III. Quantum solutions of perturbations

1. Quantization of scalar perturbation
   a. Expansion in terms of operators
   b. Choice of vacuum states
   c. Particle creation from vacuum
2. Solutions of scalar perturbation
   a. Asymptotic solutions
   b. General solutions
3. Tensor perturbation
4. Power spectrum
5. Simple example: Quadratic potential

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1. Quantization of scalar perturbation

- Our starting point is the quadratic action for scalar perturbation:

$$S_2^{(s)} = \int d^4x a^3 \epsilon m_{\rm Pl}^2 \left[\dot{\mathcal{R}}^2 - \frac{(\nabla \mathcal{R})^2}{a^2} \right]$$

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- We introduce a new parameter z and rescale the perturbation $\mathcal{R}$:

$$z \equiv \frac{a\phi_0'}{\mathcal{H}} \quad \text{and} \quad u \equiv z\mathcal{R} = a\left(\delta\phi - \frac{\phi_0'}{\mathcal{H}}\varphi\right)$$ "Sasaki-Mukhanov variable"

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Reminder: Throughout the lectures, unless explicit, we set $c = \hbar = 1$ (natural unit)

a. Expansion in terms of operators

- This action is of the same form as that of a free field with time-varying mass: Considering the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and the effective mass $m_{\mathrm{eff}}^2 \equiv -z''/z$, we can rewrite the action

$$S_2 = \int d^4x \left(-\frac{1}{2}\eta^{\mu\nu}\partial\mu u \partial_\nu u - \frac{1}{2}m_{\mathrm{eff}}^2 u^2 \right)$$

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- Thus, we can apply the standard wisdom for the quantization of a free field, or a harmonic oscillator:
  1. Promote the canonical conjugate pair to operators, and
  2. Impose canonical commutation relations

1) Promoting conjugate pair to operators

- First find the conjugate momentum of u:

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- First find the conjugate momentum of u:

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- Promote both u and Π_u to operators: $\hat{u}$ and $\widehat{\Pi}_u$
- Being a canonical field, decompose the Fourier mode of $\hat{u}$ in terms of the creation and annihilation operators:

$$\widehat{u}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \widehat{u}(\tau, \boldsymbol{k}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left[a_{\boldsymbol{k}} u_k(\tau) + a^\dagger_{-\boldsymbol{k}} u_k^*(\tau) \right]$$

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2) Imposing canonical commutation relations

- The creation and annihilation operators satisfy the canonical commutation relations:

$$\left[a_{\boldsymbol{k}}, a_{\boldsymbol{q}}^{\dagger}\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{q})\,, \ \text{otherwise zero}$$

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- With these, the equal-time commutation relation $\left[\hat{u}(\tau, \boldsymbol{x}), \widehat{\Pi}_u(\tau, \boldsymbol{y})\right]$ gives:

$$\begin{aligned}\left[\widehat{u}(\tau, \boldsymbol{x}), \widehat{\Pi}_u(\tau, \boldsymbol{y})\right] &= \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\boldsymbol{q}\cdot\boldsymbol{y}} \Big\{ \left[a_{\boldsymbol{k}}, a_{-\boldsymbol{q}}^{\dagger}\right] u_k u_q^{*\prime} - \left[a_{\boldsymbol{q}}, a_{-\boldsymbol{k}}^{\dagger}\right] u_q' u_k^* \\ &\qquad + \left[a_{\boldsymbol{k}}, a_{\boldsymbol{q}}\right] u_k u_q' + \left[a_{-\boldsymbol{k}}^{\dagger}, a_{-\boldsymbol{q}}^{\dagger}\right] u_k^* u_q^{*\prime} \Big\} \\ &= \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \left(u_k u_k^{*\prime} - u_k' u_k^* \right)\end{aligned}$$

- Being a canonical conjugate pair, we already know

$$\left[\widehat{u}(\tau, \boldsymbol{x}), \widehat{\Pi}_u(\tau, \boldsymbol{y})\right] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$$

- This imposes the normalization (or "Wronskian") of the "mode function" $u_k(\tau)$:

$$u_k u_k^{*\prime} - u_k' u_k^* = i$$

- Indeed, once this normalization is imposed, the equal-time canonical commutation relation $\left[\hat{u}(\tau, \boldsymbol{x}), \widehat{\Pi}_u(\tau, \boldsymbol{y})\right] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$ is satisfied

b. Choice of vacuum states

- We worked out the standard commutation relations for operators, but still we need to determine the mode function $u(\tau, \boldsymbol{k})$
- Determining $u(\tau, \boldsymbol{k})$ amounts to fix the vacuum state $|0\rangle_{\mathrm{vac}}$ defined by

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- In the Minkowski space, the vacuum state is such that the Hamiltonian operator of the system is minimized and is uniquely defined
- However, in FRW universe the background is time-evolving, it is not clear how and when to define the vacuum
- In fact we have only 1 situation where time dependence is not relevant – $k \gg aH$, where frequency becomes time-independent

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$$\frac{z''}{z} = 2a^2H^2\left(1 - \frac{\epsilon}{2} + \frac{3}{4}\eta - \frac{\epsilon\eta}{4} + \frac{\eta^2}{8} + \frac{\dot{\eta}}{4H}\right)$$

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- Thus, for $k \gg aH$, $\omega_k^2(\tau) \approx k^2$, thus we can straightly apply the standard procedure to find the mode function solution
- Note that…
  1. $k \gg aH$ means $H^{-1} \gg ak^{-1} \sim \lambda$, so the (physical) wavelength of interest is much smaller than the (physical) Hubble radius, “sub-horizon”
  2. $k \ll aH$ means the opposite, “super-horizon”

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- The corresponding Hamiltonian operator (promoting u and Π_u to operators)

$$\widehat{\mathfrak{H}} = \int d^3x \frac{1}{2}\left[\widehat{\Pi}_u^2 + (\nabla \widehat{u})^2\right]$$

$$= \int \frac{d^3k}{(2\pi)^3}\left\{a_{\boldsymbol{k}} a_{-\boldsymbol{k}}\left(\widehat{u}_k'^{\,2} + k^2 \widehat{u}_k^2\right) + c.c. + \left[2a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + (2\pi)^3 \delta^{(3)}(0)\right]\left(\left|\widehat{u}_k'\right|^2 + k^2\left|\widehat{u}_k\right|^2\right)\right\}$$

