IV. Inflationary evolution of quantum state

1. Preliminary: Inverted harmonic oscillator
    a. Classical inverted harmonic oscillator
    b. Quantum inverted harmonic oscillator
2. Reformulation of quantum fluctuations
3. Hamiltonian operator revisited
    a. Hamiltonian in factorized form
    b. What does the Hamiltonian do on quantum state?

IV. Inflationary evolution of quantum state

1. Preliminary: Inverted harmonic oscillator
   a. Classical inverted harmonic oscillator
   b. Quantum inverted harmonic oscillator
2. Reformulation of quantum fluctuations
3. Hamiltonian operator revisited
   a. Hamiltonian in factorized form
   b. What does the Hamiltonian do on quantum state?

a. Classical inverted harmonic oscillator

- We first consider a simple "inverted" harmonic oscillator, with (for convenience) unit mass and unit spring constant:

$$H = \frac{p^2}{2} - \frac{q^2}{2}$$

a. Classical inverted harmonic oscillator

- We first consider a simple "inverted" harmonic oscillator, with (for convenience) unit mass and unit spring constant:

$$H = \frac{p^2}{2} - \frac{q^2}{2}$$

- The Hamiltonian equations of motion read immediately:

$$\dot{q} = \frac{\partial H}{\partial p} = p$$

$$\dot{p} = -\frac{\partial H}{\partial q} = q$$

a. Classical inverted harmonic oscillator

- We first consider a simple "inverted" harmonic oscillator, with (for convenience) unit mass and unit spring constant:

$$H = \frac{p^2}{2} - \frac{q^2}{2}$$

- The Hamiltonian equations of motion read immediately:

$$\dot{q} = \frac{\partial H}{\partial p} = p$$

$$\dot{p} = -\frac{\partial H}{\partial q} = q$$

$$\therefore \ \ddot{q} - q = 0 \quad \text{so that} \quad q(t) \sim e^{\pm t}$$

- Note that $p(t)$ has exactly the same form as $q(t)$: $p(t) \sim e^{\pm t}$

- Note that $p(t)$ has exactly the same form as $q(t)$: $p(t) \sim e^{\pm t}$
- Therefore, most general solutions for $q(t)$ and $p(t)$ are

$$q(t) = A\cosh t + B\sinh t = \frac{A+B}{2}e^{t} + \frac{A-B}{2}e^{-t}$$
$$p(t) = C\cosh t + D\sinh t = \frac{C+D}{2}e^{t} + \frac{C-D}{2}e^{-t}$$

- Note that $p(t)$ has exactly the same form as $q(t)$: $p(t) \sim e^{\pm t}$
- Therefore, most general solutions for $q(t)$ and $p(t)$ are

$$q(t) = A\cosh t + B\sinh t = \frac{A+B}{2}e^{t} + \frac{A-B}{2}e^{-t}$$

$$p(t) = C\cosh t + D\sinh t = \frac{C+D}{2}e^{t} + \frac{C-D}{2}e^{-t}$$

$$\therefore\ q(0) = A\,,\ p(0) = C$$

- Note that $p(t)$ has exactly the same form as $q(t)$: $p(t) \sim e^{\pm t}$
- Therefore, most general solutions for $q(t)$ and $p(t)$ are

$$q(t) = A\cosh t + B\sinh t = \frac{A+B}{2}e^{t} + \frac{A-B}{2}e^{-t}$$

$$p(t) = C\cosh t + D\sinh t = \frac{C+D}{2}e^{t} + \frac{C-D}{2}e^{-t}$$

$$\therefore\ \ q(0) = A\,,\ p(0) = C$$

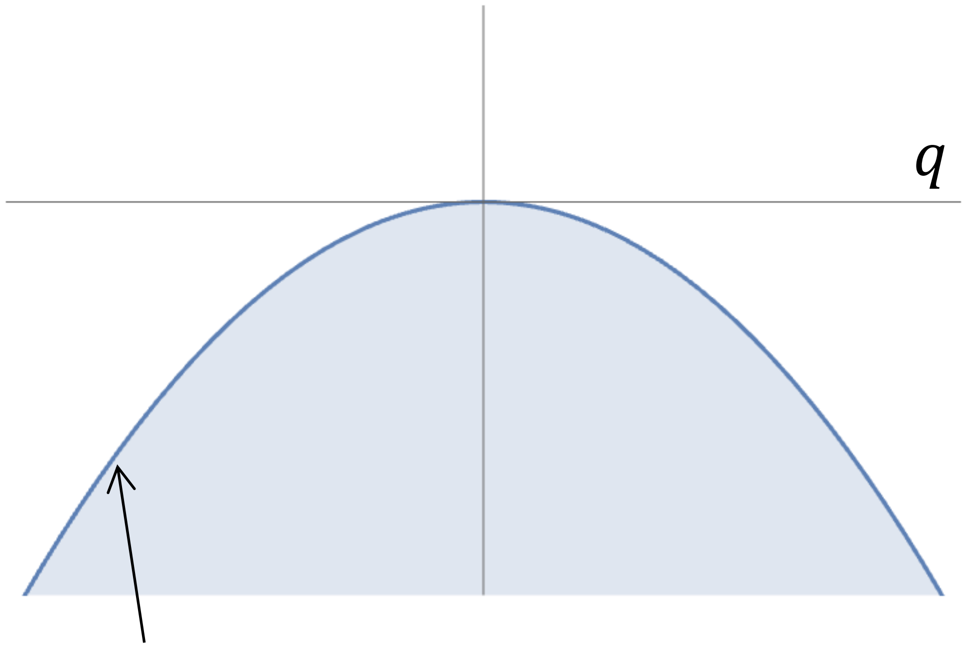
- Further, from the 2 Hamiltonian equations $A = D$ and $B = C$

$$\therefore\ \ q(t) = q(0)\cosh t + p(0)\sinh t$$

$$p(t) = p(0)\cosh t + q(0)\sinh t$$

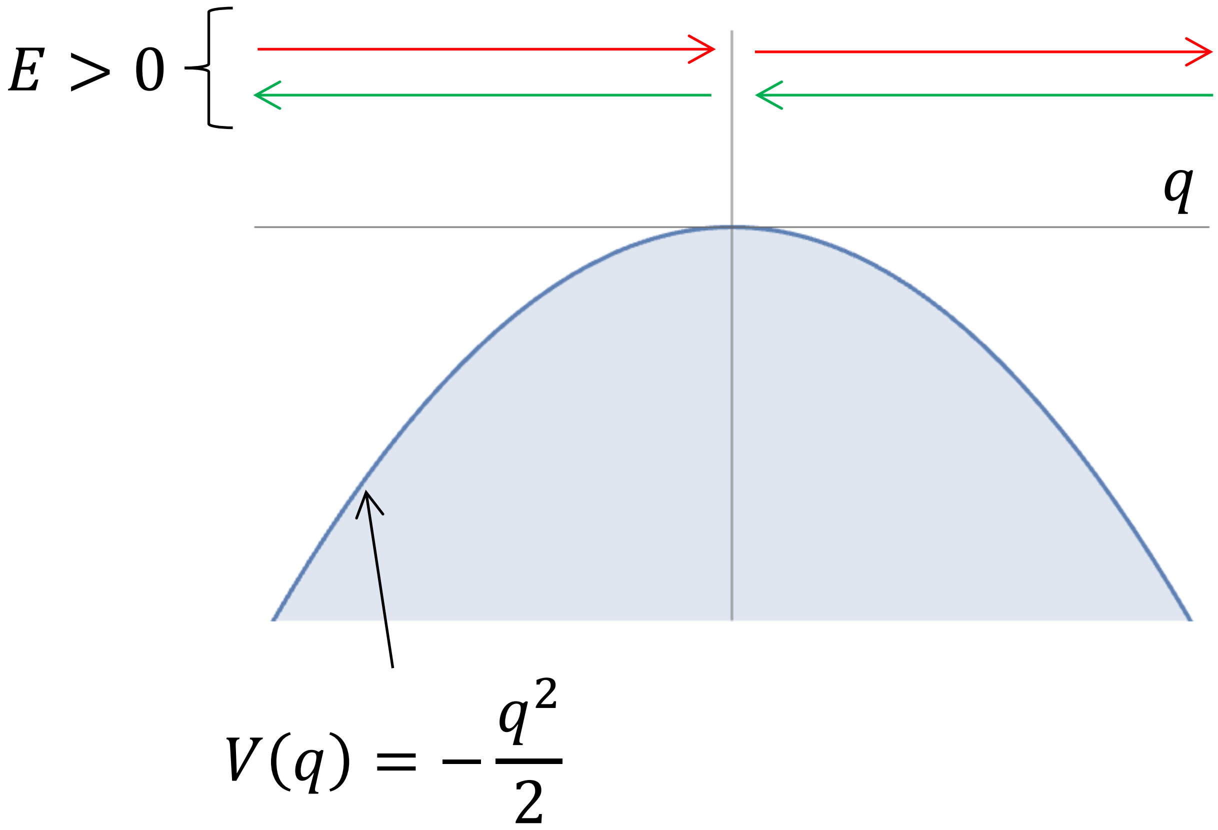
- How does the phase portrait look like?

- How does the phase portrait look like?

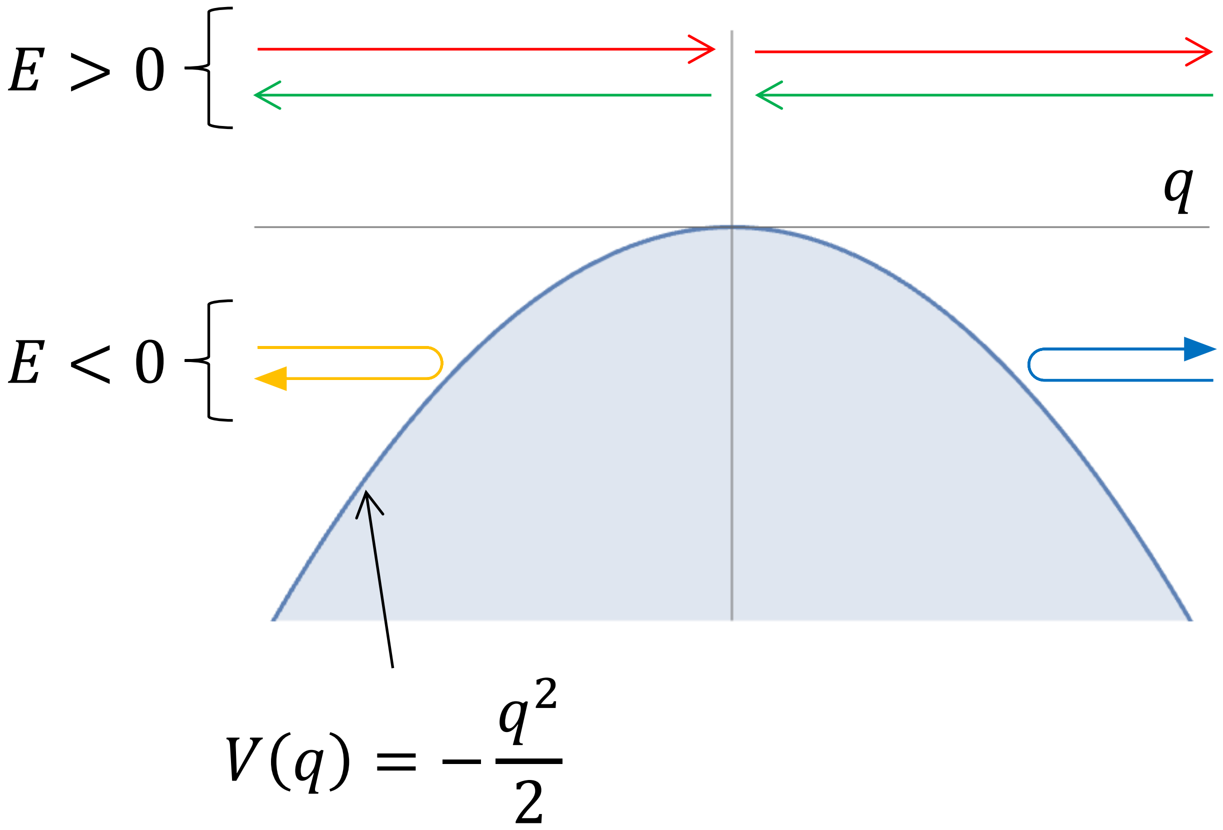


$$V(q) = -\frac{q^2}{2}$$

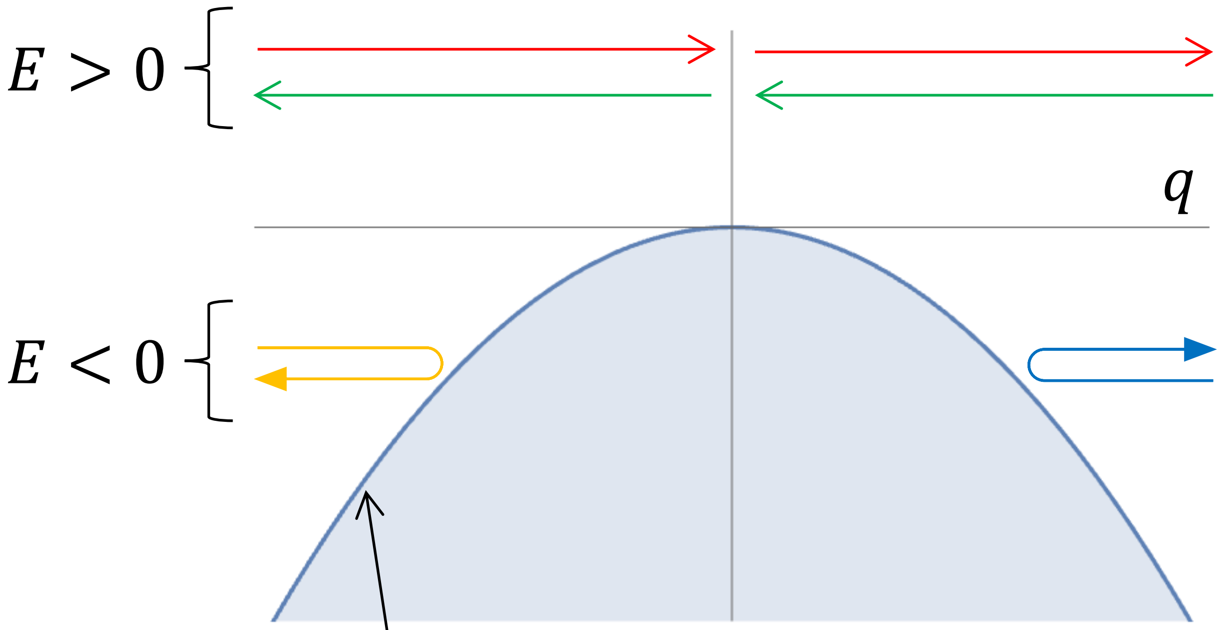
- How does the phase portrait look like?



