}(\tau_1 - \tau) T e^{-i\int_{\tau_0}^{\tau} H_{0,\mathcal{E}} d\tau'} \middle| \mathcal{E} \right\rangle$$

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- These have desired forms: Interested in cubic interaction $H_{\text{int}} \sim \mathcal{S}\mathcal{E}^2$,
  1) They are appropriate linear combinations of the system operators appearing in the interaction Hamiltonian, and
  2) They contain self-correlation of the environment operators

- Summing up, we finally find

$$\frac{d\rho_S^{(\mathrm{red})}}{d\tau} = -i\Big[H_{0,\mathcal{S}}, \rho_{(0)}^{(\mathrm{red})} + \rho_{(1)}^{(\mathrm{red})} + \rho_{(2)}^{(\mathrm{red})}\Big] - i\Big[H_{\mathrm{int}}^{(\mathrm{eff1})} + H_{\mathrm{int}}^{(\mathrm{eff2})}, \rho_{\mathrm{free},\mathcal{S}}\Big]$$

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3. The last term comes from the pure commutator contributions between the density matrix and the interaction Hamiltonian, responsible for **non-unitary evolution**
4. Some matrix elements of the reduced density matrix decays exponentially, and instead other components (usually off-diagonal ones) show up
5. The whole system is in the mixed state, or decoherence occurs

c. Decoherence of cosmological perturbation

- Now we insert the explicit form of the cubic order interaction Hamiltonian
- The Lindblad terms are, after computing out the system sector, written in terms of the squeezed states:

$$U_{0,\mathcal{S}}|0\rangle_{\mathcal{S}} \sim \sum_{n=0}^{\infty} \left(a^{\dagger}_{-\boldsymbol{k}\mathcal{S}} a^{\dagger}_{\boldsymbol{k}\mathcal{S}} \right)^{n} |0\rangle_{\mathcal{S}}$$

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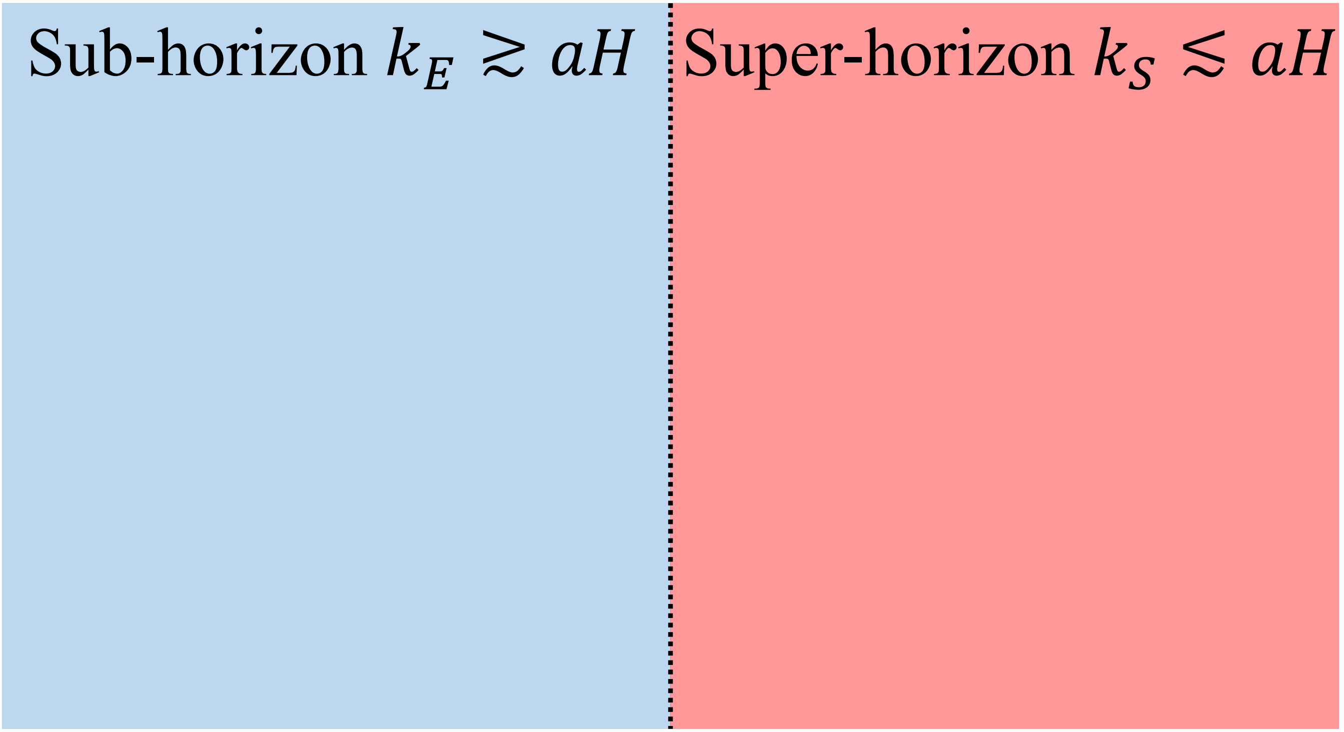
- That is, as the Lindblad terms contain 2 cubic order Hamiltonian, they are schematically of the form:

$$\frac{d\rho_{\rm red}}{d\tau} \sim \sum_{m,n\leq 6} \rho_{mn}U_{0,\mathcal{S}}a^{\dagger}_{1,\mathcal{S}}a^{\dagger}_{2,\mathcal{S}}\cdots a^{\dagger}_{m,\mathcal{S}}|0\rangle_{\mathcal{S}}{}_{\mathcal{S}}\langle 0|a^{\dagger}_{1',\mathcal{S}}a^{\dagger}_{1',\mathcal{S}}\cdots a^{\dagger}_{n,\mathcal{S}}U^{\dagger}_{0,\mathcal{S}}$$

- The reduced density matrix equation is described by a 7×7 matrix, with each row and column distinguished by the # of ops acting on $|0\rangle_{\mathcal{S}}$

- What kind of configurations are possible?



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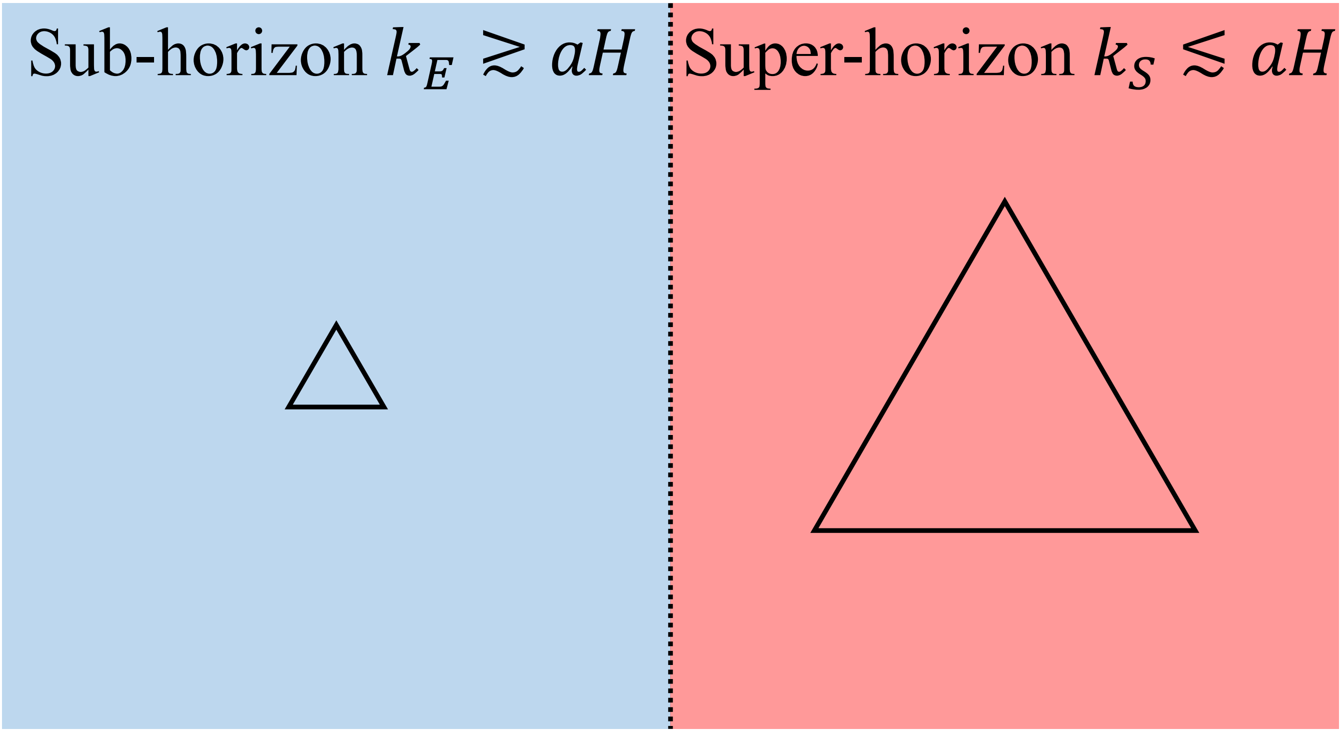