- Evaluating the expectation value of $\widehat{\mathfrak{H}}$ w.r.t. vacuum $|0\rangle_0$ gives

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{}_0\langle 0|\widehat{\mathfrak{H}}|0\rangle_0 = \frac{1}{2}\int d^3k\delta^{(3)}(0)\left(|\widehat{u}_k'|^2 + k^2|\widehat{u}_k|^2\right)
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- Thus we expect the mode function u_k takes the plane wave form:

$$ u_k = \psi_k e^{i\theta_k} $$

- Without losing generality, we take both ψ_k and θ_k real, and we assume ψ_k constant, as the maximum amplitude in Minkowski space is preserved

- Normalization $u_k {u_k^*}' - u_k' u_k^* = i$ gives

$$\psi_k e^{i\theta_k} \cdot \psi_k e^{-i\theta_k} (-i\theta_k') - \psi_k e^{i\theta_k} i\theta_k' \cdot \psi_k e^{-i\theta_k} = -2i\psi_k^2\theta_k' = i$$

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- Then the expression in the expectation value of the Hamiltonian becomes

$$\left|\widehat{u}_k'\right|^2 + k^2\left|\widehat{u}_k\right|^2 = \psi_k^2{\theta_k'}^2 + k^2\psi_k^2 = \frac{1}{4\psi_k^2} + k^2\psi_k^2$$

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$$2k^2\psi_k - \frac{1}{2\psi_k^3} = 0 \qquad \therefore\ \psi_k^4 = \frac{1}{4k^2}$$

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$$\therefore\ \psi_k = \frac{1}{\sqrt{2k}},\ \theta_k = -k\tau$$

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}$$

Massless scalar field solution!
$(E_k = \omega_k = k)$

- With the mode function solution a la massless scalar field, $\mathfrak{H}$ is written as

$$\widehat{\mathfrak{H}} = \int \frac{d^3k}{(2\pi)^3} \left[a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}} + \frac{1}{2}(2\pi)^3 \delta^{(3)}(0) \right] \omega_k$$

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Precisely the Hamiltonian of a harmonic oscillator!

c. Particle creation from vacuum

- This mode function solution, or the vacuum state, does not remain as the solution (or the vacuum) all the time, because the frequency is time dependent – $\omega_k^2 = \omega_k^2(\tau)$, so is the Hamiltonian $\hat{\mathfrak{H}} = \hat{\mathfrak{H}}(\tau)$
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- Thus, the mode function (or vacuum) that once minimizes the Hamiltonian is no longer does so at some later time
- Let $\tau_1 > \tau_0$, and expand $\hat{u}(\tau, \boldsymbol{k})$ i.t.o. new operators and mode function:

$$\widehat{u}(\tau, \boldsymbol{k}) = b_{\boldsymbol{k}} v_k(\tau) + b^{\dagger}_{-\boldsymbol{k}} v_k^*(\tau) \quad \text{with} \quad b_{\boldsymbol{k}}|0\rangle_1 = 0$$

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- In general, the new mode function v_k is related to the old mode function u_k via a linear transformation (and new operators $b_{\boldsymbol{k}}$ and $b^\dagger_{-\boldsymbol{k}}$ as well):

$$v_k = \alpha_k u_k + \beta_k u_k^*$$

"Bogoliubov transformation"

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- The notion of vacuum is dependent on time, and there is **no unique vacuum state** throughout all the time!
- This has a profound effect: Consider the expectation value of the number operator $N_{\boldsymbol{k}}^{(b)} \equiv b_{\boldsymbol{k}}^\dagger b_{\boldsymbol{k}}$ w.r.t. the vacuum at τ_0, $|0\rangle_0$:

$${}_0\langle 0|N_k^{(b)}|0\rangle_0 = {}_0\langle 0|\left(\alpha_k a_{\boldsymbol{k}}^\dagger - \beta_k a_{-\boldsymbol{k}}\right)\left(\alpha_k^* a_{\boldsymbol{k}} - \beta_k^* a_{-\boldsymbol{k}}^\dagger\right)|0\rangle_0 = (2\pi)^3|\beta_k|^2\delta^{(3)}(0)$$

- Even if we have started with a vacuum state $|0\rangle_0$ at τ_0 (that contains no particle) at a later time $\tau_1 > \tau_0$ we find that $|0\rangle_0$ contains a non-vanishing number of b-particles, or the perturbations of b-field

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- That is, **we have something out of nothing**!
- This is how quantum fluctuations are generated in gravitational background ("gravitational particle production")

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a. Asymptotic solutions

- We can think of 2 limiting cases for the frequency: Either k^2 is dominant over $z''/z \sim a^2H^2$, or the other way round

$$u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k \longrightarrow \begin{cases} u_k'' + k^2 u_k = 0 & \text{for } k \gg aH \\ u_k'' - \dfrac{z''}{z} u_k = 0 & \text{for } k \ll aH \end{cases}$$

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- Previously, we have seen k versus aH in terms of the physical length scales
- On equal footing, we can ask: Is the typical (comoving) length scale for a mode $1/k$ much smaller or larger than the comoving Hubble horizon $1/(aH)$? Super- ($k \ll aH$) or sub- ($k \gg aH$) horizon limit

a. Asymptotic solutions

- We can think of 2 limiting cases for the frequency: Either k^2 is dominant over $z''/z \sim a^2H^2$, or the other way round

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k \longrightarrow \begin{cases} u_k'' + k^2 u_k = 0 & \text{for } k \gg aH \\ u_k'' - \dfrac{z''}{z} u_k = 0 & \text{for } k \ll aH \end{cases}$$

- Previously, we have seen k versus aH in terms of the physical length scales
- On equal footing, we can ask: Is the typical (comoving) length scale for a mode $1/k$ much smaller or larger than the comoving Hubble horizon $1/(aH)$? Super- ($k \ll aH$) or sub- ($k \gg aH$) horizon limit
- For sub-horizon modes, the solution is a plane wave (we already saw it)
- For super-horizon modes:

$$\frac{u_k''}{u_k} = \frac{z''}{z} \quad \rightarrow \quad u_k \propto z$$

- What does it mean that on super-horizon scales $u_k \propto z$?