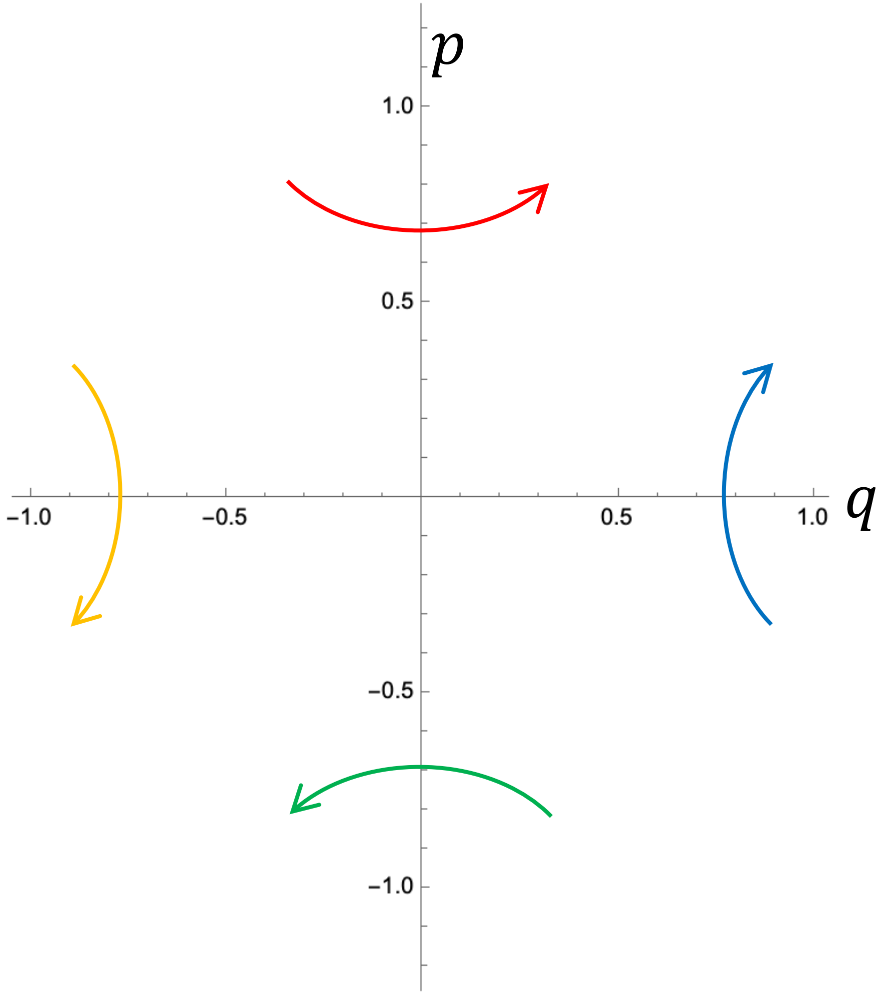
- How does the phase portrait look like?



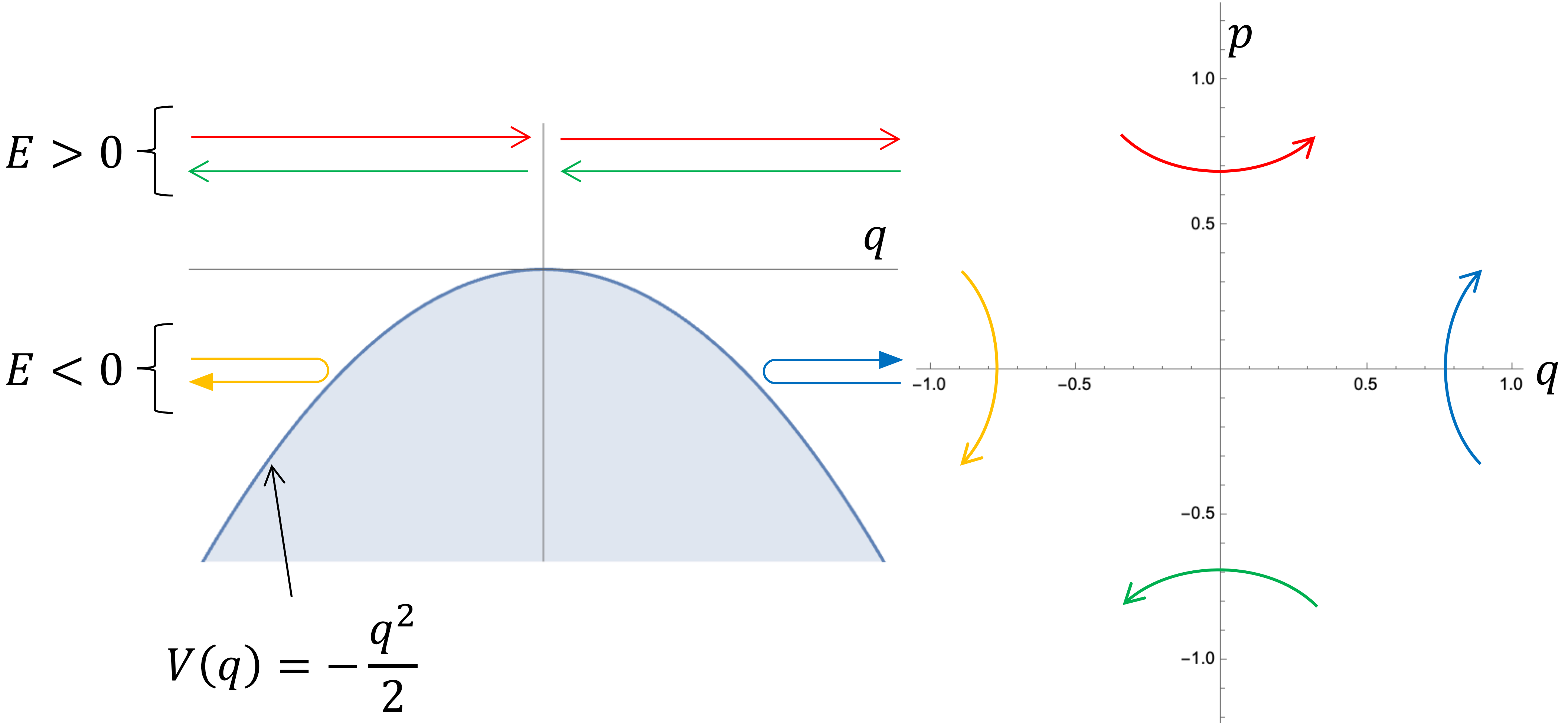
- How does the phase portrait look like?



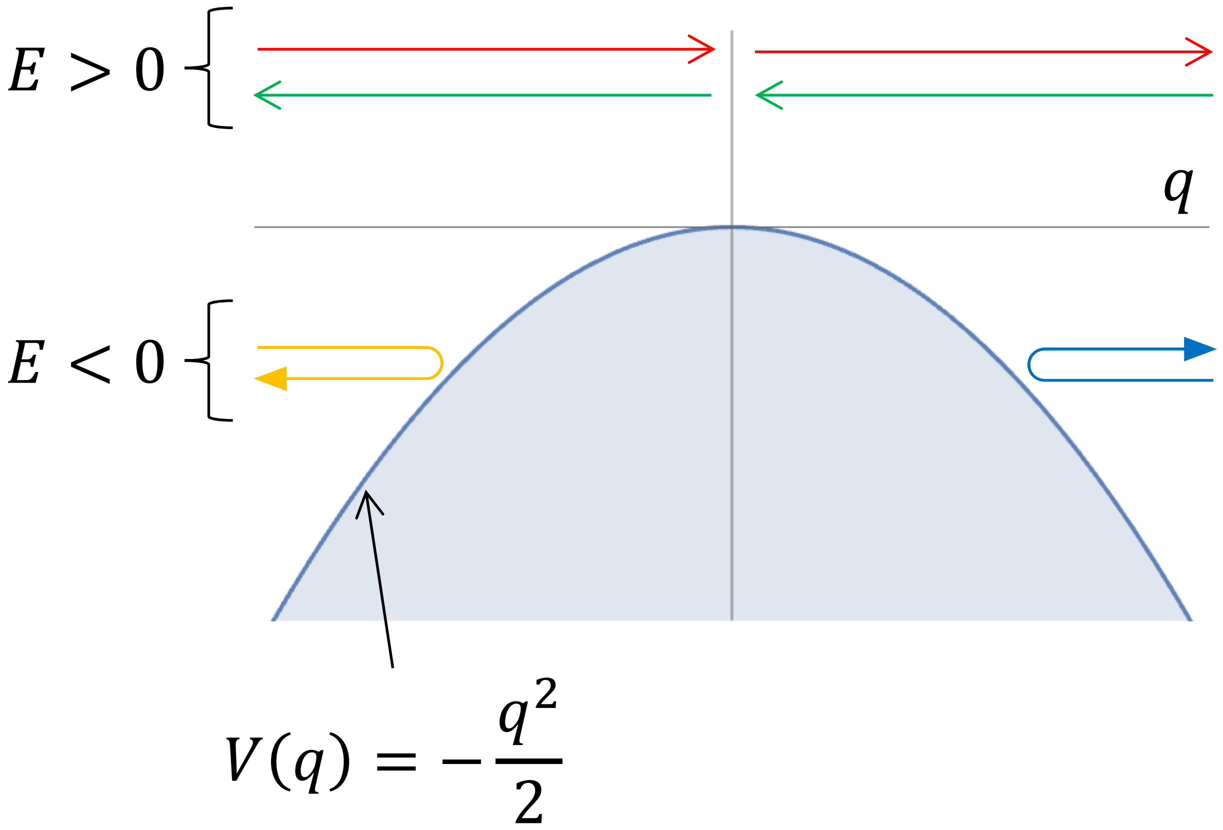
$$V(q) = -\frac{q^2}{2}$$

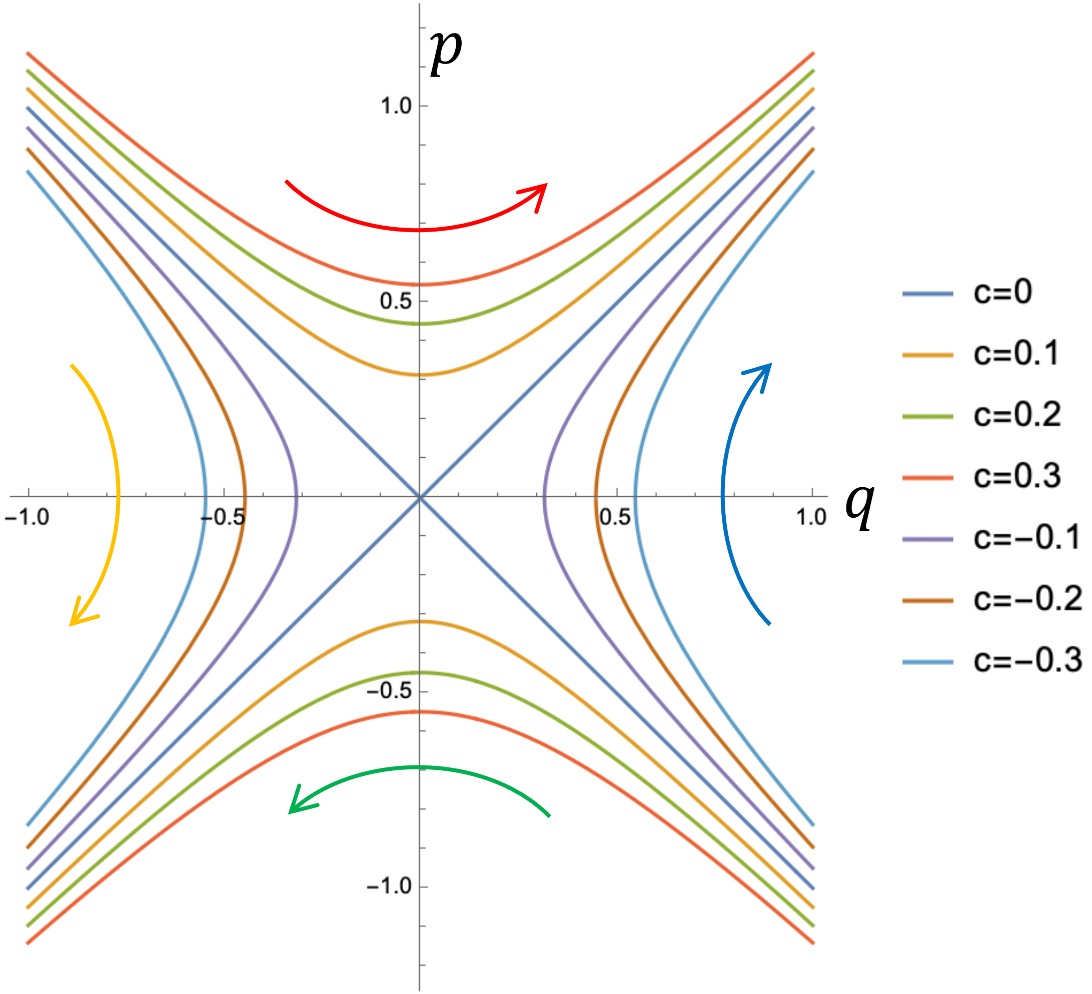


- How does the phase portrait look like?
- Since $p = \dot{q}$, we can write $\dot{p} = (dp/dq)\dot{q} = (dp/dq)p = q$, or simply $pdp = qdq$, which results in $p^2 - q^2 = c$ (c is a constant)