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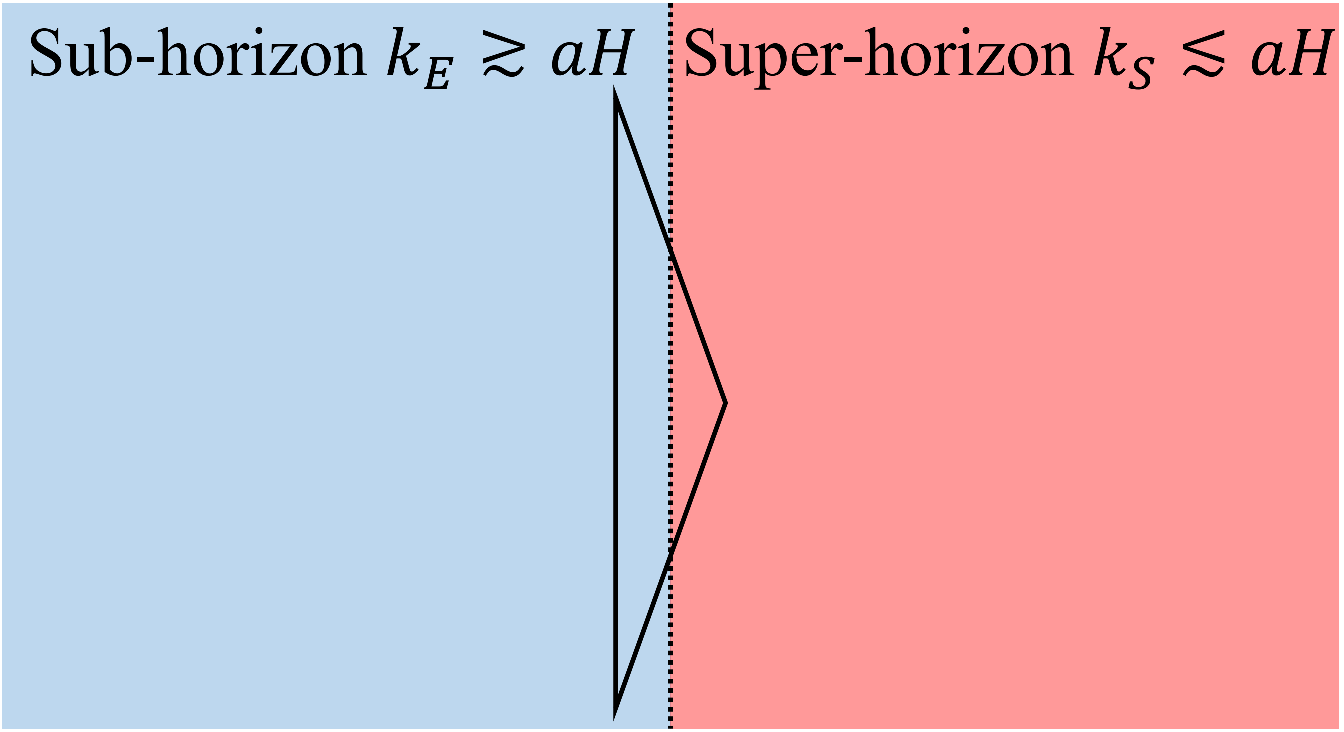