$$\mathcal{R}_k = \frac{u_k}{z} = \mathrm{constant}$$

- That is, $\mathcal{R}_k$ (the Fourier mode of $\mathcal{R}$) is **conserved**
- This is another reason why we consider $\mathcal{R}$: It is conserved on very large scales during inflation, until it reenters the horizon after inflation
- Other perturbations, e.g. $\delta\phi$, keep evolving on super-horizon

b. General solution

- We have seen the solutions of perturbation equations in the 2 limiting cases: Super- and sub-horizon limits

$$u_k \longrightarrow \begin{cases} \dfrac{e^{-ik\tau}}{\sqrt{2k}} & (k \gg aH) \\ z & (k \ll aH) \end{cases}$$

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- Thus we want a general solution that satisfies…
  1. (Of course) the equation of motion
  2. Normalization condition $u_k {u_k^*}' - u_k' u_k^* = i$
  3. The above boundary conditions at early ($-k\tau \gg 1$) and late ($-k\tau \to 0$) times

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$$\equiv \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)$$

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- If we solve this equation assuming that ν is constant (a good approximation!) we find the solution in terms of the Bessel functions:

$$u(\tau) = c_1\sqrt{-\tau}J_\nu(-k\tau) + c_2\sqrt{-\tau}Y_\nu(-k\tau)$$

- But for later convenience (it will become clear very soon) instead of the Bessel functions we use the Hankel functions:

$$u(\tau) = \sqrt{-\tau}\left[c_1(k)H_\nu^{(1)}(-k\tau) + c_2(k)H_\nu^{(2)}(-k\tau)\right]$$

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- In the sub-horizon limit $k \gg aH$, from the asymptotic form of the first kind of the Hankel function in the limit $z \equiv -k\tau \approx k/(aH) \gg 1$:

$$H_\nu^{(1)}(z) \xrightarrow[z\gg1]{} \sqrt{\frac{2}{\pi z}}e^{i(z-\pi\nu/2-\pi/4)}$$

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- By matching the sub-horizon solution $u = e^{-ik\tau}/\sqrt{2k}$, we find c_1 and c_2 as:

$$c_1 = \frac{\sqrt{\pi}}{2}e^{i(\nu+1/2)\pi/2}$$

$$c_2 = 0$$

- A particularly important and simple case is $\nu = 3/2$
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- Then we exactly have

$$\frac{z''}{z} = 2a^2H^2 \quad \text{and} \quad \tau = \frac{-1}{aH}$$

- The corresponding first kind Hankel function of order 3/2 is

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- We can find the mode function solution in perfect de Sitter space as

$$u_k(\tau) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau}$$

III. Quantum solutions of perturbations

1. Quantization of scalar perturbation
   a. Expansion in terms of operators
   b. Choice of vacuum states
   c. Particle creation from vacuum
2. Solutions of scalar perturbation
   a. Asymptotic solutions
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3. Tensor perturbation
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5. Simple example: Quadratic potential

3. Tensor perturbation

- Before we closely study the quantization and the mode function solution of tensor perturbation, let us first consider the asymptotic solution in the large-scale limit:

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\Delta}{a^2}h_{ij} = 0$$

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$$\frac{\Delta}{a^2} \sim -\frac{k^2}{a^2}$$

- Thus, on very large scales the amplitude of the tensor perturbation remains constant
- Having seen the constant large-scale solution, now let us study more closely the quantization and general solution of tensor perturbation

- To incorporate the 2 physical d.o.f. of tensor perturbation, we introduce the polarization tensor $e_{ij}^{(s)}(\boldsymbol{k})$, with s denoting 2 different polarization states:

$$h_{ij}(\tau, \boldsymbol{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{s=1}^{2} h_{(s)}(\tau, \boldsymbol{k}) e_{ij}^{(s)}(\boldsymbol{k})$$

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- The properties of the polarization tensor are

$$\begin{aligned}
e_{ij}^{(s)} &= e_{ji}^{(s)} && \text{(symmetry bet spatial indices)} \\
\delta^{ij} e_{ij}^{(s)} &= 0 && \text{(traceless)} \\
k^i e_{ij}^{(s)} &= 0 && \text{(transverse)} \\
e_{ij}^{(s)} e_{ij}^{(s')^*} &= 2\delta_{ss'} && \text{(2 indep pol)} \\
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$$\frac{k_l}{k} \epsilon^{ilk} e_{jk}^{(s)} = -ise^{i\,(s)}_{\;j}$$ if we choose circular polarizations

- The the tensor quadratic action (in the Fourier space) becomes

$$S_2^{(t)} = \int d\tau \frac{a^2 m_{\rm Pl}^2}{4} \int \frac{d^3k}{(2\pi)^3} \sum_s \left({h'_{(s)}}^2 - k^2 h_{(s)}^2 \right)$$

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1. 2 copies of identical action for each polarization state
2. The effective mass is not z''/z but a''/a – much simpler than that of curvature pert!

- We proceed the quantization of perturbation as before, that is:
  1. Find the conj momentum $\Pi_s \equiv \delta\mathcal{L}/\delta v_s'$ and promote v_s and Π_s to ops:

$$\Pi_s = \frac{\delta\mathcal{L}}{\delta v_s'} = v_s'$$

$$\widehat{v}_s(\tau, \boldsymbol{k}) = a_{\boldsymbol{k}}^{(s)} v_s(k, \tau) + a_{-\boldsymbol{k}}^{(s)\,\dagger} v_s^*(k, \tau)$$

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Different polarization states are independent

  3. Minimize the Hamiltonian operator at early times ($k \gg aH$):

$$v_s(k,\tau) \xrightarrow[k \gg aH]{} \frac{e^{-ik\tau}}{\sqrt{2k}}$$

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Exact!

$$\mu^2 = \frac{9}{4} + 3\epsilon + \cdots$$

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Exact!