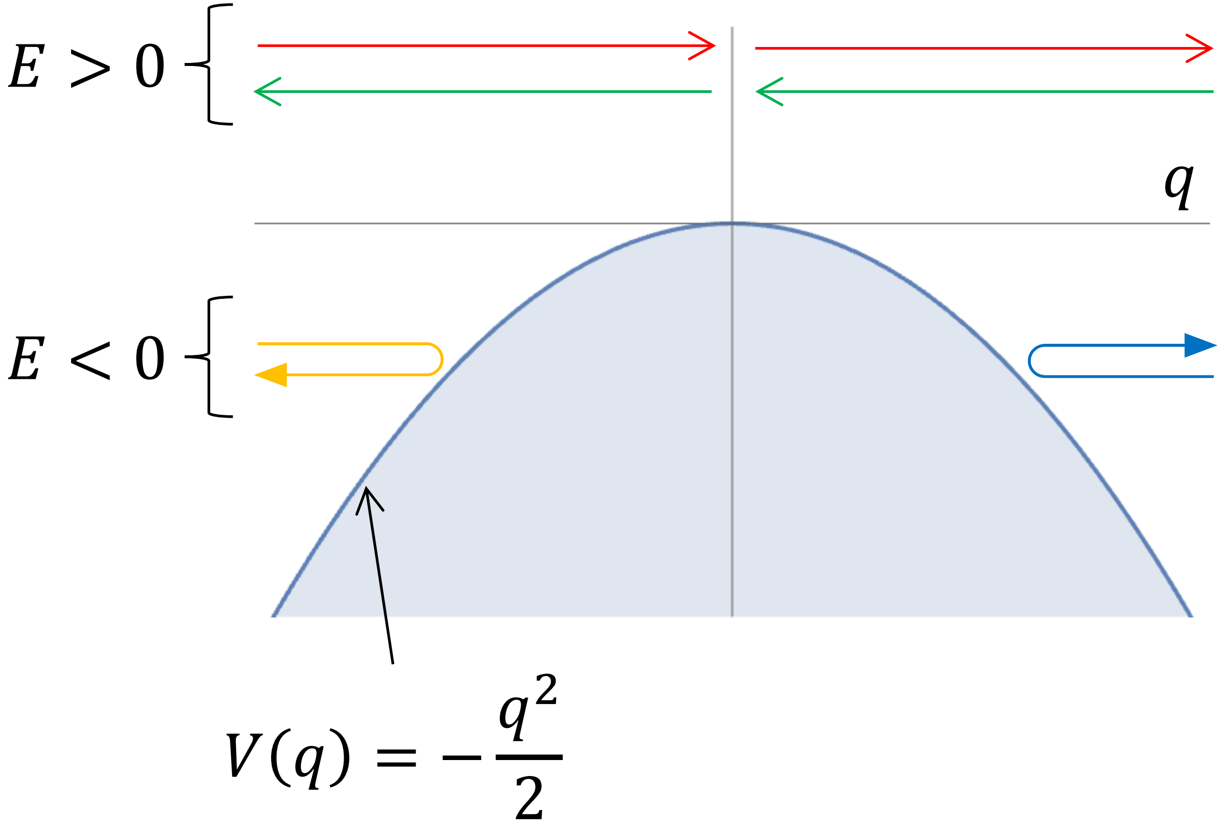


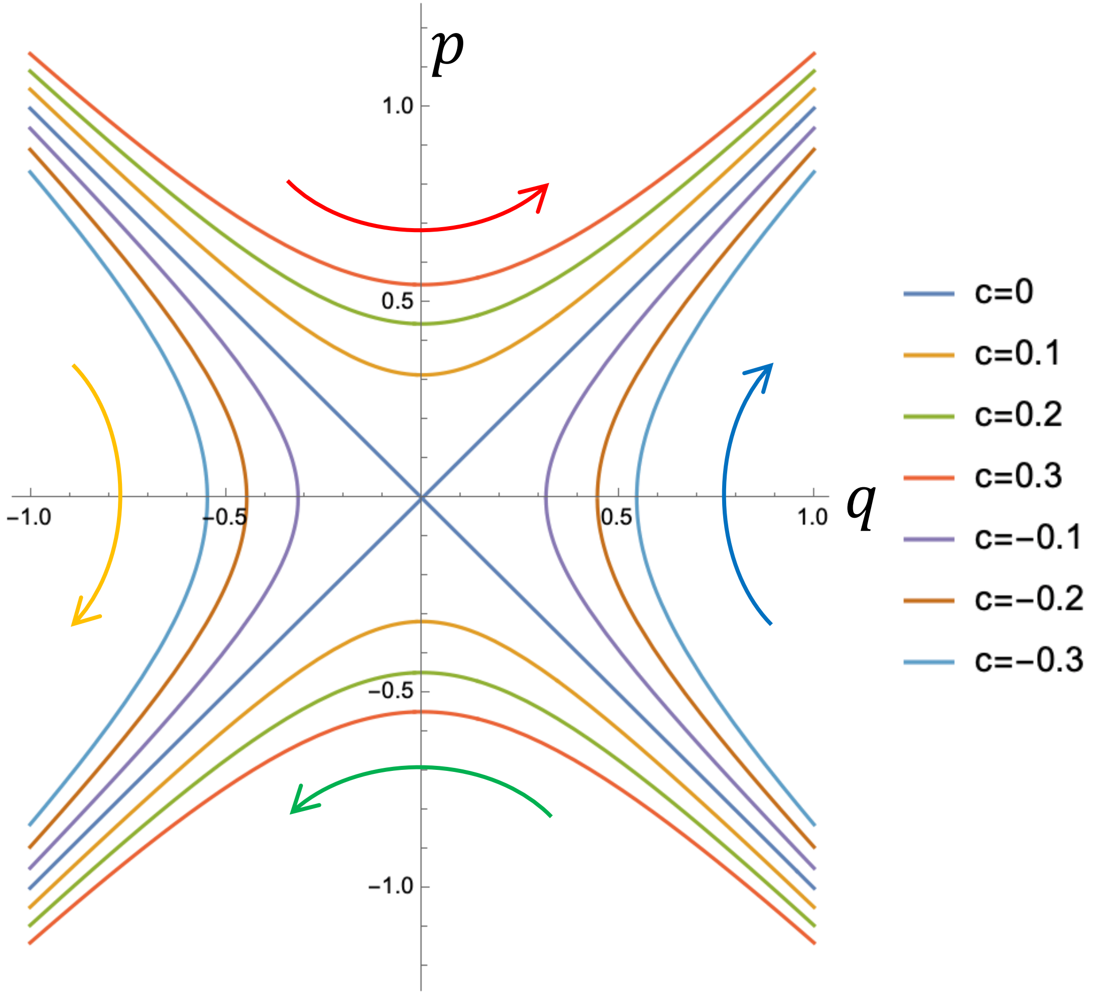
- How does the phase portrait look like?
- Since $p = \dot{q}$, we can write $\dot{p} = (dp/dq)\dot{q} = (dp/dq)p = q$, or simply $pdp = qdq$, which results in $p^2 - q^2 = c$ (c is a constant)





- How does the phase portrait look like?
- Since $p = \dot{q}$, we can write $\dot{p} = (dp/dq)\dot{q} = (dp/dq)p = q$, or simply $pdp = qdq$, which results in $p^2 - q^2 = c$ (c is a constant)
- Obviously, the phase trajectory depends on the energy of the oscillator





- One very convenient choice of an alternative coordinate system is

$$b_\pm \equiv \frac{p \pm q}{\sqrt{2}}$$ (Nothing but 45° rotation of axes)

- One very convenient choice of an alternative coordinate system is

$$b_\pm \equiv \frac{p \pm q}{\sqrt{2}}$$

(Nothing but $45°$ rotation of axes)

- Equations for $b_\pm$ read from the Hamiltonian equations:

$$\begin{aligned} \dot{b}_+ - \dot{b}_- &= b_+ + b_- \\ \dot{b}_+ + \dot{b}_- &= b_+ - b_- \end{aligned} \quad \therefore \quad \begin{aligned} \dot{b}_+ &= b_+ \\ \dot{b}_- &= -b_- \end{aligned} \quad \therefore \quad \begin{aligned} b_+(t) &= b_+(0)e^{t} \\ b_-(t) &= b_-(0)e^{-t} \end{aligned}$$

- Compared to solutions of $q(t)$ and $p(t)$, $b_\pm(0) = [p(0) \pm q(0)]/\sqrt{2}$ (of course!)

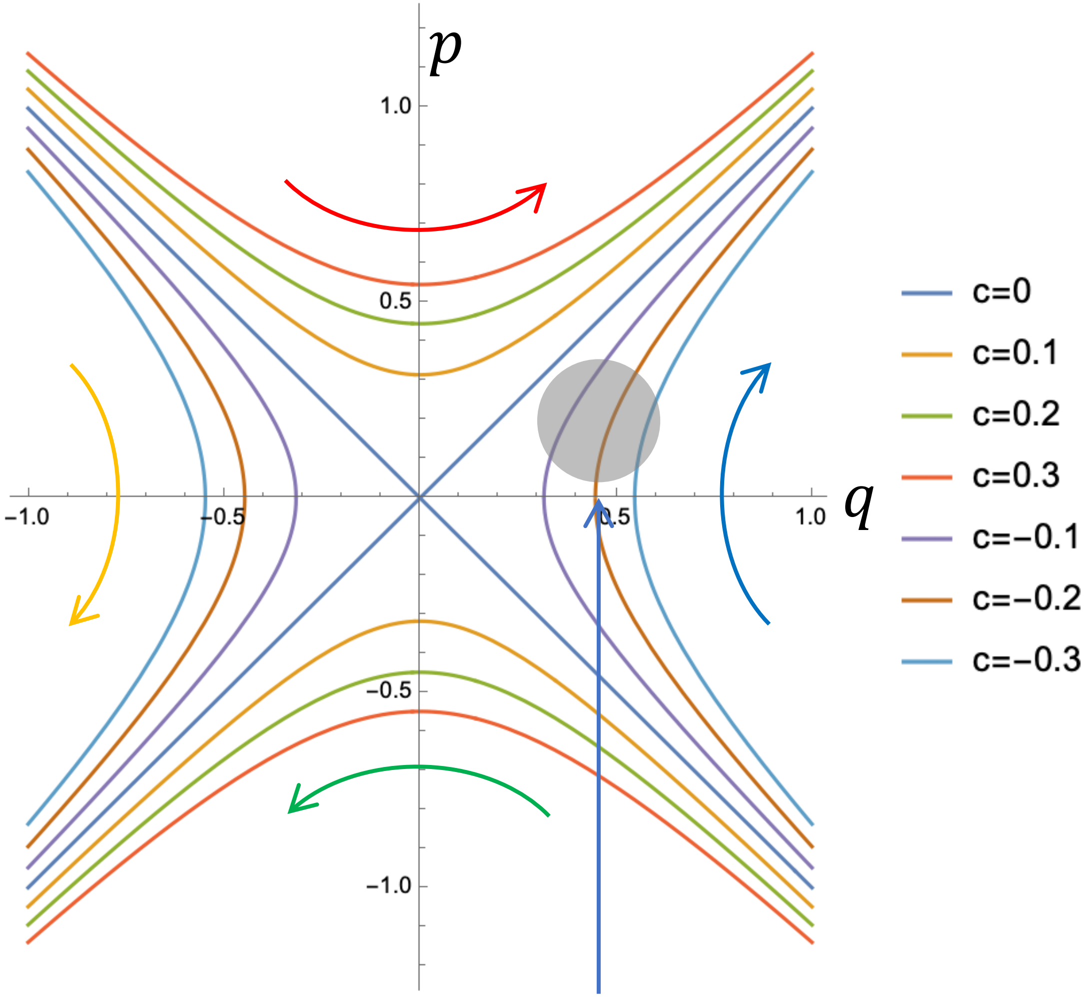
- One very convenient choice of an alternative coordinate system is

$$b_\pm \equiv \frac{p \pm q}{\sqrt{2}}$$ (Nothing but $45°$ rotation of axes)

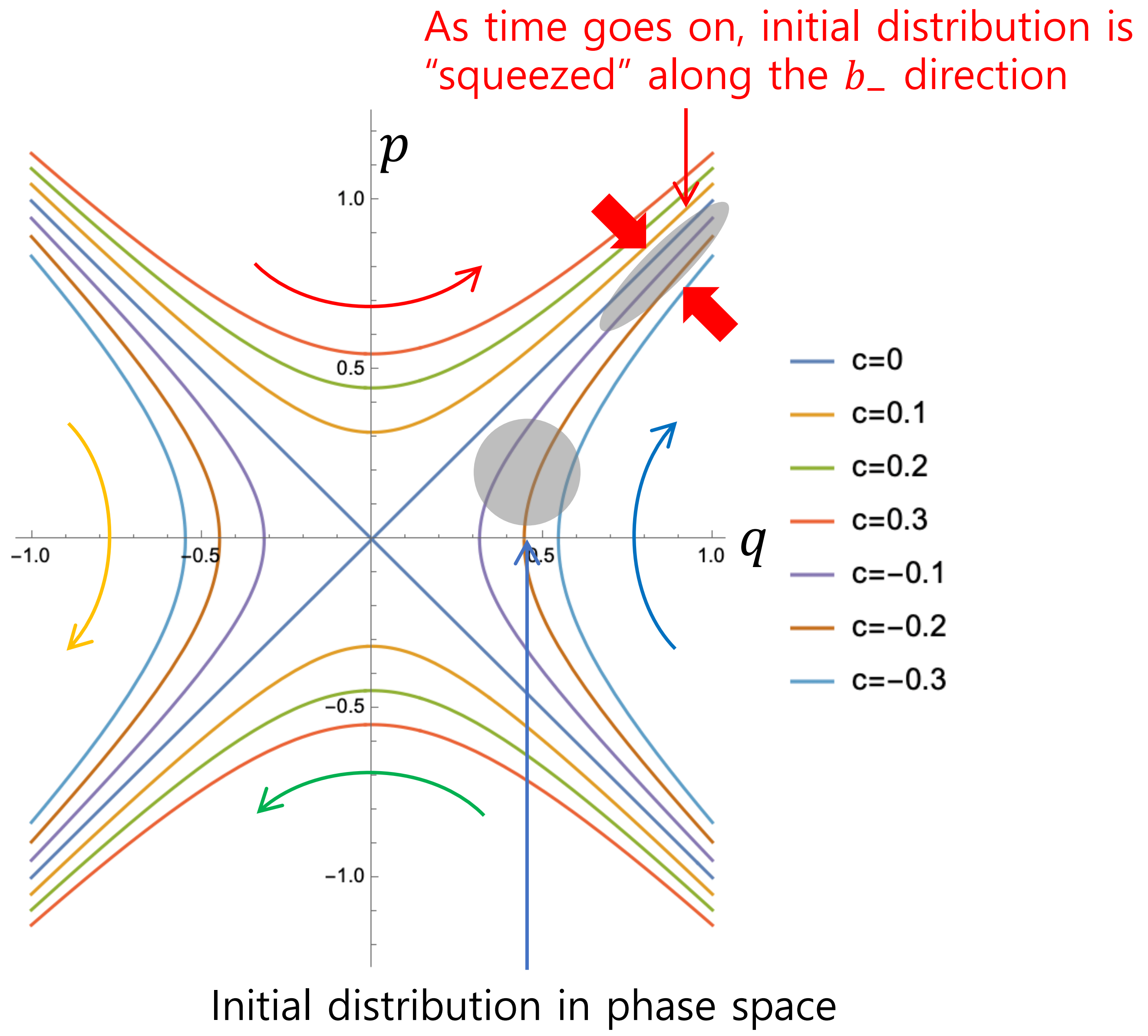
- Equations for $b_\pm$ read from the Hamiltonian equations:

$$\begin{aligned}\dot{b}_+ - \dot{b}_- &= b_+ + b_- \\ \dot{b}_+ + \dot{b}_- &= b_+ - b_-\end{aligned} \quad \therefore \quad \begin{aligned}\dot{b}_+ &= b_+ \\ \dot{b}_- &= -b_-\end{aligned} \quad \therefore \quad \begin{aligned}b_+(t) &= b_+(0)e^{t} \\ b_-(t) &= b_-(0)e^{-t}\end{aligned}$$

- Compared to solutions of $q(t)$ and $p(t)$, $b_\pm(0) = [p(0) \pm q(0)]/\sqrt{2}$ (of course!)
- As time goes on, b_+ becomes exponentially large (p and q grow exponentially) and b_- exponentially small (difference between p and q exp approaches zero)
- Thus, any distribution in phase space eventually become "squeezed" along the axis $p = q$ as the system evolves



Initial distribution in phase space


As time goes on, initial distribution is
"squeezed" along the b_- direction
p
1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0
q
c=0
c=0.1
c=0.2
c=0.3
c=−0.1
c=−0.2
c=−0.3
Initial distribution in phase space

b. Quantum inverted harmonic oscillator

- Now consider the quantum description of an inverted harmonic oscillator:

$$\widehat{H} = \frac{\widehat{P}^2}{2} - \frac{\widehat{Q}^2}{2}$$

b. Quantum inverted harmonic oscillator

- Now consider the quantum description of an inverted harmonic oscillator:

$$\widehat{H} = \frac{\widehat{P}^2}{2} - \frac{\widehat{Q}^2}{2}$$

- We decompose the momentum and position operators in terms of creation and annihilation operators as usual:

$$\widehat{Q} = \sqrt{\frac{\hbar}{2}}(a + a^\dagger)$$

$$\widehat{P} = -i\sqrt{\frac{\hbar}{2}}(a - a^\dagger)$$

b. Quantum inverted harmonic oscillator

- Now consider the quantum description of an inverted harmonic oscillator:

$$\widehat{H} = \frac{\widehat{P}^2}{2} - \frac{\widehat{Q}^2}{2}$$

- We decompose the momentum and position operators in terms of creation and annihilation operators as usual:

$$\widehat{Q} = \sqrt{\frac{\hbar}{2}}(a + a^\dagger)$$

$$\widehat{P} = -i\sqrt{\frac{\hbar}{2}}(a - a^\dagger)$$

- Indeed this gives the canonical commutation relations:

$$\left.\begin{aligned} a &= \frac{\widehat{Q} + i\widehat{P}}{\sqrt{2\hbar}} \\ a^\dagger &= \frac{\widehat{Q} - i\widehat{P}}{\sqrt{2\hbar}} \end{aligned}\right\} \quad \therefore \ [a, a^\dagger] = -\frac{i}{\hbar}[\widehat{Q}, \widehat{P}] = 1$$

- The Hamiltonian then is written in terms of creation and annihilation ops as:

$$\widehat{H} = \frac{1}{2}\left[-\frac{\hbar}{2}\left(a - a^\dagger\right)^2 - \frac{\hbar}{2}\left(a + a^\dagger\right)^2 \right] = -\frac{\hbar}{2}\left(a^2 + {a^\dagger}^2\right)$$

- The Hamiltonian then is written in terms of creation and annihilation ops as:

$$\widehat{H} = \frac{1}{2}\left[-\frac{\hbar}{2}\left(a - a^\dagger\right)^2 - \frac{\hbar}{2}\left(a + a^\dagger\right)^2\right] = -\frac{\hbar}{2}\left(a^2 + {a^\dagger}^2\right)$$