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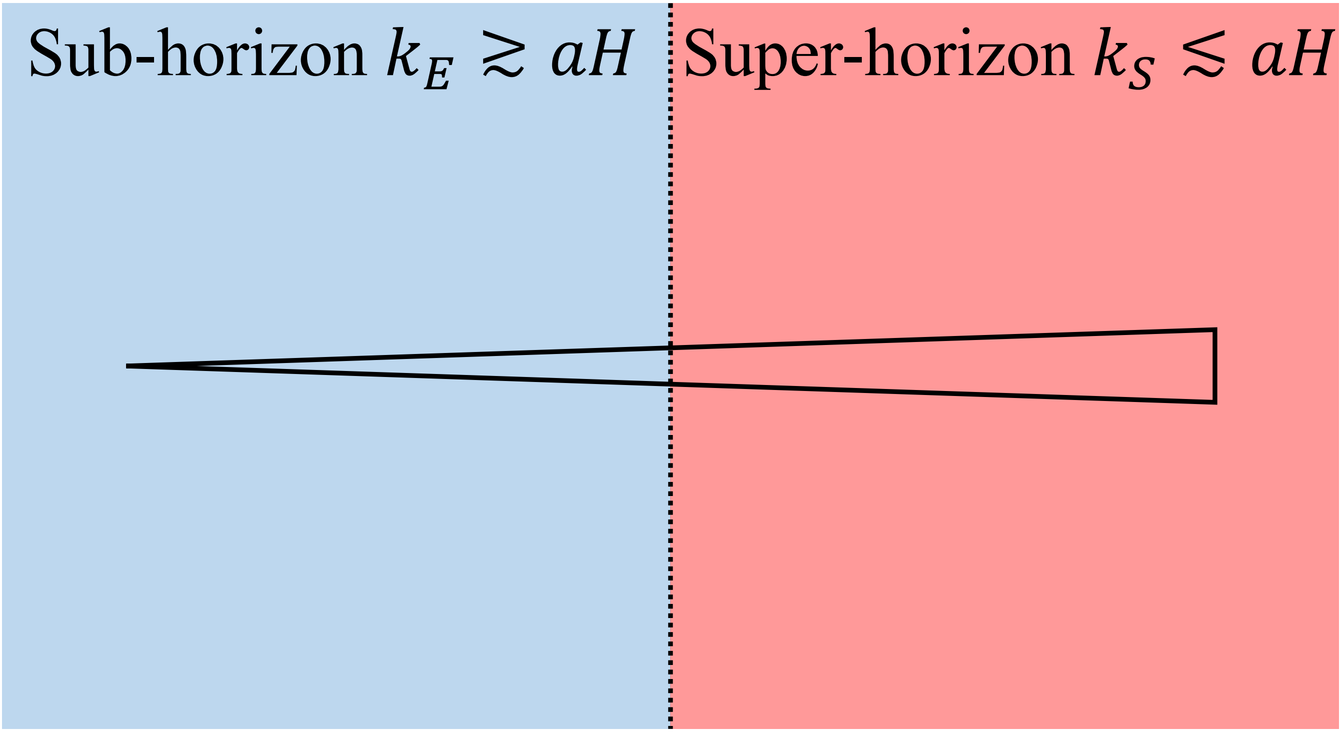

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1. All modes are in the environment or system sector: 0 ($\mathcal{EEE}$) or absorbed ($\mathcal{SSS}$)
2. 2 system and 1 environment: No clear distinction, disappears in the enfolded limit
3. 1 system and 2 environment: 0 if perfectly squeezed limit, only non-zero otherwise

- In the mildly squeezed configuration, the matrix notation of Lindblad eq looks:

$$\frac{d\rho_{\rm red}}{d\tau} = -\left(\begin{array}{ccc|c} \mathfrak{E}_{00} & 0 & \mathfrak{E}_{02} & \\ 0 & \mathfrak{E}_{11} & 0 & 0_{3\times 4} \\ \mathfrak{E}_{20} & 0 & 0 & \\ \hline & 0_{4\times 3} & & 0_{4\times 4} \end{array}\right)$$