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- Again, matching the early-time solution $v = e^{-ik\tau}/\sqrt{2k}$ gives Hankel fct sol:

$$v(\tau, k) = \frac{\sqrt{\pi}}{2} e^{i(\mu+1/2)\pi/2}\sqrt{-\tau}H_\mu^{(1)}(-k\tau)$$

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- Perturbations are the fluctuations around a certain value, ″mean″, and their values (around the mean) follow a certain probability distribution
- Thus, it is somewhat pointless to talk about their definite values at individual points
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- Thus, the first non-trivial statistical information comes from 2-point correlation function, or its Fourier transform, power spectrum

$$\xi(r \equiv |\boldsymbol{x} - \boldsymbol{y}|) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} P(k)$$

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Because of homogeneity & isotropy

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- We are working in the Heisenberg picture, so the quantum operator $\hat{u}$ is time-evolving, i.e. $\hat{u} = \hat{u}(\tau, k)$
- Thus, the ensemble average can be taken with respect to some state of the operator – in our case, the initial state – the vacuum!
- This is essentially not too different from the standard QM problem of computing the VEV of an operator at some time $t > 0$, given an initial state: $\langle X(t) \rangle = \langle \psi(t)|X|\psi(t)\rangle = \left\langle \psi(0) \middle| U^\dagger(t) X U(t) \middle| \psi(0) \right\rangle = \langle \psi(0)|X(t)|\psi(0)\rangle$ with $|\psi(0)\rangle = |0\rangle$

- From the metric $a^2\left[(1+\mathcal{R})\delta_{ij}+h_{ij}\right]$, both $\mathcal{R}$ and h_{ij} are dimensionless, thus their Fourier components have a mass dimension -3
- Thus, $\langle\mathcal{R}\mathcal{R}\rangle$ overall has dimension -6, and the delta function has dimension -3
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- Sometimes $\mathcal{P}_{\mathcal{R}}(k)$ is denoted by $\Delta^2_{\mathcal{R}}(k)$
- In terms of the canonical field $\hat{u}$, we can write

$$\mathcal{R}(\tau,\boldsymbol{k}) = \frac{\widehat{u}(\tau,\boldsymbol{k})}{z} = \frac{1}{z}\Big[a_{\boldsymbol{k}}u_k(\tau) + a^\dagger_{-\boldsymbol{k}}u^*_k(\tau)\Big]$$

- Thus (suppressing the time dependence) the power spectrum can be written in terms of the mode function $u_k(\tau)$ as:

$$\begin{aligned}\left\langle \mathcal{R}(\boldsymbol{k})\mathcal{R}(\boldsymbol{q})\right\rangle &= \left\langle \frac{1}{z}\left(a_{\boldsymbol{k}}u_k + a^{\dagger}_{-\boldsymbol{k}}u^*_k\right)\frac{1}{z}\left(a_{\boldsymbol{q}}u_q + a^{\dagger}_{-\boldsymbol{q}}u^*_q\right)\right\rangle \\ &= \frac{1}{z^2}\left\langle a_{\boldsymbol{k}}a_{\boldsymbol{q}}u_k u_q + a_{\boldsymbol{k}}a^{\dagger}_{-\boldsymbol{q}}u_k u^*_q + a^{\dagger}_{-\boldsymbol{k}}a_{\boldsymbol{q}}u^*_k u_q + a^{\dagger}_{-\boldsymbol{k}}a^{\dagger}_{-\boldsymbol{q}}u^*_k u^*_q\right\rangle\end{aligned}$$

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- For general case ($\nu \neq 3/2$) the Hankel function approaches

$$H_\nu^{(1)}(z) \underset{z \ll 1}{\longrightarrow} \sqrt{\frac{2}{\pi}} e^{-i\pi/2} 2^{\nu - 3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} z^{-\nu}$$

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- Thus, the power spectrum is written as

$$\nu = \frac{3}{2} + \epsilon + \frac{\eta}{2} + \cdots$$

$$\begin{aligned}
\mathcal{P}_\mathcal{R}(k) &= \lim_{-k\tau\to 0} \frac{k^3}{2\pi^2} \left| \frac{u_k}{z} \right|^2 \\
&= \lim_{-k\tau\to 0} 2^{2\nu-3} \left[\frac{\Gamma(\nu)}{\Gamma(3/2)} \right]^2 (1+\epsilon)^{1-2\nu} \left(\frac{H}{2\pi} \right)^2 \left(\frac{H}{\dot{\phi}_0} \right)^2 \left(\frac{k}{aH} \right)^{3-2\nu} \\
&= \lim_{-k\tau\to 0} \left[1 + 2(\alpha - 1)\epsilon + \alpha\eta \right] \left(\frac{H}{2\pi} \right)^2 \left(\frac{H}{\dot{\phi}_0} \right)^2 \left(\frac{k}{aH} \right)^{-2\epsilon-\eta}
\end{aligned}$$

$$\alpha \equiv 2 - \log 2 - \gamma \approx 0.729637$$

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$$\mathcal{P}_{\mathcal{R}}(k) = \Big[1 + 2(\alpha - 1)\epsilon + \alpha\eta\Big]\left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}_0}\right)^2 \Bigg|_{k=aH}$$

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- Another important property is how it scales with different k, viz. if it has a larger amplitude on larger or smaller length scales
- As a simple ansatz, we take a power-law form

$$\mathcal{P}_{\mathcal{R}} = A_{\mathcal{R}}\left(\frac{k}{k_\star}\right)^{n_{\mathcal{R}}-1}$$