- For later convenience, we change the form of $\widehat{H}$ as:

$$\begin{aligned}
\widehat{H} &= i\frac{\hbar}{2}\left(ia^2 + i{a^\dagger}^2\right) \\
&= i\frac{\hbar}{2}\left(a^2 e^{i\pi/2} - {a^\dagger}^2 e^{-i\pi/2}\right) \\
&= i\frac{\hbar}{2}\left(a^2 e^{2i\pi/4} - {a^\dagger}^2 e^{-2i\pi/4}\right) \\
&= i\frac{\hbar}{2}\left(a^2 e^{2i\pi/4} - h.c.\right)
\end{aligned}$$

- Thus, if we have started at $t = 0$ from the vacuum state $|0\rangle$ that is annihilated by a in such a way that $a|0\rangle = 0$, the time evolution tells us the state $|\psi(t)\rangle$ at $t > 0$ is

- Thus, if we have started at $t = 0$ from the vacuum state $|0\rangle$ that is annihilated by a in such a way that $a|0\rangle = 0$, the time evolution tells us the state $|\psi(t)\rangle$ at $t > 0$ is

$$|\psi(t)\rangle \equiv e^{-iHt/\hbar}|0\rangle = \exp\left[\frac{t}{2}\left(a^2 e^{2i\pi/4} - h.c.\right)\right]|0\rangle$$

- Thus, if we have started at $t = 0$ from the vacuum state $|0\rangle$ that is annihilated by a in such a way that $a|0\rangle = 0$, the time evolution tells us the state $|\psi(t)\rangle$ at $t > 0$ is

$$|\psi(t)\rangle \equiv e^{-iHt/\hbar}|0\rangle = \underbrace{\exp\left[\frac{t}{2}\left(a^2 e^{2i\pi/4} - h.c.\right)\right]}_{\equiv \mathcal{S},\ \text{“squeezing” operator}}|0\rangle$$

- Thus, if we have started at $t = 0$ from the vacuum state $|0\rangle$ that is annihilated by a in such a way that $a|0\rangle = 0$, the time evolution tells us the state $|\psi(t)\rangle$ at $t > 0$ is

$$|\psi(t)\rangle \equiv e^{-iHt/\hbar}|0\rangle = \underbrace{\exp\left[\frac{t}{2}\left(a^2 e^{2i\pi/4} - h.c.\right)\right]}_{\equiv \mathcal{S},\ \text{"squeezing" operator}}|0\rangle$$

- As the squeezing is caused by time, instead of t we introduce "squeezing parameter" r and, instead of $-\pi/4$ we introduce "squeezing phase" ϕ:

$$\mathcal{S} \equiv \exp\left[\frac{r}{2}\left(a^2 e^{-2i\phi} - h.c.\right)\right]$$

- Thus, if we have started at $t = 0$ from the vacuum state $|0\rangle$ that is annihilated by a in such a way that $a|0\rangle = 0$, the time evolution tells us the state $|\psi(t)\rangle$ at $t > 0$ is

$$|\psi(t)\rangle \equiv e^{-iHt/\hbar}|0\rangle = \underbrace{\exp\left[\frac{t}{2}\left(a^2 e^{2i\pi/4} - h.c.\right)\right]}_{\equiv\, \mathcal{S},\ \text{“squeezing” operator}}|0\rangle$$

- As the squeezing is caused by time, instead of t we introduce “squeezing parameter” r and, instead of $-\pi/4$ we introduce “squeezing phase” ϕ:

$$\mathcal{S} \equiv \exp\left[\frac{r}{2}\left(a^2 e^{-2i\phi} - h.c.\right)\right]$$

- Of course, $\mathcal{S}^\dagger = \mathcal{S}^{-1}$ [In fact $\mathcal{S}$ is another name for $U(t) \equiv e^{-iHt/\hbar}$]

- Now we make use of the following general operator relation:

$$e^{a\widehat{\mathcal{O}}}\widehat{\mathcal{P}}e^{-a\widehat{\mathcal{O}}} = \widehat{\mathcal{P}} + a\left[\widehat{\mathcal{O}},\widehat{\mathcal{P}}\right] + \frac{a^2}{2!}\left[\widehat{\mathcal{O}},\left[\widehat{\mathcal{O}},\widehat{\mathcal{P}}\right]\right] + \cdots$$

- Now we make use of the following general operator relation:

$$e^{a\widehat{\mathcal{O}}}\widehat{\mathcal{P}}e^{-a\widehat{\mathcal{O}}} = \widehat{\mathcal{P}} + a\left[\widehat{\mathcal{O}}, \widehat{\mathcal{P}}\right] + \frac{a^2}{2!}\left[\widehat{\mathcal{O}}, \left[\widehat{\mathcal{O}}, \widehat{\mathcal{P}}\right]\right] + \cdots$$

$$a = -r$$

$$\widehat{\mathcal{P}} = a$$

$$\widehat{\mathcal{O}} = e^{-2i\phi}\frac{a^2}{2} - e^{2i\phi}\frac{{a^\dagger}^2}{2}$$

- Now we make use of the following general operator relation:

$$e^{a\widehat{\mathcal{O}}}\widehat{\mathcal{P}}e^{-a\widehat{\mathcal{O}}} = \widehat{\mathcal{P}} + a\left[\widehat{\mathcal{O}},\widehat{\mathcal{P}}\right] + \frac{a^2}{2!}\left[\widehat{\mathcal{O}},\left[\widehat{\mathcal{O}},\widehat{\mathcal{P}}\right]\right] + \cdots$$

$$a = -r$$

$$\widehat{\mathcal{P}} = a$$

$$\widehat{\mathcal{O}} = e^{-2i\phi}\frac{a^2}{2} - e^{2i\phi}\frac{{a^\dagger}^2}{2}$$

$$\begin{aligned}\left[\widehat{\mathcal{O}},\widehat{\mathcal{P}}\right] &= \frac{e^{-2i\phi}}{2}\left[a^2, a\right] - \frac{e^{2i\phi}}{2}\left[{a^\dagger}^2, a\right] = -\frac{e^{2i\phi}}{2}\left\{a^\dagger\left[a^\dagger, a\right] + \left[a^\dagger, a\right]a^\dagger\right\} \\ &= e^{2i\phi}a^\dagger\end{aligned}$$

$$\begin{aligned}\left[\widehat{\mathcal{O}},\left[\widehat{\mathcal{O}},\widehat{\mathcal{P}}\right]\right] &= e^{2i\phi}\left[\widehat{\mathcal{O}}, a^\dagger\right] = \frac{1}{2}\left[a^2, a^\dagger\right] - \frac{e^{4i\phi}}{2}\left[{a^\dagger}^2, a^\dagger\right] \\ &= a = \widehat{\mathcal{P}}\end{aligned}$$

$$\left[\widehat{\mathcal{O}},\left[\widehat{\mathcal{O}},\left[\widehat{\mathcal{O}},\widehat{\mathcal{P}}\right]\right]\right] = \left[\widehat{\mathcal{O}},\widehat{\mathcal{P}}\right]$$

$$\vdots$$

- Thus we find $\mathcal{S}^\dagger a \mathcal{S}$ as

$$\begin{aligned}
\mathcal{S}^\dagger a \mathcal{S} &= \exp\left[-r\left(e^{-2i\phi}\frac{a^2}{2} - e^{2i\phi}\frac{{a^\dagger}^2}{2}\right)\right] a \exp\left[r\left(e^{-2i\phi}\frac{a^2}{2} - e^{2i\phi}\frac{{a^\dagger}^2}{2}\right)\right] \\
&= a - re^{2i\phi}a^\dagger + \frac{r^2}{2!}a - \frac{r^3}{3!}e^{2i\phi}a^\dagger + \cdots \\
&= a\left(1 + \frac{r^2}{2!} + \cdots\right) - a^\dagger e^{2i\phi}\left(r + \frac{r^3}{3!} + \cdots\right) \\
&= a\cosh r - a^\dagger e^{2i\phi}\sinh r
\end{aligned}$$

- Thus we find $\mathcal{S}^\dagger a \mathcal{S}$ as

$$\mathcal{S}^\dagger a \mathcal{S} = \exp\left[-r\left(e^{-2i\phi}\frac{a^2}{2} - e^{2i\phi}\frac{{a^\dagger}^2}{2}\right)\right] a \exp\left[r\left(e^{-2i\phi}\frac{a^2}{2} - e^{2i\phi}\frac{{a^\dagger}^2}{2}\right)\right]$$

$$= a - re^{2i\phi}a^\dagger + \frac{r^2}{2!}a - \frac{r^3}{3!}e^{2i\phi}a^\dagger + \cdots$$

$$= a\left(1 + \frac{r^2}{2!} + \cdots\right) - a^\dagger e^{2i\phi}\left(r + \frac{r^3}{3!} + \cdots\right)$$

$$= a\cosh r - a^\dagger e^{2i\phi}\sinh r$$

- Taking adjoint of both sides gives

$$\left(\mathcal{S}^\dagger a \mathcal{S}\right)^\dagger = \mathcal{S}^\dagger a^\dagger \mathcal{S} = a^\dagger \cosh r - a e^{-2\phi}\sinh r$$

- This immediately gives $\mathcal{S}^\dagger \hat{P}|\psi(t)\rangle$ as

$$\begin{aligned}
\mathcal{S}^\dagger \widehat{P}|\psi(t)\rangle &= \mathcal{S}^\dagger \widehat{P}\mathcal{S}|0\rangle \\
&= \mathcal{S}^\dagger \cdot -i\sqrt{\frac{\hbar}{2}}\left(a - a^\dagger\right) \cdot \mathcal{S}|0\rangle \\
&= -i\sqrt{\frac{\hbar}{2}}\left(\mathcal{S}^\dagger a \mathcal{S} - \mathcal{S}^\dagger a^\dagger \mathcal{S}\right)|0\rangle \\
&= -i\sqrt{\frac{\hbar}{2}}\Big[\big(\cosh r a - e^{2i\phi}\sinh r a^\dagger\big) - \big(\cosh r a^\dagger - e^{-2i\phi}\sinh r a\big)\Big]|0\rangle \\
&= i\sqrt{\frac{\hbar}{2}}\big(\cosh r + e^{2i\phi}\sinh r\big)a^\dagger|0\rangle
\end{aligned}$$

- This immediately gives $\mathcal{S}^\dagger \hat{P}|\psi(t)\rangle$ as

$$\begin{aligned}
\mathcal{S}^\dagger \widehat{P}|\psi(t)\rangle &= \mathcal{S}^\dagger \widehat{P}\mathcal{S}|0\rangle \\
&= \mathcal{S}^\dagger \cdot -i\sqrt{\frac{\hbar}{2}}\left(a - a^\dagger\right) \cdot \mathcal{S}|0\rangle \\
&= -i\sqrt{\frac{\hbar}{2}}\Big(\mathcal{S}^\dagger a \mathcal{S} - \mathcal{S}^\dagger a^\dagger \mathcal{S}\Big)|0\rangle \\
&= -i\sqrt{\frac{\hbar}{2}}\Big[\big(\cosh r a - e^{2i\phi}\sinh r a^\dagger\big) - \big(\cosh r a^\dagger - e^{-2i\phi}\sinh r a\big)\Big]|0\rangle \\
&= i\sqrt{\frac{\hbar}{2}}\big(\cosh r + e^{2i\phi}\sinh r\big)a^\dagger|0\rangle
\end{aligned}$$

- In the same way, we find $\mathcal{S}^\dagger \hat{Q}|\psi(t)\rangle$ as

$$\mathcal{S}^\dagger \widehat{Q}|\psi(t)\rangle = \mathcal{S}^\dagger \cdot \sqrt{\frac{\hbar}{2}}\left(a + a^\dagger\right) \cdot \mathcal{S}|0\rangle = \sqrt{\frac{\hbar}{2}}\big(\cosh r - e^{2i\phi}\sinh r\big)a^\dagger|0\rangle$$

- This immediately gives $\mathcal{S}^\dagger \hat{P}|\psi(t)\rangle$ as

$$\begin{aligned}
\mathcal{S}^\dagger \widehat{P}|\psi(t)\rangle &= \mathcal{S}^\dagger \widehat{P}\mathcal{S}|0\rangle \\
&= \mathcal{S}^\dagger \cdot -i\sqrt{\frac{\hbar}{2}}(a-a^\dagger)\cdot\mathcal{S}|0\rangle \\
&= -i\sqrt{\frac{\hbar}{2}}\left(\mathcal{S}^\dagger a\mathcal{S} - \mathcal{S}^\dagger a^\dagger\mathcal{S}\right)|0\rangle \\
&= -i\sqrt{\frac{\hbar}{2}}\Big[\big(\cosh r a - e^{2i\phi}\sinh r a^\dagger\big) - \big(\cosh r a^\dagger - e^{-2i\phi}\sinh r a\big)\Big]|0\rangle \\
&= i\sqrt{\frac{\hbar}{2}}\big(\cosh r + e^{2i\phi}\sinh r\big)a^\dagger|0\rangle
\end{aligned}$$

$$\left\langle \widehat{P}\right\rangle \equiv \left\langle \psi(t)\middle|\widehat{P}\middle|\psi(t)\right\rangle = \left\langle 0\middle|\mathcal{S}^\dagger\widehat{P}\mathcal{S}\middle|0\right\rangle = 0$$

$$\left\langle \widehat{Q}\right\rangle = 0$$

- In the same way, we find $\mathcal{S}^\dagger \hat{Q}|\psi(t)\rangle$ as

$$\mathcal{S}^\dagger \widehat{Q}|\psi(t)\rangle = \mathcal{S}^\dagger \cdot \sqrt{\frac{\hbar}{2}}(a+a^\dagger)\cdot\mathcal{S}|0\rangle = \sqrt{\frac{\hbar}{2}}\big(\cosh r - e^{2i\phi}\sinh r\big)a^\dagger|0\rangle$$

- Then we have:

$$\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle = \left\langle \psi(t) \left| \widehat{P}\mathcal{S} \cdot \mathcal{S}^\dagger \widehat{P} \right| \psi(t) \right\rangle$$
$$= \frac{\hbar}{2} \langle 0 | a \left(\cosh r + e^{-2i\phi} \sinh r \right) \left(\cosh r + e^{2i\phi} \sinh r \right) a^\dagger | 0 \rangle$$

- Then we have:

$$\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle = \left\langle \psi(t) \left| \widehat{P}\mathcal{S} \cdot \mathcal{S}^\dagger \widehat{P} \right| \psi(t) \right\rangle$$
$$= \frac{\hbar}{2} \langle 0 | a \underbrace{\left(\cosh r + e^{-2i\phi} \sinh r \right) \left(\cosh r + e^{2i\phi} \sinh r \right)}_{=\cosh(2r) + \cos(2\phi)\sinh(2r)} a^\dagger | 0 \rangle$$

- Then we have:

$$\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle = \left\langle \psi(t) \left| \widehat{P}\mathcal{S} \cdot \mathcal{S}^\dagger \widehat{P} \right| \psi(t) \right\rangle$$

$$= \frac{\hbar}{2} \langle 0| a \underbrace{\left(\cosh r + e^{-2i\phi} \sinh r \right) \left(\cosh r + e^{2i\phi} \sinh r \right)}_{= \cosh(2r) + \cos(2\phi) \sinh(2r)} a^\dagger |0\rangle$$

$$= \frac{\hbar}{2} \left[\cosh(2r) + \cos(2\phi) \sinh(2r) \right]$$

- Then we have:

$$
\begin{aligned}
\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle &= \left\langle \psi(t) \left| \widehat{P} \mathcal{S} \cdot \mathcal{S}^\dagger \widehat{P} \right| \psi(t) \right\rangle \\
&= \frac{\hbar}{2} \langle 0 | a \underbrace{\left(\cosh r + e^{-2i\phi} \sinh r \right) \left(\cosh r + e^{2i\phi} \sinh r \right)}_{= \cosh(2r) + \cos(2\phi) \sinh(2r)} a^\dagger | 0 \rangle \\
&= \frac{\hbar}{2} \left[\cosh(2r) + \cos(2\phi) \sinh(2r) \right] \\
\left\langle \psi(t) \left| \widehat{Q}^2 \right| \psi(t) \right\rangle &= \frac{\hbar}{2} \left[\cosh(2r) - \cos(2\phi) \sinh(2r) \right]
\end{aligned}
$$