$$\begin{aligned} \mathfrak{E}_{00} &= \frac{2}{(2\pi)^3}\delta^{(3)}(\boldsymbol{q})\frac{H^2}{m_{\rm Pl}^2}\frac{4}{3\tau^4}8\pi^2\mathcal{C}_{\mathcal{SE}} \\ \mathfrak{E}_{11} &= -2\delta^{(3)}(\boldsymbol{q}_{ab})\delta_{\lambda_a\lambda_b}\frac{H^2}{m_{\rm Pl}^2}\frac{4}{3\tau^4}2\pi\frac{\mathcal{C}_{\mathcal{E}}}{q^3} \qquad (\boldsymbol{q}_a = -\boldsymbol{q}_b \equiv \boldsymbol{q}) \\ \mathfrak{E}_{20} &= -3\delta^{(3)}(\boldsymbol{q}_{ab})\delta_{\lambda_a\lambda_b}12\frac{H^2}{m_{\rm Pl}^2}\frac{4}{3\tau^4}2\pi e^{2iq\tau}\frac{\mathcal{C}_{\mathcal{E}}}{q^3} \\ &= \mathfrak{E}_{02}^* \end{aligned}$$

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$$\mathcal{C}_{\mathcal{S}\mathcal{E}} \approx 0.577148 - \frac{32}{45}\log\varepsilon \quad (\varepsilon \ll 1 : \text{IR cutoff})$$

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$$\mathcal{C}_{\mathcal{SE}} = \int_{\varepsilon\ll 1}^{1} \bar{q}^2 d\bar{q} \frac{\mathcal{C}_{\mathcal{E}}}{\bar{q}^3} \quad \left(\bar{q} \equiv \frac{q}{aH}\right)$$

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- Note that the trace of $\mathfrak{E}_{00}$ and $\mathfrak{E}_{11}$ guarantees no divergence
- $\mathfrak{E}_{20}$ and $\mathfrak{E}_{02}$: 2-particle and no excitation for in- and out-states
  1. $e^{-iq\tau}$ and $e^{iq\tau}$ are the phase factors for in- and out-states
  2. Phase factor cancels in $\mathfrak{E}_{11}$, but squared in $\mathfrak{E}_{20}$ and $\mathfrak{E}_{02}$

- The 00 element of ρ_{red}, i.e. evolution w.r.t. $U_{0,\mathcal{S}}|0\rangle_{\mathcal{S}}$

$$\rho_{\mathrm{red}}\Big|_{00} = 1 - \frac{2}{(2\pi)^3}\delta^{(3)}(\boldsymbol{q})\frac{H^2}{m_{\mathrm{Pl}}^2}\frac{4}{9|\tau|^3}8\pi^2\mathcal{C}_{\mathcal{S}\mathcal{E}}$$

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1. Unitary evolution in the system sector
2. Non-unitary evolution through the Lindblad terms
   - $\rho_{\mathrm{red}}|_{00}$ is reduced from 1
   - The probability of keeping the pure squeezed is reduced
   - Classical probability for other processes involving 2 excitations

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2. Non-unitary evolution through the Lindblad terms
   - $\rho_{\text{red}}|_{00}$ is reduced from 1
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3. Exponentiating $\rho_{\text{red}}|_{00}$ with $(2\pi)^3\delta^{(3)}(\boldsymbol{q})$ = comoving 3D volume V

$$\rho_{\text{red}}\big|_{00} = \exp\left(-V\int \underbrace{\frac{\Delta_{\mathcal{R}}^2}{(2\pi)^2}\frac{2}{3\tau^4}r\mathcal{C}_{\mathcal{S}\mathcal{E}}}_{\equiv\Gamma}\, d\tau\right)$$

Γ grows as the physical volume a^3, is suppressed as $r\mathcal{C}_{\mathcal{S}\mathcal{E}} \sim H^2/m_{\text{Pl}}^2$

- For an interval ΔN, the change in $\rho_{\mathrm{dec}}|_{00}$ is

$$\exp\left[-\frac{\Delta_{\mathcal{R}}^2}{(2\pi)^2}\frac{2}{9}e^{3\Delta N}r\mathcal{C}_{\mathcal{S}\mathcal{E}}\right]$$

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- We may call this duration as "**decoherence time**": By this time $\rho_{\text{dec}}|_{00}$ is appreciably reduced from 1, and other states are populated
- Typically $5 < \Delta N_{\text{dec}} < 10$ for a wide range of r and $\mathcal{C}_{\mathcal{SE}}$
- We can proceed similarly for scalar perturbation and find $10 < \Delta N_{\text{dec}} < 15$