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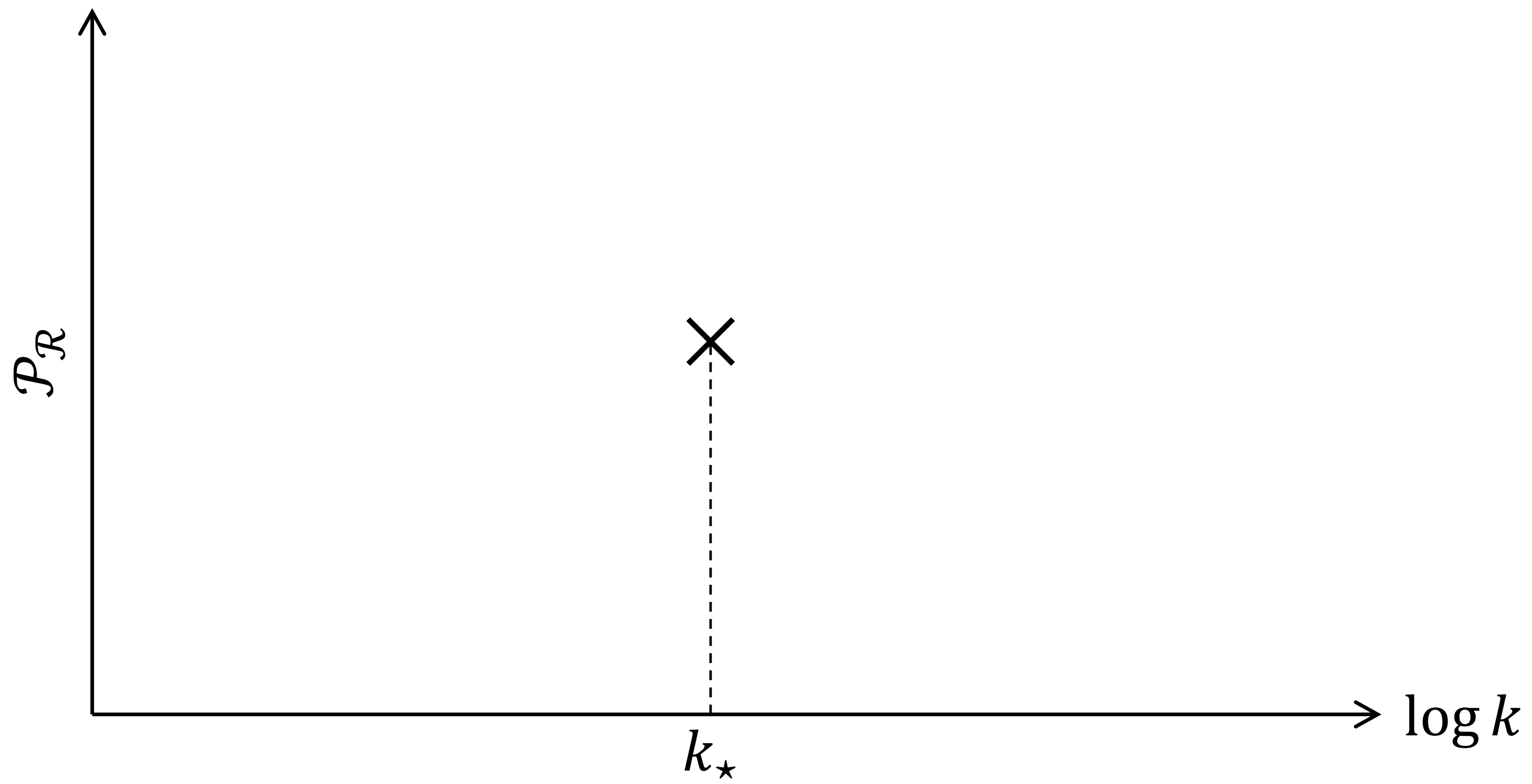

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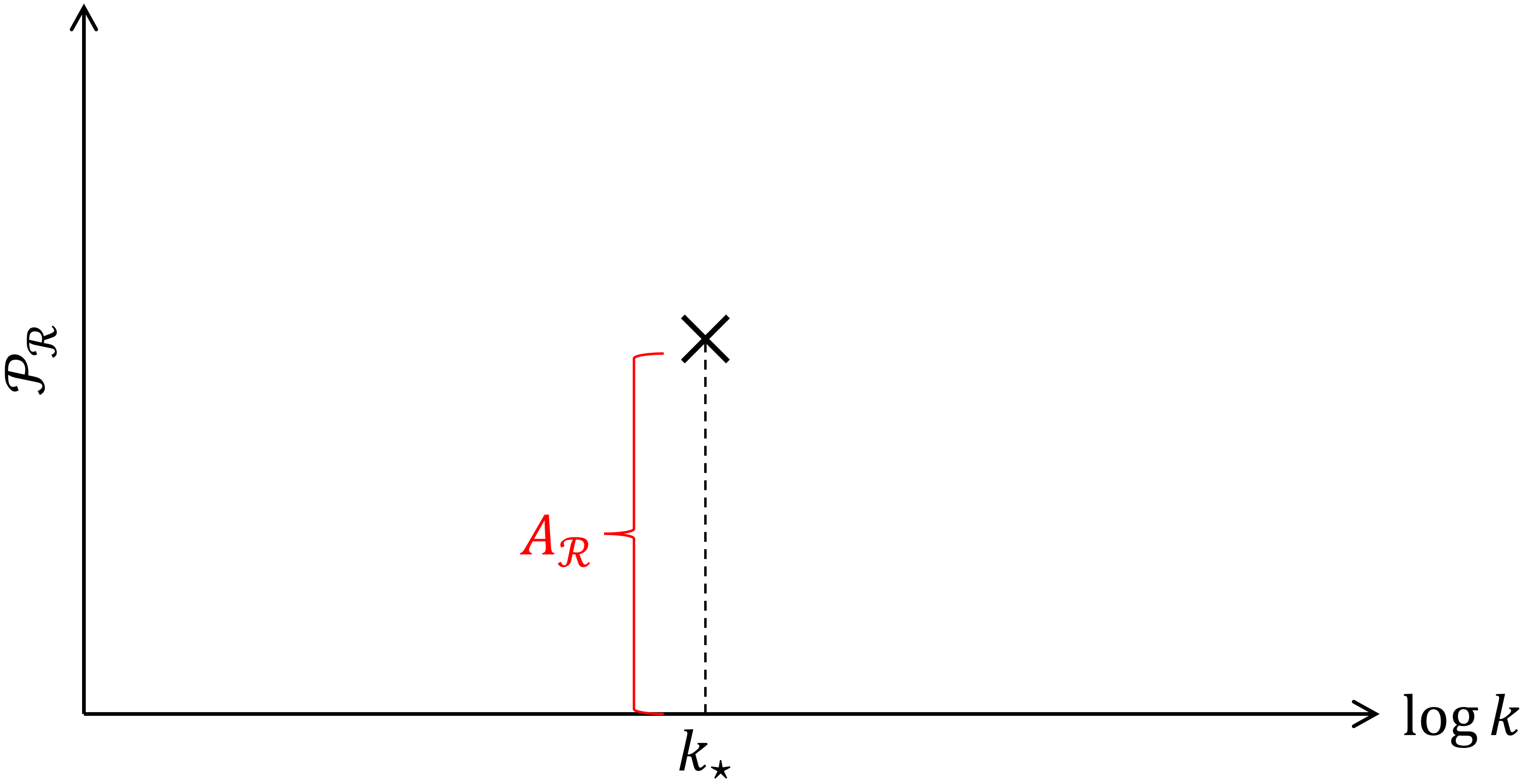