- Then we have:

$$
\begin{aligned}
\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle &= \left\langle \psi(t) \left| \widehat{P}\mathcal{S} \cdot \mathcal{S}^\dagger \widehat{P} \right| \psi(t) \right\rangle \\
&= \frac{\hbar}{2} \langle 0 | a \underbrace{\left(\cosh r + e^{-2i\phi} \sinh r \right) \left(\cosh r + e^{2i\phi} \sinh r \right)}_{= \cosh(2r) + \cos(2\phi) \sinh(2r)} a^\dagger | 0 \rangle \\
&= \frac{\hbar}{2} \left[\cosh(2r) + \cos(2\phi) \sinh(2r) \right] \\
\left\langle \psi(t) \left| \widehat{Q}^2 \right| \psi(t) \right\rangle &= \frac{\hbar}{2} \left[\cosh(2r) - \cos(2\phi) \sinh(2r) \right]
\end{aligned}
$$

- Then the uncertainty relationship is:

$$
\left[\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle \left\langle \psi(t) \left| \widehat{Q}^2 \right| \psi(t) \right\rangle \right]^{1/2} = \frac{\hbar}{2} \left[1 + \sin^2(2\phi) \sinh^2(2r) \right]^{1/2}
$$

- Then we have:

$$\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle = \left\langle \psi(t) \left| \widehat{P}\mathcal{S} \cdot \mathcal{S}^\dagger \widehat{P} \right| \psi(t) \right\rangle$$

$$= \frac{\hbar}{2} \langle 0 | a \underbrace{\left(\cosh r + e^{-2i\phi} \sinh r \right) \left(\cosh r + e^{2i\phi} \sinh r \right)}_{= \cosh(2r) + \cos(2\phi) \sinh(2r)} a^\dagger | 0 \rangle$$

$$= \frac{\hbar}{2} \left[\cosh(2r) + \cos(2\phi) \sinh(2r) \right]$$

$$\left\langle \psi(t) \left| \widehat{Q}^2 \right| \psi(t) \right\rangle = \frac{\hbar}{2} \left[\cosh(2r) - \cos(2\phi) \sinh(2r) \right]$$

- Then the uncertainty relationship is:

$$\underbrace{\left[\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle \left\langle \psi(t) \left| \widehat{Q}^2 \right| \psi(t) \right\rangle \right]^{1/2}}_{= \Delta\widehat{Q} \Delta\widehat{P}} = \frac{\hbar}{2} \left[1 + \sin^2(2\phi) \sinh^2(2r) \right]^{1/2}$$

- Then we have:

$$
\begin{aligned}
\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle &= \left\langle \psi(t) \left| \widehat{P}\mathcal{S} \cdot \mathcal{S}^\dagger \widehat{P} \right| \psi(t) \right\rangle \\
&= \frac{\hbar}{2} \langle 0 | a \underbrace{\left(\cosh r + e^{-2i\phi} \sinh r \right) \left(\cosh r + e^{2i\phi} \sinh r \right)}_{=\cosh(2r)+\cos(2\phi)\sinh(2r)} a^\dagger | 0 \rangle \\
&= \frac{\hbar}{2} \left[\cosh(2r) + \cos(2\phi) \sinh(2r) \right] \\
\left\langle \psi(t) \left| \widehat{Q}^2 \right| \psi(t) \right\rangle &= \frac{\hbar}{2} \left[\cosh(2r) - \cos(2\phi) \sinh(2r) \right]
\end{aligned}
$$

- Then the uncertainty relationship is:

$$
\underbrace{\left[\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle \left\langle \psi(t) \left| \widehat{Q}^2 \right| \psi(t) \right\rangle \right]^{1/2}}_{= \Delta \widehat{Q} \Delta \widehat{P}} = \underbrace{\frac{\hbar}{2} \left[1 + \sin^2(2\phi) \sinh^2(2r) \right]^{1/2}}_{\xrightarrow[r=t\gg 1]{} \frac{\hbar}{4} e^{2t}}
$$

- Then we have:

$$\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle = \left\langle \psi(t) \left| \widehat{P}\mathcal{S} \cdot \mathcal{S}^\dagger \widehat{P} \right| \psi(t) \right\rangle$$

$$= \frac{\hbar}{2} \langle 0 | a \underbrace{\left(\cosh r + e^{-2i\phi} \sinh r \right) \left(\cosh r + e^{2i\phi} \sinh r \right)}_{=\cosh(2r)+\cos(2\phi)\sinh(2r)} a^\dagger | 0 \rangle$$

$$= \frac{\hbar}{2} \left[\cosh(2r) + \cos(2\phi) \sinh(2r) \right]$$

$$\left\langle \psi(t) \left| \widehat{Q}^2 \right| \psi(t) \right\rangle = \frac{\hbar}{2} \left[\cosh(2r) - \cos(2\phi) \sinh(2r) \right]$$

- Then the uncertainty relationship is:

$$\underbrace{\left[\left\langle \psi(t) \left| \widehat{P}^2 \right| \psi(t) \right\rangle \left\langle \psi(t) \left| \widehat{Q}^2 \right| \psi(t) \right\rangle \right]^{1/2}}_{= \Delta \widehat{Q} \Delta \widehat{P}} = \underbrace{\frac{\hbar}{2} \left[1 + \sin^2(2\phi) \sinh^2(2r) \right]^{1/2}}_{\xrightarrow[r=t \gg 1]{} \frac{\hbar}{4} e^{2t}}$$

Uncertainty spreads rapidly in $\hat{Q}$ and $\hat{P}$!

- However, consider the following:

$$\mathcal{S}^\dagger \widehat{P} \cos\phi |\psi(t)\rangle = i\sqrt{\frac{\hbar}{2}}\big(\cosh r + e^{2i\phi}\sinh r\big)\cos\phi a^\dagger |0\rangle$$

$$\mathcal{S}^\dagger \widehat{Q} \sin\phi |\psi(t)\rangle = \sqrt{\frac{\hbar}{2}}\big(\cosh r - e^{2i\phi}\sinh r\big)\sin\phi a^\dagger |0\rangle$$

- However, consider the following:

$$\mathcal{S}^\dagger \widehat{P}\cos\phi|\psi(t)\rangle = i\sqrt{\frac{\hbar}{2}}\big(\cosh r + e^{2i\phi}\sinh r\big)\cos\phi a^\dagger|0\rangle$$

$$\mathcal{S}^\dagger \widehat{Q}\sin\phi|\psi(t)\rangle = \sqrt{\frac{\hbar}{2}}\big(\cosh r - e^{2i\phi}\sinh r\big)\sin\phi a^\dagger|0\rangle$$

$$\therefore\ \mathcal{S}^\dagger\left(\widehat{P}\cos\phi - \widehat{Q}\sin\phi\right)|\psi(t)\rangle = \sqrt{\frac{\hbar}{2}} i e^{i\phi} e^{r} a^\dagger|0\rangle$$

$$\text{Similarly } \mathcal{S}^\dagger\left(\widehat{P}\sin\phi + \widehat{Q}\cos\phi\right)|\psi(t)\rangle = \sqrt{\frac{\hbar}{2}} e^{i\phi} e^{-r} a^\dagger|0\rangle$$

- However, consider the following:

$$\mathcal{S}^\dagger \widehat{P}\cos\phi|\psi(t)\rangle = i\sqrt{\frac{\hbar}{2}}\big(\cosh r + e^{2i\phi}\sinh r\big)\cos\phi a^\dagger|0\rangle$$

$$\mathcal{S}^\dagger \widehat{Q}\sin\phi|\psi(t)\rangle = \sqrt{\frac{\hbar}{2}}\big(\cosh r - e^{2i\phi}\sinh r\big)\sin\phi a^\dagger|0\rangle$$

$$\therefore\ \mathcal{S}^\dagger\left(\widehat{P}\cos\phi - \widehat{Q}\sin\phi\right)|\psi(t)\rangle = \sqrt{\frac{\hbar}{2}} i e^{i\phi} e^{r} a^\dagger|0\rangle$$

$$\text{Similarly } \mathcal{S}^\dagger\left(\widehat{P}\sin\phi + \widehat{Q}\cos\phi\right)|\psi(t)\rangle = \sqrt{\frac{\hbar}{2}} e^{i\phi} e^{-r} a^\dagger|0\rangle$$

$$\therefore\ \left\langle \psi(t) \middle| \left(\widehat{P}\cos\phi - \widehat{Q}\sin\phi\right)^2 \middle| \psi(t) \right\rangle = \frac{\hbar}{2}e^{2r}$$

$$\left\langle \psi(t) \middle| \left(\widehat{P}\sin\phi + \widehat{Q}\cos\phi\right)^2 \middle| \psi(t) \right\rangle = \frac{\hbar}{2}e^{-2r}$$

- Several important remarks in order:
    1. For $\phi = -\pi/4$, $\left(\hat{P}\cos\phi - \hat{Q}\sin\phi\right)^2 = (\Delta b_+)^2$, $\left(\hat{P}\sin\phi + \hat{Q}\cos\phi\right)^2 = (\Delta b_-)^2$
    2. The uncertainty in one direction (b_+) increases exponentially, while that in the other direction (b_-) decreases exponentially
    3. This behaviour mirrors that of phase space trajectories for the classical system – squeezing!

- Several important remarks in order:
  1. For $\phi = -\pi/4$, $\left(\hat{P}\cos\phi - \hat{Q}\sin\phi\right)^2 = (\Delta b_+)^2$, $\left(\hat{P}\sin\phi + \hat{Q}\cos\phi\right)^2 = (\Delta b_-)^2$
  2. The uncertainty in one direction (b_+) increases exponentially, while that in the other direction (b_-) decreases exponentially
  3. This behaviour mirrors that of phase space trajectories for the classical system – squeezing!
- We will observe similar properties in the evolution of the quantum states for cosmological perturbations!

IV. Inflationary evolution of quantum state

1. Preliminary: Inverted harmonic oscillator
   a. Classical inverted harmonic oscillator
   b. Quantum inverted harmonic oscillator
2. Reformulation of quantum fluctuations
3. Hamiltonian operator revisited
   a. Hamiltonian in factorized form
   b. What does the Hamiltonian do on quantum state?

- We begin with the quadratic action

$$S_2^{(s)} = \int d^4x a^3 \epsilon m_{\rm Pl}^2 \left[\mathcal{R}'^2 - \left(\nabla\mathcal{R}\right)^2 \right]$$

$$S_2^{(t)} = \int d^4x \frac{a^2 m_{\rm Pl}^2}{8} \left(h^{ij\prime} h'_{ij} - h^{ij,k} h_{ij,k} \right)$$

- We begin with the quadratic action

$$S_2^{(s)} = \int d^4x a^3 \epsilon m_{\rm Pl}^2 \left[\mathcal{R}'^2 - \left(\nabla \mathcal{R}\right)^2 \right]$$

$$S_2^{(t)} = \int d^4x \frac{a^2 m_{\rm Pl}^2}{8} \left(h^{ij\prime} h'_{ij} - h^{ij,k} h_{ij,k} \right)$$

- In terms of the Fourier modes:

$$\mathcal{R}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \mathcal{R}(\tau, \boldsymbol{k})$$

$$h_{ij}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{s=1}^{2} h_{(s)}(\tau, \boldsymbol{k}) e_{ij}^{(s)}(\hat{\boldsymbol{k}})$$

- We begin with the quadratic action

$$S_2^{(s)} = \int d^4x a^3 \epsilon m_{\mathrm{Pl}}^2 \left[\mathcal{R}'^2 - (\nabla\mathcal{R})^2 \right]$$

$$S_2^{(t)} = \int d^4x \frac{a^2 m_{\mathrm{Pl}}^2}{8} \left(h^{ij\prime} h'_{ij} - h^{ij,k} h_{ij,k} \right)$$

- In terms of the Fourier modes:

$$\mathcal{R}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \mathcal{R}(\tau, \boldsymbol{k})$$

$$h_{ij}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{s=1}^{2} h_{(s)}(\tau, \boldsymbol{k}) e_{ij}^{(s)}(\hat{\boldsymbol{k}})$$

- From now on we only consider tensor perturbations because 1) both scalar and tensor actions have exactly the same structure, and 2) tensor is simpler

- The action in terms of the Fourier modes (with more explicit in momentum):

$$
\begin{aligned}
S_2^{(t)} &= \int d\tau \frac{a^2 m_{\rm Pl}^2}{4} \int \frac{d^3k}{(2\pi)^3} \sum_s \left[h'_{(s)}(\tau, \boldsymbol{k}) h'_{(s)}(\tau, -\boldsymbol{k}) - k^2 h_{(s)}(\tau, \boldsymbol{k}) h_{(s)}(\tau, -\boldsymbol{k}) \right] \\
&= \sum_s \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{2} \left[v_{\boldsymbol{k}}^{(s)\prime} v_{-\boldsymbol{k}}^{(s)\prime} - \frac{a'}{a} v_{\boldsymbol{k}}^{(s)\prime} v_{-\boldsymbol{k}}^{(s)} - \frac{a'}{a} v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)\prime} - k^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} + \left(\frac{a'}{a} \right)^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} \right]
\end{aligned}
$$

$$
v_{\boldsymbol{k}}^{(s)}(\tau) \equiv \frac{a m_{\rm Pl}}{\sqrt{2}} h_{(s)}(\tau, \boldsymbol{k})
$$

- The action in terms of the Fourier modes (with more explicit in momentum):

$$S_2^{(t)} = \int d\tau \frac{a^2 m_{\rm Pl}^2}{4} \int \frac{d^3k}{(2\pi)^3} \sum_s \left[h'_{(s)}(\tau, \boldsymbol{k}) h'_{(s)}(\tau, -\boldsymbol{k}) - k^2 h_{(s)}(\tau, \boldsymbol{k}) h_{(s)}(\tau, -\boldsymbol{k}) \right]$$

$$= \sum_s \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{2} \left[v_{\boldsymbol{k}}^{(s)\prime} v_{-\boldsymbol{k}}^{(s)\prime} \boxed{- \frac{a'}{a} v_{\boldsymbol{k}}^{(s)\prime} v_{-\boldsymbol{k}}^{(s)} - \frac{a'}{a} v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)\prime}} - k^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} + \left(\frac{a'}{a} \right)^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} \right]$$

$$v_{\boldsymbol{k}}^{(s)}(\tau) \equiv \frac{a m_{\rm Pl}}{\sqrt{2}} h_{(s)}(\tau, \boldsymbol{k})$$

- The action in terms of the Fourier modes (with more explicit in momentum):

$$S_2^{(t)} = \int d\tau \frac{a^2 m_{\rm Pl}^2}{4} \int \frac{d^3k}{(2\pi)^3} \sum_s \left[h'_{(s)}(\tau, \boldsymbol{k}) h'_{(s)}(\tau, -\boldsymbol{k}) - k^2 h_{(s)}(\tau, \boldsymbol{k}) h_{(s)}(\tau, -\boldsymbol{k}) \right]$$

$$= \sum_s \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{2} \left[v_{\boldsymbol{k}}^{(s)\prime} v_{-\boldsymbol{k}}^{(s)\prime} \boxed{- \frac{a'}{a} v_{\boldsymbol{k}}^{(s)\prime} v_{-\boldsymbol{k}}^{(s)} - \frac{a'}{a} v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)\prime}} - k^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} + \left(\frac{a'}{a} \right)^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} \right]$$

$$v_{\boldsymbol{k}}^{(s)}(\tau) \equiv \frac{a m_{\rm Pl}}{\sqrt{2}} h_{(s)}(\tau, \boldsymbol{k})$$