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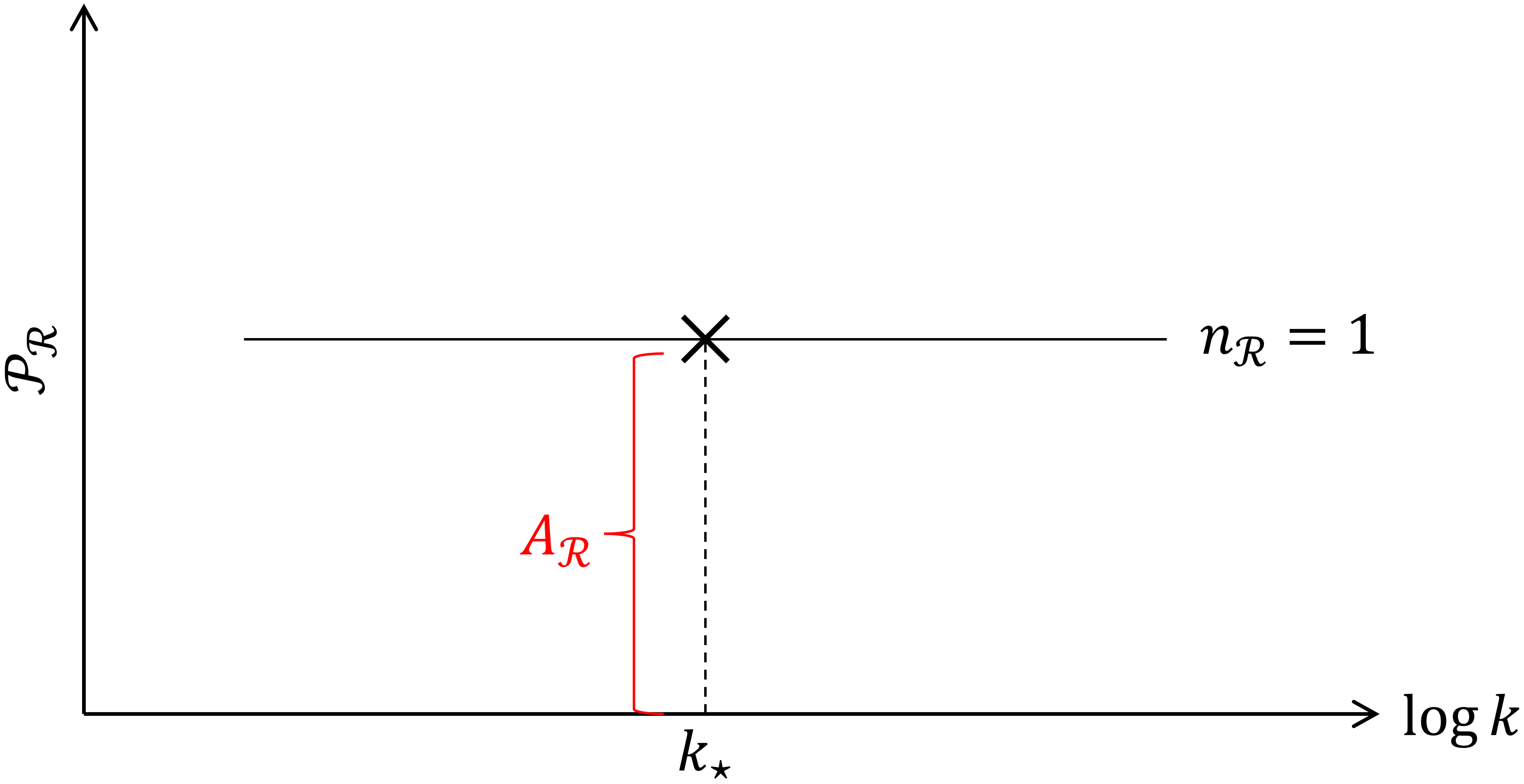

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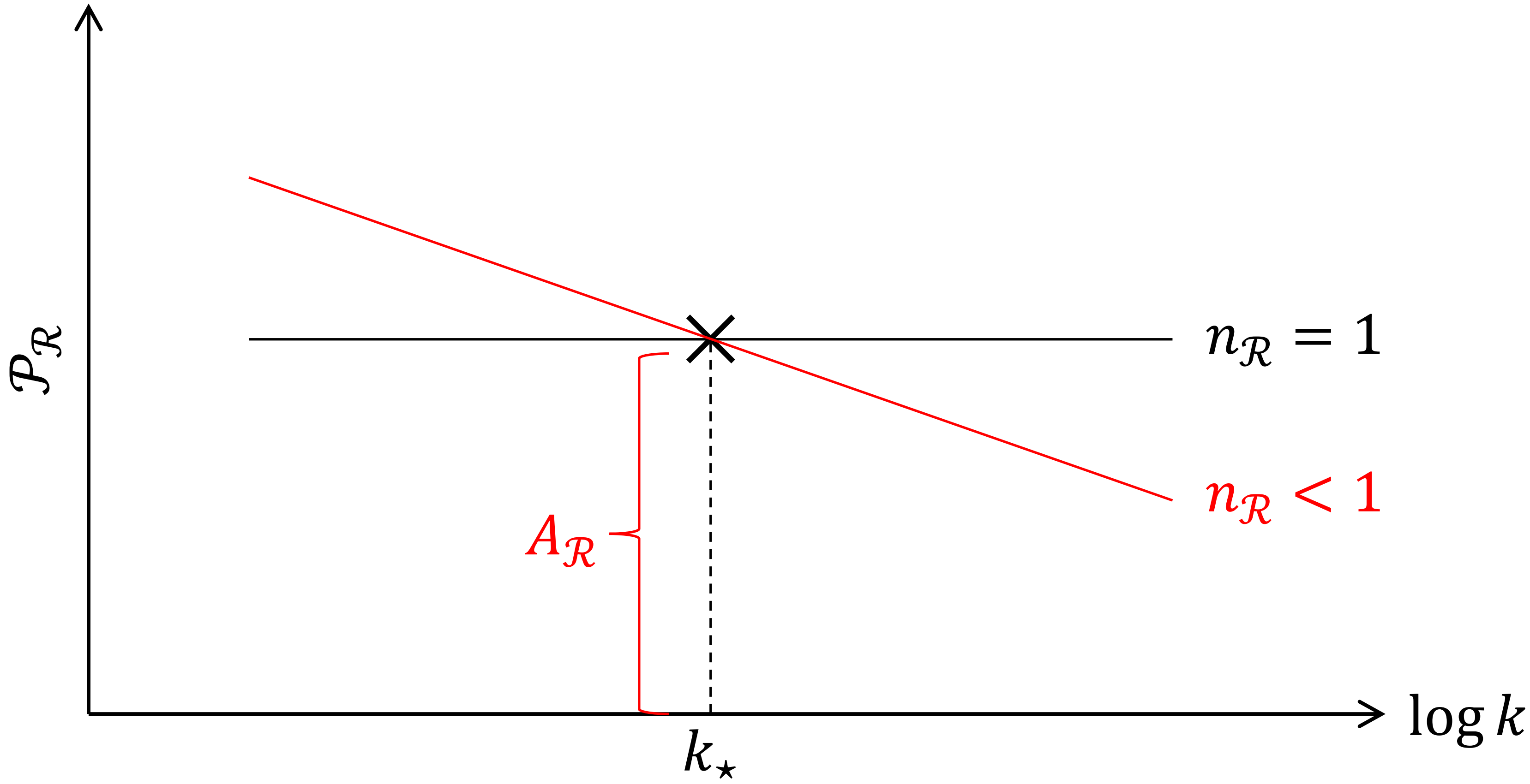

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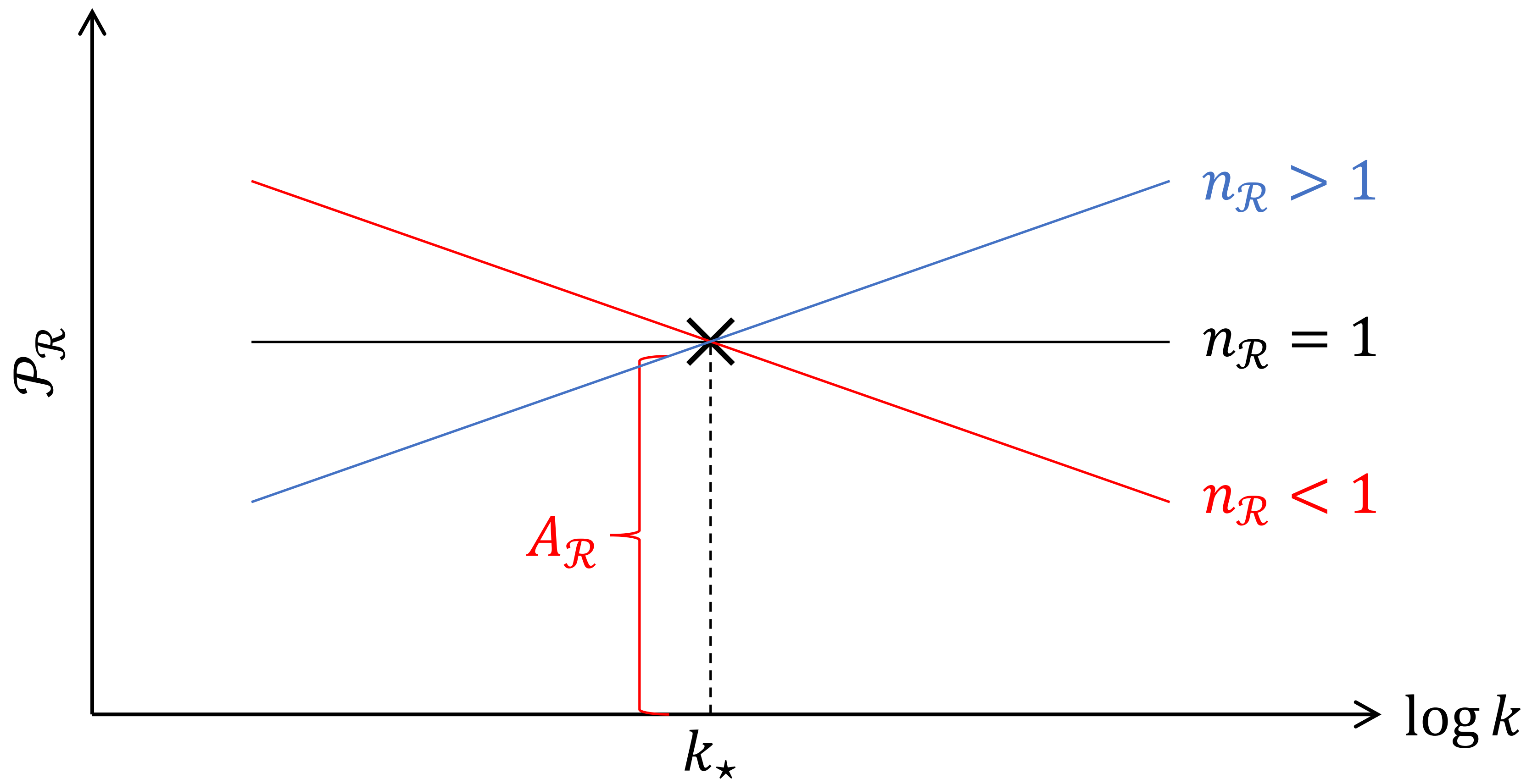