- Then the canonical conjugate momentum of $v_{\boldsymbol{k}}^{(s)}$ is given by

$$\pi_{\boldsymbol{k}}^{(s)} \equiv \frac{\delta \mathcal{L}}{\delta v_{\boldsymbol{k}}^{(s)\prime}} = v_{\boldsymbol{k}}^{(s)\prime} - \frac{a'}{a} v_{\boldsymbol{k}}^{(s)}$$

- Then we can find the Hamiltonian from the quadratic action as:

$$H = \frac{1}{2} \sum_s \int \frac{d^3k}{(2\pi)^3} \left[\pi_{\boldsymbol{k}}^{(s)} \pi_{-\boldsymbol{k}}^{(s)} + k^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} + \frac{a'}{a} \left(\pi_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} + v_{\boldsymbol{k}}^{(s)} \pi_{-\boldsymbol{k}}^{(s)} \right) \right]$$

- Then we can find the Hamiltonian from the quadratic action as:

$$H = \frac{1}{2}\sum_s \int \frac{d^3k}{(2\pi)^3}\left[\underbrace{\pi_{\boldsymbol{k}}^{(s)}\pi_{-\boldsymbol{k}}^{(s)} + k^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)}}_{\text{These exist in Minkowski space}} + \underbrace{\frac{a'}{a}\left(\pi_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} + v_{\boldsymbol{k}}^{(s)}\pi_{-\boldsymbol{k}}^{(s)}\right)}_{\text{Were it not for gravity (scale factor!) these terms are absent!}}\right]$$

- Then we can find the Hamiltonian from the quadratic action as:

$$H = \frac{1}{2}\sum_s \int \frac{d^3k}{(2\pi)^3}\left[\underbrace{\pi_{\boldsymbol{k}}^{(s)}\pi_{-\boldsymbol{k}}^{(s)} + k^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)}}_{\text{These exist in Minkowski space}} + \underbrace{\frac{a'}{a}\left(\pi_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} + v_{\boldsymbol{k}}^{(s)} \pi_{-\boldsymbol{k}}^{(s)}\right)}_{\substack{\text{Were it not for gravity (scale factor!)} \\ \text{these terms are absent!}}}\right]$$

- Now we promote the conjugate pair to time-dependent operators:

$$\widehat{\pi}_{\boldsymbol{k}}^{(s)}(\tau) = a_{\boldsymbol{k}}^{(s)}(\tau) u_k + a_{-\boldsymbol{k}}^{(s)\,\dagger}(\tau) u_k^*$$

$$\widehat{v}_{\boldsymbol{k}}^{(s)}(\tau) = a_{\boldsymbol{k}}^{(s)}(\tau) v_k + a_{-\boldsymbol{k}}^{(s)\,\dagger}(\tau) v_k^*$$

- Then we can find the Hamiltonian from the quadratic action as:

$$H = \frac{1}{2}\sum_s \int \frac{d^3k}{(2\pi)^3}\left[\underbrace{\pi_{\boldsymbol{k}}^{(s)}\pi_{-\boldsymbol{k}}^{(s)} + k^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)}}_{\text{These exist in Minkowski space}} + \underbrace{\frac{a'}{a}\left(\pi_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)} + v_{\boldsymbol{k}}^{(s)}\pi_{-\boldsymbol{k}}^{(s)}\right)}_{\text{Were it not for gravity (scale factor!) these terms are absent!}}\right]$$

- Now we promote the conjugate pair to time-dependent operators:

$$\widehat{\pi}_{\boldsymbol{k}}^{(s)}(\tau) = a_{\boldsymbol{k}}^{(s)}(\tau) u_k + a_{-\boldsymbol{k}}^{(s)\dagger}(\tau) u_k^*$$

$$\widehat{v}_{\boldsymbol{k}}^{(s)}(\tau) = a_{\boldsymbol{k}}^{(s)}(\tau) v_k + a_{-\boldsymbol{k}}^{(s)\dagger}(\tau) v_k^*$$

- An important difference from the conventional approach is that the time dependence is given to the creation and annihilation operators, $a_{\boldsymbol{k}}^{(s)} = a_{\boldsymbol{k}}^{(s)}(\tau)$ and $a_{-\boldsymbol{k}}^{(s)\dagger} = a_{-\boldsymbol{k}}^{(s)\dagger}(\tau)$, while the mode functions are set up at an initial time τ_0

- We impose the canonical commutation relations:

$$\left[\widehat{v}^{(s)}(\tau,\boldsymbol{x}),\widehat{\pi}^{(s')}(\tau,\boldsymbol{y})\right]=i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta_{ss'}$$

$$\left[a_{\boldsymbol{k}}^{(s)}(\tau),a_{\boldsymbol{q}}^{(s')\dagger}(\tau)\right]=(2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q})\delta_{ss'}$$

- These imposes the relation between (initial) mode functions v_k and u_k:

$$u_k^* v_k - u_k v_k^* = i$$

- We impose the canonical commutation relations:

$$\left[\widehat{v}^{(s)}(\tau, \boldsymbol{x}), \widehat{\pi}^{(s')}(\tau, \boldsymbol{y})\right] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\delta_{ss'}$$

$$\left[a_{\boldsymbol{k}}^{(s)}(\tau), a_{\boldsymbol{q}}^{(s')\dagger}(\tau)\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{k} - \boldsymbol{q})\delta_{ss'}$$

- These imposes the relation between (initial) mode functions v_k and u_k:

$$u_k^* v_k - u_k v_k^* = i$$

- To set up the initial conditions at $\tau = \tau_0$, we assume that the modes are deep inside the horizon so that the effects of gravity can be neglected

- We impose the canonical commutation relations:

$$\left[\widehat{v}^{(s)}(\tau, \boldsymbol{x}), \widehat{\pi}^{(s')}(\tau, \boldsymbol{y})\right] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\delta_{ss'}$$

$$\left[a_{\boldsymbol{k}}^{(s)}(\tau), a_{\boldsymbol{q}}^{(s')\dagger}(\tau)\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{k} - \boldsymbol{q})\delta_{ss'}$$

- These imposes the relation between (initial) mode functions v_k and u_k:

$$u_k^* v_k - u_k v_k^* = i$$

- To set up the initial conditions at $\tau = \tau_0$, we assume that the modes are deep inside the horizon so that the effects of gravity can be neglected
- This amounts to considering only the part of the full Hamiltonian that is free from gravitational effects:

$$H \to \frac{1}{2}\sum_s \int \frac{d^3k}{(2\pi)^3}\left[\pi_{\boldsymbol{k}}^{(s)}\pi_{-\boldsymbol{k}}^{(s)} + k^2 v_{\boldsymbol{k}}^{(s)} v_{-\boldsymbol{k}}^{(s)}\right]$$

- Plugging the operator expansion and taking the expectation value with respect to the initial vacuum state that is annihilated by $a_{\boldsymbol{k}}^{(s)}$, we have (now we omit the polarization index for simplicity)

$$\langle H(\tau_0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}(2\pi)^3\delta^{(3)}(0)\left[|u_k|^2 + k^2|v_k|^2\right]$$

- Plugging the operator expansion and taking the expectation value with respect to the initial vacuum state that is annihilated by $a_{\boldsymbol{k}}^{(s)}$, we have (now we omit the polarization index for simplicity)

$$\langle H(\tau_0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}(2\pi)^3\delta^{(3)}(0)\left[|u_k|^2 + k^2|v_k|^2\right]$$

- The vacuum state, or equivalently the mode function, is chosen to minimize this expectation value of the Hamiltonian
- We may proceed formally by using the Cauchy-Schwarz inequality, but let us be satisfied with a more heuristic, handwaving approach

- Plugging the operator expansion and taking the expectation value with respect to the initial vacuum state that is annihilated by $a_{\boldsymbol{k}}^{(s)}$, we have (now we omit the polarization index for simplicity)

$$\langle H(\tau_0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}(2\pi)^3\delta^{(3)}(0)\left[|u_k|^2 + k^2|v_k|^2\right]$$

- The vacuum state, or equivalently the mode function, is chosen to minimize this expectation value of the Hamiltonian
- We may proceed formally by using the Cauchy-Schwarz inequality, but let us be satisfied with a more heuristic, handwaving approach
- Without losing generality, let us assume that v_k is real
- Then, because of the condition $u_k^* v_k - u_k v_k^* = i$, u_k should be purely imaginary so let us write $u_k = -i\tilde{u}_k$

- Then the condition is reduced to $\tilde{u}_k v_k = 1/2$

- Then the condition is reduced to $\tilde{u}_k v_k = 1/2$
- Thus, we can find v_k that minimizes $|u_k|^2 + k^2|v_k|^2 = 1/\left(4v_k^2\right) + k^2 v_k^2$ as we did previously, i.e. 1st derivative = 0 and 2nd derivative > 0:

$$v_k = \frac{1}{\sqrt{2k}}$$

$$u_k = -i\widetilde{u}_k = \frac{-i}{2v_k} = -i\sqrt{\frac{k}{2}}$$

- Then the condition is reduced to $\tilde{u}_k v_k = 1/2$
- Thus, we can find v_k that minimizes $|u_k|^2 + k^2|v_k|^2 = 1/\left(4v_k^2\right) + k^2 v_k^2$ as we did previously, i.e. 1st derivative = 0 and 2nd derivative > 0:

$$v_k = \frac{1}{\sqrt{2k}}$$

$$u_k = -i\widetilde{u}_k = \frac{-i}{2v_k} = -i\sqrt{\frac{k}{2}}$$

- Thus, the canonical variables are

$$\widehat{\pi}_{\boldsymbol{k}}(\tau) = -i\sqrt{\frac{k}{2}}\left[a_{\boldsymbol{k}}(\tau) - a^{\dagger}_{-\boldsymbol{k}}(\tau)\right]$$

$$\widehat{v}_{\boldsymbol{k}}(\tau) = \frac{1}{\sqrt{2}}\left[a_{\boldsymbol{k}}(\tau) + a^{\dagger}_{-\boldsymbol{k}}(\tau)\right]$$

- Then the condition is reduced to $\tilde{u}_k v_k = 1/2$
- Thus, we can find v_k that minimizes $|u_k|^2 + k^2|v_k|^2 = 1/\left(4v_k^2\right) + k^2 v_k^2$ as we did previously, i.e. 1st derivative = 0 and 2nd derivative > 0:

$$v_k = \frac{1}{\sqrt{2k}}$$

$$u_k = -i\widetilde{u}_k = \frac{-i}{2v_k} = -i\sqrt{\frac{k}{2}}$$

- Thus, the canonical variables are

$$\widehat{\pi}_{\boldsymbol{k}}(\tau) = -i\sqrt{\frac{k}{2}}\left[a_{\boldsymbol{k}}(\tau) - a^{\dagger}_{-\boldsymbol{k}}(\tau)\right]$$
$$\widehat{v}_{\boldsymbol{k}}(\tau) = \frac{1}{\sqrt{2}}\left[a_{\boldsymbol{k}}(\tau) + a^{\dagger}_{-\boldsymbol{k}}(\tau)\right]$$

- Barring the time dependence, these solutions are of the same form as those for the time-independent textbook QM harmonic oscillator!

- We can derive the Hamiltonian equations from the Hamiltonian:

$$
\begin{aligned}
v'_{\boldsymbol{k}} &= \frac{\partial H}{\partial \pi_{\boldsymbol{k}}} = \pi_{\boldsymbol{k}} + \frac{a'}{a} v_{\boldsymbol{k}} \\
\pi'_{\boldsymbol{k}} &= -\frac{\partial H}{\partial v_{\boldsymbol{k}}} = -\left(k^2 v_{\boldsymbol{k}} + \frac{a'}{a} \pi_{\boldsymbol{k}} \right)
\end{aligned}
$$

- We can derive the Hamiltonian equations from the Hamiltonian:

$$
\begin{aligned}
v'_{\boldsymbol{k}} &= \frac{\partial H}{\partial \pi_{\boldsymbol{k}}} = \pi_{\boldsymbol{k}} + \frac{a'}{a} v_{\boldsymbol{k}} \\
\pi'_{\boldsymbol{k}} &= -\frac{\partial H}{\partial v_{\boldsymbol{k}}} = -\left(k^2 v_{\boldsymbol{k}} + \frac{a'}{a} \pi_{\boldsymbol{k}} \right) \\
\therefore \;\; v''_{\boldsymbol{k}} &= \pi'_{\boldsymbol{k}} + \frac{a'}{a} v'_{\boldsymbol{k}} + \left[\frac{a''}{a} - \left(\frac{a'}{a} \right)^2 \right] v_{\boldsymbol{k}} \\
&= -\left(k^2 v_{\boldsymbol{k}} + \frac{a'}{a} \pi_{\boldsymbol{k}} \right) + \frac{a'}{a} \left(\pi_{\boldsymbol{k}} + \frac{a'}{a} v_{\boldsymbol{k}} \right) + \left[\frac{a''}{a} - \left(\frac{a'}{a} \right)^2 \right] v_{-\boldsymbol{k}} \\
&= -\left(k^2 - \frac{a''}{a} \right) v_{\boldsymbol{k}}
\end{aligned}
$$

- As we know the conjugate variables $v_{\boldsymbol{k}}$ and $\pi_{\boldsymbol{k}}$ in terms of the creation and annihilation operators $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$, the Hamiltonian equations can be written as the equations for $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$:

$$v_{\boldsymbol{k}}' = \pi_{\boldsymbol{k}} + \frac{a'}{a} v_{\boldsymbol{k}}$$

$$\rightarrow \quad {a_{\boldsymbol{k}}^\dagger + a_{-\boldsymbol{k}}^\dagger}' = -ik\left(a_{\boldsymbol{k}} - a_{-\boldsymbol{k}}^\dagger\right) + \frac{a'}{a}\left(a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^\dagger\right)$$

$$\pi_{\boldsymbol{k}}' = -\left(k^2 v_{\boldsymbol{k}} + \frac{a'}{a}\pi_{\boldsymbol{k}}\right)$$

$$\rightarrow \quad {a_{\boldsymbol{k}}^\dagger - a_{-\boldsymbol{k}}^\dagger}' = -ik\left(a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^\dagger\right) - \frac{a'}{a}\left(a_{\boldsymbol{k}} - a_{-\boldsymbol{k}}^\dagger\right)$$

- As we know the conjugate variables $v_{\boldsymbol{k}}$ and $\pi_{\boldsymbol{k}}$ in terms of the creation and annihilation operators $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}'}$, the Hamiltonian equations can be written as the equations for $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$:

$$v'_{\boldsymbol{k}} = \pi_{\boldsymbol{k}} + \frac{a'}{a} v_{\boldsymbol{k}}$$

$$\rightarrow \quad {a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}}' = -ik\left(a_{\boldsymbol{k}} - a^\dagger_{-\boldsymbol{k}}\right) + \frac{a'}{a}\left(a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}\right)$$

$$\pi'_{\boldsymbol{k}} = -\left(k^2 v_{\boldsymbol{k}} + \frac{a'}{a}\pi_{\boldsymbol{k}}\right)$$

$$\rightarrow \quad {a_{\boldsymbol{k}} - a^\dagger_{-\boldsymbol{k}}}' = -ik\left(a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}\right) - \frac{a'}{a}\left(a_{\boldsymbol{k}} - a^\dagger_{-\boldsymbol{k}}\right)$$

$$\therefore \; a'_{\boldsymbol{k}} = -ika_{\boldsymbol{k}} + \frac{a'}{a} a^\dagger_{-\boldsymbol{k}}$$

$${a^\dagger_{-\boldsymbol{k}}}' = ika^\dagger_{-\boldsymbol{k}} + \frac{a'}{a} a_{\boldsymbol{k}}$$

- These are 1st-order coupled differential equations for $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}'}$, the general solutions are given by the linear combinations of the initial conditions at τ_0:

$$a_{\boldsymbol{k}}(\tau) = \alpha_k(\tau) a_{\boldsymbol{k}}(\tau_0) + \beta_k(\tau) a^\dagger_{-\boldsymbol{k}}(\tau_0)$$
$$a^\dagger_{-\boldsymbol{k}}(\tau) = \alpha^*_k(\tau) a^\dagger_{-\boldsymbol{k}}(\tau_0) + \beta^*_k(\tau) a_{\boldsymbol{k}}(\tau_0)$$

We rediscover the Bogoliubov transformation!

- These are 1st-order coupled differential equations for $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$, the general solutions are given by the linear combinations of the initial conditions at τ_0:

$$a_{\boldsymbol{k}}(\tau) = \alpha_k(\tau) a_{\boldsymbol{k}}(\tau_0) + \beta_k(\tau) a^\dagger_{-\boldsymbol{k}}(\tau_0)$$

$$a^\dagger_{-\boldsymbol{k}}(\tau) = \alpha_k^*(\tau) a^\dagger_{-\boldsymbol{k}}(\tau_0) + \beta_k^*(\tau) a_{\boldsymbol{k}}(\tau_0)$$

We rediscover the Bogoliubov transformation!

- The canonical equal-time commutation relation $\left[a_{\boldsymbol{k}}, a_{\boldsymbol{q}}^\dagger\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{q})$ leads to the relation between α_k and β_k:

$$\left[a_{\boldsymbol{k}}(\tau), a_{\boldsymbol{q}}^\dagger(\tau)\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{q}) \left[|\alpha_k(\tau)|^2 - |\beta_k(\tau)|^2\right]$$

$$\therefore \ |\alpha_k|^2 - |\beta_k|^2 = 1$$

- Motivated by this form, we can parametrize α_k and β_k i.t.o. hyperbolic fcts:

$$\alpha_k = e^{-i\theta_k} \cosh r_k$$
$$\beta_k = e^{i(\theta_k + 2\phi_k)} \sinh r_k$$

- Without losing generality, we take the time-dependent functions $r_k(\tau)$ (squeezing parameter), $\theta_k(\tau)$ (squeezing phase) and $\phi_k(\tau)$ (relative phase difference) all real

- Motivated by this form, we can parametrize α_k and β_k i.t.o. hyperbolic fcts:

$$\alpha_k = e^{-i\theta_k} \cosh r_k$$

$$\beta_k = e^{i(\theta_k + 2\phi_k)} \sinh r_k$$

- Without losing generality, we take the time-dependent functions $r_k(\tau)$ (squeezing parameter), $\theta_k(\tau)$ (squeezing phase) and $\phi_k(\tau)$ (relative phase difference) all real

- The equations for r_k, θ_k and ϕ_k can be derived from those for $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$:

$$r_k' = \frac{a'}{a} \cos(2\phi_k)$$

$$\theta_k' = k + \frac{a'}{a} \sin(2\phi_k) \tanh r_k$$

$$\phi_k' = -k - \frac{a'}{a} \sin(2\phi_k) \coth r_k$$

- We can derive the solutions for r_k, θ_k and ϕ_k in de Sitter limit:

$$r_k = \sinh^{-1}\left[\frac{1}{2(-k\tau)}\right]$$

$$\theta_k = k\tau - \tan^{-1}\left[\frac{1}{2(-k\tau)}\right]$$

$$\phi_k = \frac{3}{4}\pi + \frac{1}{2}\tan^{-1}\left[\frac{1}{2(-k\tau)}\right]$$

- We can derive the solutions for r_k, θ_k and ϕ_k in de Sitter limit:

$$r_k = \sinh^{-1}\left[\frac{1}{2(-k\tau)}\right]$$

We have chosen $r_k \gg 1$ as $-k\tau \to 0$

$$\theta_k = k\tau - \tan^{-1}\left[\frac{1}{2(-k\tau)}\right]$$

$$\phi_k = \frac{3}{4}\pi + \frac{1}{2}\tan^{-1}\left[\frac{1}{2(-k\tau)}\right]$$

We have chosen $\log(-1) = 3i\pi$

- We can derive the solutions for r_k, θ_k and ϕ_k in de Sitter limit:

$$r_k = \sinh^{-1}\left[\frac{1}{2(-k\tau)}\right]$$

We have chosen $r_k \gg 1$ as $-k\tau \to 0$

$$\theta_k = k\tau - \tan^{-1}\left[\frac{1}{2(-k\tau)}\right]$$

$$\phi_k = \frac{3}{4}\pi + \frac{1}{2}\tan^{-1}\left[\frac{1}{2(-k\tau)}\right]$$

We have chosen $\log(-1) = 3i\pi$

- Derivation of these solutions are a bit complicated and we will not cover it…

IV. Inflationary evolution of quantum state

1. Preliminary: Inverted harmonic oscillator
   a. Classical inverted harmonic oscillator
   b. Quantum inverted harmonic oscillator
2. Reformulation of quantum fluctuations
3. Hamiltonian operator revisited
   a. Hamiltonian in factorized form
   b. What does the Hamiltonian do on quantum state?

a. Hamiltonian in factorized form

- We can write the Hamiltonian "density" purely in terms of $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^{\dagger}$:

$$\mathcal{H} = \frac{1}{2}\left[\pi_{\boldsymbol{k}}\pi_{-\boldsymbol{k}} + k^2 v_{\boldsymbol{k}} v_{-\boldsymbol{k}} + \frac{a'}{a}\left(\pi_{\boldsymbol{k}} v_{-\boldsymbol{k}} + v_{\boldsymbol{k}}\pi_{-\boldsymbol{k}}\right)\right]$$

$$= \frac{1}{2}\left[k\left(a_{\boldsymbol{k}} a_{\boldsymbol{k}}^{\dagger} + a^{\dagger}_{-\boldsymbol{k}} a_{-\boldsymbol{k}}\right) + i\frac{a'}{a}\left(-a_{\boldsymbol{k}} a_{-\boldsymbol{k}} + a^{\dagger}_{-\boldsymbol{k}} a_{\boldsymbol{k}}^{\dagger}\right)\right]$$

a. Hamiltonian in factorized form

- We can write the Hamiltonian "density" purely in terms of $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$:

$$\mathcal{H} = \frac{1}{2}\left[\pi_{\boldsymbol{k}}\pi_{-\boldsymbol{k}} + k^2 v_{\boldsymbol{k}} v_{-\boldsymbol{k}} + \frac{a'}{a}\Big(\pi_{\boldsymbol{k}} v_{-\boldsymbol{k}} + v_{\boldsymbol{k}}\pi_{-\boldsymbol{k}}\Big)\right]$$
$$= \frac{1}{2}\left[k\Big(a_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger + a_{-\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}\Big) + i\frac{a'}{a}\Big(-a_{\boldsymbol{k}} a_{-\boldsymbol{k}} + a_{-\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}^\dagger\Big)\right]$$

- We may use the commutation relation to change the order of $a_{\boldsymbol{k}} a_{\boldsymbol{k}'}^\dagger$ because after all we are interested in the evolution of the (initial vacuum) state so that the operations by the part without a'/a keep the vacuum

$$\mathcal{H} = \frac{1}{2}\left\{k\Big[(2\pi)^3\delta^{(3)}(0) + a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}\Big] + i\frac{a'}{a}\Big(-a_{\boldsymbol{k}} a_{-\boldsymbol{k}} + a_{-\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}^\dagger\Big)\right\}$$

a. Hamiltonian in factorized form

- We can write the Hamiltonian "density" purely in terms of $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^{\dagger}$:

$$\mathcal{H} = \frac{1}{2}\left[\pi_{\boldsymbol{k}}\pi_{-\boldsymbol{k}} + k^2 v_{\boldsymbol{k}} v_{-\boldsymbol{k}} + \frac{a'}{a}\left(\pi_{\boldsymbol{k}} v_{-\boldsymbol{k}} + v_{\boldsymbol{k}}\pi_{-\boldsymbol{k}}\right)\right]$$
$$= \frac{1}{2}\left[k\left(a_{\boldsymbol{k}} a_{\boldsymbol{k}}^{\dagger} + a_{-\boldsymbol{k}}^{\dagger} a_{-\boldsymbol{k}}\right) + i\frac{a'}{a}\left(-a_{\boldsymbol{k}} a_{-\boldsymbol{k}} + a_{-\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}}^{\dagger}\right)\right]$$

- We may use the commutation relation to change the order of $a_{\boldsymbol{k}} a_{\boldsymbol{k}'}^{\dagger}$, because after all we are interested in the evolution of the (initial vacuum) state so that the operations by the part without a'/a keep the vacuum

$$\mathcal{H} = \frac{1}{2}\left\{k\left[(2\pi)^3\delta^{(3)}(0) + a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}} + a_{-\boldsymbol{k}}^{\dagger} a_{-\boldsymbol{k}}\right] + \underbrace{i\frac{a'}{a}\left(-a_{\boldsymbol{k}} a_{-\boldsymbol{k}} + a_{-\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}}^{\dagger}\right)}_{\text{Due to t-dep BG (gravity)}}\right\}$$

- For clarity, let us concentrate on a single k-mode
- This means the Hamiltonian we consider is $\mathcal{H}$, rather than $H = \int d^3k/(2\pi)^3 \mathcal{H}$
- Equivalently, we may "descretize" the momentum $\int d^3k/(2\pi)^3(\ldots) \to L^{-3}\sum_k(\ldots)$ where L^3 is the volume into which the mode of our interest permeates

- For clarity, let us concentrate on a single k-mode
- This means the Hamiltonian we consider is $\mathcal{H}$, rather than $H = \int d^3k/(2\pi)^3\mathcal{H}$
- Equivalently, we may "descretize" the momentum $\int d^3k/(2\pi)^3(\dots) \to L^{-3}\sum_k(\dots)$ where L^3 is the volume into which the mode of our interest permeates
- With a given volume L^3, we may now isolate the volume from the canonical creation and annihilation operators in such a way that (note that previously we have $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$ mass dimension 3/2)

$$a_{\boldsymbol{k}} \to L^{3/2}a_{\boldsymbol{k}} \equiv \hat{a}_{\boldsymbol{k}}$$

- For clarity, let us concentrate on a single k-mode
- This means the Hamiltonian we consider is $\mathcal{H}$, rather than $H = \int d^3k/(2\pi)^3 \mathcal{H}$
- Equivalently, we may "descretize" the momentum $\int d^3k/(2\pi)^3(\dots) \to L^{-3} \sum_k (\dots)$ where L^3 is the volume into which the mode of our interest permeates
- With a given volume L^3, we may now isolate the volume from the canonical creation and annihilation operators in such a way that (note that previously we have $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$ mass dimension 3/2)

$$a_{\boldsymbol{k}} \to L^{3/2} a_{\boldsymbol{k}} \equiv \hat{a}_{\boldsymbol{k}}$$

- Then the new dimensionless operators satisfy

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}}^\dagger\right] = 1$$

- Thus, the Hamiltonian density and the Hamiltonian operator can be written as

$$\widehat{\mathcal{H}} = L^3 \times \frac{1}{2}\left\{k\left[1 + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right] + i\frac{a'}{a}\left(-\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)\right\}$$

$$\widehat{H} = L^{-3}\sum_k \widehat{\mathcal{H}} = \sum_k \frac{1}{2}\left\{k\left[1 + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right] + i\frac{a'}{a}\left(-\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)\right\}$$

$$\equiv \sum_k \widehat{\mathcal{H}}_k$$

- Thus, the Hamiltonian density and the Hamiltonian operator can be written as

$$\widehat{\mathcal{H}} = L^3 \times \frac{1}{2}\left\{k\left[1 + \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}}\right] + i\frac{a'}{a}\left(-\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}^\dagger\right)\right\}$$

$$\widehat{H} = L^{-3}\sum_k \widehat{\mathcal{H}} = \sum_k \frac{1}{2}\left\{k\left[1 + \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}}\right] + i\frac{a'}{a}\left(-\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}^\dagger\right)\right\}$$

$$\equiv \sum_k \widehat{\mathcal{H}}_k$$

- $\widehat{\mathcal{H}}$: Hamiltonian "density" in the sense that it only accounts for a single k-mode, but it includes the whole volume over which that k-mode is effective
- $\widehat{H}$: Hamiltonian per unit volume, but the contributions of all k-mode is considered
- $\widehat{\mathcal{H}}_k$: Hamiltonian of a single k-mode per unit volume

- Thus, the Hamiltonian density and the Hamiltonian operator can be written as

$$\widehat{\mathcal{H}} = L^3 \times \frac{1}{2}\left\{k\left[1 + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right] + i\frac{a'}{a}\left(-\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)\right\}$$

$$\widehat{H} = L^{-3}\sum_k \widehat{\mathcal{H}} = \sum_k \boxed{\frac{1}{2}\left\{k\left[1 + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right] + i\frac{a'}{a}\left(-\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)\right\}}$$

$$\equiv \sum_k \widehat{\mathcal{H}}_k$$

- $\widehat{\mathcal{H}}$: Hamiltonian "density" in the sense that it only accounts for a single k-mode, but it includes the whole volume over which that k-mode is effective
- $\widehat{H}$: Hamiltonian per unit volume, but the contributions of all k-mode is considered
- $\widehat{\mathcal{H}}_k$: Hamiltonian of a single k-mode per unit volume

- After long calculations (I will not show you… basically solving the Schroedinger equation) we find the time evolution operator is factorized:

$$\widehat{U}_{\boldsymbol{k}} = \widehat{R}_{\boldsymbol{k}} \widehat{S}_{\boldsymbol{k}}$$

$$\widehat{R}_{\boldsymbol{k}} = \exp\left[-i\theta_k \left(1 + \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}} \right) \right]$$

$$\widehat{S}_{\boldsymbol{k}} = \exp\left[\frac{r_k}{2} \left(\hat{a}_{\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}} e^{-2i\phi_k} - \hat{a}^\dagger_{-\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k}} e^{2i\phi_k} \right) \right]$$

- After long calculations (I will not show you… basically solving the Schroedinger equation) we find the time evolution operator is factorized:

$$\widehat{U}_{\boldsymbol{k}} = \widehat{R}_{\boldsymbol{k}}\widehat{S}_{\boldsymbol{k}}$$

$$\widehat{R}_{\boldsymbol{k}} = \exp\left[-i\theta_k\left(1 + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)\right]$$

$$\widehat{S}_{\boldsymbol{k}} = \exp\left[\frac{r_k}{2}\left(\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}e^{-2i\phi_k} - \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}e^{2i\phi_k}\right)\right]$$

- The operator $\widehat{U}_{\boldsymbol{k}}$ is called "rotation operator" and $\hat{S}_{\boldsymbol{k}}$ "squeezing operator" for the reasons that will be obvious soon

b. What does the Hamiltonian to on quantum state?

- The action of $\hat{R}_{\boldsymbol{k}}$ on the vacuum state $|0\rangle$ is:

$$\widehat{R}_{\boldsymbol{k}}|0\rangle = \exp\left[-i\theta_k\left(1 + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)\right]|0\rangle$$

b. What does the Hamiltonian to on quantum state?

- The action of $\hat{R}_{\boldsymbol{k}}$ on the vacuum state $|0\rangle$ is:

$$\widehat{R}_{\boldsymbol{k}}|0\rangle = \exp\left[-i\theta_k\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)\right]|0\rangle \longleftarrow e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$$

b. What does the Hamiltonian to on quantum state?

- The action of $\widehat{R}_{\boldsymbol{k}}$ on the vacuum state $|0\rangle$ is:

$$\widehat{R}_{\boldsymbol{k}}|0\rangle = \exp\left[-i\theta_k\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)\right]|0\rangle \longleftarrow e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$$

$$= \left[1-i\theta_k\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)+\frac{1}{2}(-i\theta_k)^2\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)^2+\cdots\right]|0\rangle$$

b. What does the Hamiltonian to on quantum state?

- The action of $\hat{R}_{\boldsymbol{k}}$ on the vacuum state $|0\rangle$ is:

$$\widehat{R}_{\boldsymbol{k}}|0\rangle = \exp\left[-i\theta_k\left(1+\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}}+\hat{a}_{-\boldsymbol{k}}^\dagger\hat{a}_{-\boldsymbol{k}}\right)\right]|0\rangle$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$$

$$= \left[1 - i\theta_k\left(1+\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}}+\hat{a}_{-\boldsymbol{k}}^\dagger\hat{a}_{-\boldsymbol{k}}\right)+\frac{1}{2}(-i\theta_k)^2\left(1+\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}}+\hat{a}_{-\boldsymbol{k}}^\dagger\hat{a}_{-\boldsymbol{k}}\right)^2+\cdots\right]|0\rangle$$

$a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

b. What does the Hamiltonian to on quantum state?

- The action of $\hat{R}_{\boldsymbol{k}}$ on the vacuum state $|0\rangle$ is:

$$\widehat{R}_{\boldsymbol{k}}|0\rangle = \exp\left[-i\theta_k\left(1+\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}}+\hat{a}_{-\boldsymbol{k}}^\dagger\hat{a}_{-\boldsymbol{k}}\right)\right]|0\rangle \qquad \longleftarrow \qquad e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$$

$$= \left[1 - i\theta_k\left(1+\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}}+\hat{a}_{-\boldsymbol{k}}^\dagger\hat{a}_{-\boldsymbol{k}}\right) + \frac{1}{2}(-i\theta_k)^2\left(1+\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}}+\hat{a}_{-\boldsymbol{k}}^\dagger\hat{a}_{-\boldsymbol{k}}\right)^2 + \cdots\right]|0\rangle$$

$$= \left[1 - i\theta_k + \frac{1}{2}(-i\theta_k)^2 + \cdots\right]|0\rangle$$

$a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

b. What does the Hamiltonian to on quantum state?

- The action of $\hat{R}_{\boldsymbol{k}}$ on the vacuum state $|0\rangle$ is:

$$
\begin{aligned}
\widehat{R}_{\boldsymbol{k}}|0\rangle &= \exp\left[-i\theta_k\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)\right]|0\rangle \\
&= \left[1-i\theta_k\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)+\frac{1}{2}(-i\theta_k)^2\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)^2+\cdots\right]|0\rangle \\
&= \left[1-i\theta_k+\frac{1}{2}(-i\theta_k)^2+\cdots\right]|0\rangle \\
&= e^{-i\theta_k}|0\rangle
\end{aligned}
$$

$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$

$a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

b. What does the Hamiltonian to on quantum state?

- The action of $\hat{R}_{\boldsymbol{k}}$ on the vacuum state $|0\rangle$ is:

$$\hat{R}_{\boldsymbol{k}}|0\rangle = \exp\left[-i\theta_k\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)\right]|0\rangle \longleftarrow e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$$

$$= \left[1 - i\theta_k\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right) + \frac{1}{2}(-i\theta_k)^2\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)^2 + \cdots\right]|0\rangle$$

$$= \left[1 - i\theta_k + \frac{1}{2}(-i\theta_k)^2 + \cdots\right]|0\rangle$$

$a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

$$= e^{-i\theta_k}|0\rangle$$

- Thus, what the operator $\hat{R}_{\boldsymbol{k}}$ does on the vacuum state $|0\rangle$ is adding a phase angle $\theta_{\boldsymbol{k}}$, so the name "rotation operator" is given
- This is expected, as $\hat{R}_{\boldsymbol{k}}$ is coming from the part of the Hamiltonian that does not include any gravitational effects (i.e. scale factor) so it should do what is done in Minkowski space
- That is, vacuum remains vacuum under $\hat{R}_{\boldsymbol{k}}$

- Meanwhile, the squeezing operator $\hat{S}_k$ can be rewritten as (another long calculation!):

$$\widehat{S}_{\boldsymbol{k}}|0\rangle = \exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]\left[\frac{1}{\cosh(r_k/2)}\right]^{\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}+1}\exp\left[e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right]$$

- Meanwhile, the squeezing operator $\hat{S}_{\boldsymbol{k}}$ can be rewritten as (another long calculation!):

$$\widehat{S}_{\boldsymbol{k}}|0\rangle = \exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]\left[\frac{1}{\cosh(r_k/2)}\right]^{\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}+1}\exp\left[e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right]$$

- What happens when $\hat{S}_{\boldsymbol{k}}$ acts upon the vacuum state $|0\rangle$?

- Meanwhile, the squeezing operator $\hat{S}_{\boldsymbol{k}}$ can be rewritten as (another long calculation!):

$$\widehat{S}_{\boldsymbol{k}}|0\rangle = \exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]\left[\frac{1}{\cosh(r_k/2)}\right]^{\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}+1}\exp\left[e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right]$$

- What happens when $\hat{S}_{\boldsymbol{k}}$ acts upon the vacuum state $|0\rangle$?
  1. Expanding the exponential, the first part does nothing:

$$\exp\left[e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right]|0\rangle = \left[1 + e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \frac{1}{2}e^{-4i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\left(\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right)^2 + \cdots\right]|0\rangle$$

- Meanwhile, the squeezing operator $\hat{S}_k$ can be rewritten as (another long calculation!):

$$\widehat{S}_{\boldsymbol{k}}|0\rangle = \exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]\left[\frac{1}{\cosh(r_k/2)}\right]^{\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}+1}\exp\left[e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right]$$

- What happens when $\hat{S}_{\boldsymbol{k}}$ acts upon the vacuum state $|0\rangle$?
  1. Expanding the exponential, the first part does nothing:

$$\exp\left[e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right]|0\rangle = \left[1 + e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \frac{1}{2}e^{-4i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\left(\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right)^2 + \cdots\right]|0\rangle$$

$a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

- Meanwhile, the squeezing operator $\hat{S}_k$ can be rewritten as (another long calculation!):

$$\widehat{S}_{\boldsymbol{k}}|0\rangle = \exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}}^\dagger\right]\left[\frac{1}{\cosh(r_k/2)}\right]^{\hat{a}_{\boldsymbol{k}}^\dagger\hat{a}_{\boldsymbol{k}}+\hat{a}_{-\boldsymbol{k}}^\dagger\hat{a}_{-\boldsymbol{k}}+1}\exp\left[e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right]$$

- What happens when $\hat{S}_{\boldsymbol{k}}$ acts upon the vacuum state $|0\rangle$?
  1. Expanding the exponential, the first part does nothing:

$$\begin{aligned}\exp\left[e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right]|0\rangle &= \left[1+e^{-2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right.\\ &\quad\left.+\frac{1}{2}e^{-4i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\left(\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right)^2+\cdots\right]|0\rangle\\ &= |0\rangle\end{aligned}$$

$a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

2. Expanding the next part using $a^x = 1 + x \log a + x^2 (\log a)^2/2! + \cdots$ gives

$$\left[\frac{1}{\cosh(r_k/2)}\right]^{\hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}} + 1} |0\rangle = \left\{1 + \log\left[\frac{1}{\cosh(r_k/2)}\right]\left(1 + \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}}\right) \right.$$
$$\left. + \frac{1}{2!}\log^2\left[\frac{1}{\cosh(r_k/2)}\right]\left(1 + \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}}\right)^2 + \cdots \right\}|0\rangle$$

2. Expanding the next part using $a^x = 1 + x\log a + x^2(\log a)^2/2! + \cdots$ gives

$$\left[\frac{1}{\cosh(r_k/2)}\right]^{\hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}} + 1}|0\rangle = \left\{1 + \log\left[\frac{1}{\cosh(r_k/2)}\right]\left(1 + \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right) + \frac{1}{2!}\log^2\left[\frac{1}{\cosh(r_k/2)}\right]\left(1 + \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)^2 + \cdots\right\}|0\rangle$$

$a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

2. Expanding the next part using $a^x = 1 + x\log a + x^2(\log a)^2/2! + \cdots$ gives

$$\left[\frac{1}{\cosh(r_k/2)}\right]^{\hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}_{-\boldsymbol{k}}^\dagger \hat{a}_{-\boldsymbol{k}} + 1}|0\rangle = \left\{1 + \log\left[\frac{1}{\cosh(r_k/2)}\right]\left(1 + \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}_{-\boldsymbol{k}}^\dagger \hat{a}_{-\boldsymbol{k}}\right)\right.$$

$a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

$$\left. + \frac{1}{2!}\log^2\left[\frac{1}{\cosh(r_k/2)}\right]\left(1 + \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}_{-\boldsymbol{k}}^\dagger \hat{a}_{-\boldsymbol{k}}\right)^2 + \cdots\right\}|0\rangle$$

$$= |0\rangle \underbrace{\left\{1 + \log\left[\frac{1}{\cosh(r_k/2)}\right] + \frac{1}{2!}\log^2\left[\frac{1}{\cosh(r_k/2)}\right] + \cdots\right\}}_{=1/\cosh(r_k/2)}$$

2. Expanding the next part using $a^x = 1 + x\log a + x^2(\log a)^2/2! + \cdots$ gives

$$
\begin{aligned}
\left[\frac{1}{\cosh(r_k/2)}\right]^{\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}+1}|0\rangle &= \left\{1+\log\left[\frac{1}{\cosh(r_k/2)}\right]\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)\right.\\
&\left.\quad+\frac{1}{2!}\log^2\left[\frac{1}{\cosh(r_k/2)}\right]\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)^2+\cdots\right\}|0\rangle\\
&= |0\rangle\underbrace{\left\{1+\log\left[\frac{1}{\cosh(r_k/2)}\right]+\frac{1}{2!}\log^2\left[\frac{1}{\cosh(r_k/2)}\right]+\cdots\right\}}_{=1/\cosh(r_k/2)}\\
&= |0\rangle\frac{1}{\cosh(r_k/2)}
\end{aligned}
$$

$a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

3. Finally, applying the last part gives:

$$\exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]|0\rangle\frac{1}{\cosh(r_k/2)}$$
$$=\left\{1-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}+\frac{1}{2}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2+\cdots\right\}|0\rangle\frac{1}{\cosh(r_k/2)}$$

3. Finally, applying the last part gives:

$$\exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\left\{1-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}+\frac{1}{2}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2+\cdots\right\}|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\frac{1}{\cosh(r_k/2)}\left\{|0\rangle-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\ \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle\ +\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]^2\frac{1}{2}\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2|0\rangle+\cdots\right\}$$

3. Finally, applying the last part gives:

$$\exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\left\{1-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}+\frac{1}{2}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2+\cdots\right\}|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\frac{1}{\cosh(r_k/2)}\left\{|0\rangle-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\boxed{\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle}+\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]^2\frac{1}{2}\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2|0\rangle+\cdots\right\}$$

1 particle of momentum $\boldsymbol{k}$ and
another of $-\boldsymbol{k}$ are created

3. Finally, applying the last part gives:

$$
\begin{aligned}
&\exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]|0\rangle\frac{1}{\cosh(r_k/2)} \\
&=\left\{1-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}+\frac{1}{2}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2+\cdots\right\}|0\rangle\frac{1}{\cosh(r_k/2)} \\
&=\frac{1}{\cosh(r_k/2)}\left\{|0\rangle-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\boxed{\underbrace{\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle}_{\equiv|1,\boldsymbol{k};1,-\boldsymbol{k}\rangle}}+\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]^2\frac{1}{2}\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2|0\rangle+\cdots\right\}
\end{aligned}
$$

1 particle of momentum $\boldsymbol{k}$ and another of $-\boldsymbol{k}$ are created

3. Finally, applying the last part gives:

$$\exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\left\{1-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}+\frac{1}{2}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2+\cdots\right\}|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\frac{1}{\cosh(r_k/2)}\left\{|0\rangle-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\boxed{\underbrace{\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle}_{\equiv|1,\boldsymbol{k};1,-\boldsymbol{k}\rangle}}+\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]^2\boxed{\frac{1}{2}\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2|0\rangle}+\cdots\right\}$$

1 particle of momentum ***k*** and another of –***k*** are created

2 particles of momentum ***k*** and the other 2 of –***k*** are created

3. Finally, applying the last part gives:

$$\exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]|0\rangle\frac{1}{\cosh(r_k/2)}$$
$$=\left\{1-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}+\frac{1}{2}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2+\cdots\right\}|0\rangle\frac{1}{\cosh(r_k/2)}$$
$$=\frac{1}{\cosh(r_k/2)}\left\{|0\rangle-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\boxed{\underbrace{\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle}_{\equiv|1,\boldsymbol{k};1,-\boldsymbol{k}\rangle}}+\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]^2\boxed{\underbrace{\frac{1}{2}\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2|0\rangle}_{\equiv|2,\boldsymbol{k};2,-\boldsymbol{k}\rangle}}+\cdots\right\}$$

1 particle of momentum $\boldsymbol{k}$ and another of $-\boldsymbol{k}$ are created

2 particles of momentum $\boldsymbol{k}$ and the other 2 of $-\boldsymbol{k}$ are created

3. Finally, applying the last part gives:

$$\exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\left\{1-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}+\frac{1}{2}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2+\cdots\right\}|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\frac{1}{\cosh(r_k/2)}\left\{|0\rangle-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\boxed{\underbrace{\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle}_{\equiv|1,\boldsymbol{k};1,-\boldsymbol{k}\rangle}}+\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]^2\boxed{\underbrace{\frac{1}{2}\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2|0\rangle}_{\equiv|2,\boldsymbol{k};2,-\boldsymbol{k}\rangle}}+\cdots\right\}$$

1 particle of momentum $\boldsymbol{k}$ and another of $-\boldsymbol{k}$ are created

2 particles of momentum $\boldsymbol{k}$ and the other 2 of $-\boldsymbol{k}$ are created

$$=\frac{1}{\cosh(r_k/2)}\sum_{n=0}^{\infty}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]^n|n,\boldsymbol{k};n,-\boldsymbol{k}\rangle$$

3. Finally, applying the last part gives:

$$\exp\left[-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\left\{1-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}+\frac{1}{2}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2+\cdots\right\}|0\rangle\frac{1}{\cosh(r_k/2)}$$

$$=\frac{1}{\cosh(r_k/2)}\left\{|0\rangle-e^{2i\phi_k}\tanh\left(\frac{r_k}{2}\right)\boxed{\underbrace{\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle}_{\equiv|1,\boldsymbol{k};1,-\boldsymbol{k}\rangle}}+\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]^2\boxed{\underbrace{\frac{1}{2}\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^2|0\rangle}_{\equiv|2,\boldsymbol{k};2,-\boldsymbol{k}\rangle}}+\cdots\right\}$$

1 particle of momentum ***k*** and another of –***k*** are created

2 particles of momentum ***k*** and the other 2 of –***k*** are created

$$=\frac{1}{\cosh(r_k/2)}\sum_{n=0}^{\infty}\left[-e^{2i\phi_k}\tanh^2\left(\frac{r_k}{2}\right)\right]^n\underbrace{|n,\boldsymbol{k};n,-\boldsymbol{k}\rangle}_{\equiv\frac{1}{n!}\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^n}$$

- Here, $|n, \boldsymbol{k}; n, -\boldsymbol{k}\rangle$ are the state with the number of particles with the momenta $\boldsymbol{k}$ and $-\boldsymbol{k}$ being both n
- That is, the squeezing operator $\hat{S}_{\boldsymbol{k}}$ is responsible for the creation of equal numbers of particles with momenta $\boldsymbol{k}$ and $-\boldsymbol{k}$
- The numbers of particles are very very large! (sum runs up to ∞)
- **Cosmological vacuum fluctuations are amplified**, or equivalently, as time goes on quantum fluctuations are created!
- This is another equivalent way of formulating generation of perturbations during inflation