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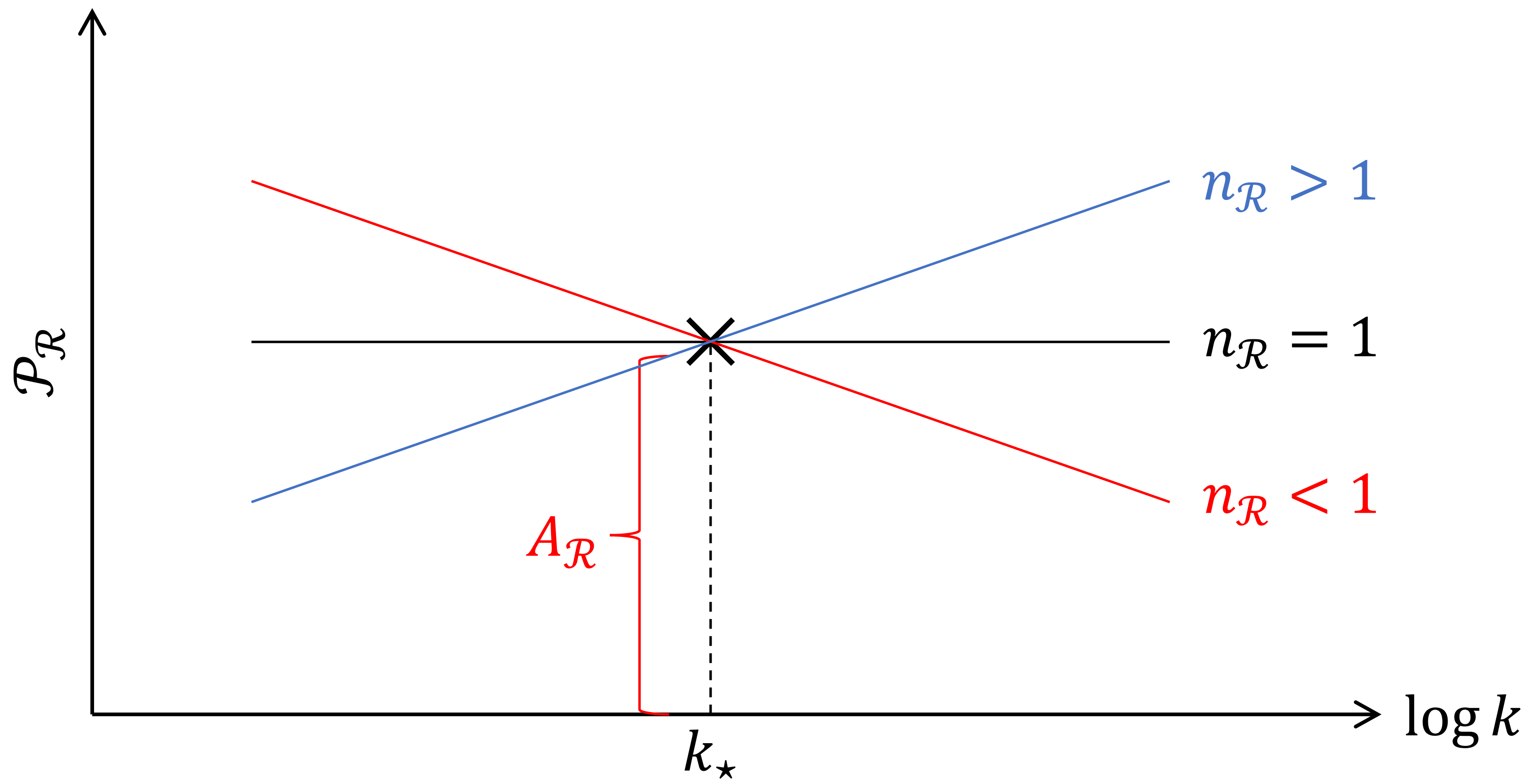

- $A_\mathcal{R}$ gives the “amplitude” of $\mathcal{P}_\mathcal{R}$ at the reference scale $k_\star$
- $n_\mathcal{R}$ tells how $\mathcal{P}_\mathcal{R}$ is “tilted” towards long- or short-wavelength regime

- Thus, from the calculated power spectrum we can read the spectral index as

$$\therefore \; n_{\mathcal{R}} - 1 \equiv \frac{d \log \mathcal{P}_{\mathcal{R}}}{d \log k} = 3 - 2\nu = -2\epsilon - \eta\big|_{k=aH}$$

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- The most recent observations on CMB by Planck satellite constrain, on the reference scale $k_\star = 0.05/\mathrm{Mpc}$, the amplitude of the power spectrum $A_{\mathcal{R}}$ and the spectral index to be:

$$A_{\mathcal{R}} = 2.0968^{+0.0296}_{-0.0292} \times 10^{-9}$$

$$n_{\mathcal{R}} = 0.9652 \pm 0.0042$$

- The power spectrum of tensor perturbation is defined by the sum of each polarization mode:

$$\sum_s \left\langle h_{(s)}(\boldsymbol{k}) h_{(s)}(\boldsymbol{q}) \right\rangle \equiv (2\pi)^3 \delta^{(3)}(\boldsymbol{k}+\boldsymbol{q}) P_h(k)$$

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- If evaluated at the moment of horizon crossing, we have

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- Note that $\mathcal{P}_h$ is directly proportional to $H^2 \propto \rho$, thus once we detect the tensor power spectrum we can determine the energy scale during inflation!

- Another important quantity related to the tensor spectrum is the so-called tensor-to-scalar ratio r, defined by

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- This relation is valid for any single field inflation model with canonical kinetic term, so it is called a **consistency relation**
- Thus if we are lucky enough to test this relation, that amounts to test **all** canonical single field inflation models **at one shot**!

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- We can write the tensor-to-scalar ratio as

$$r = \frac{8}{m_{\rm Pl}^2}\left(\frac{\dot{\phi}}{H}\right)^2 \underset{\substack{\uparrow \\ dN = Hdt}}{=} \frac{8}{m_{\rm Pl}^2}\left(\frac{d\phi}{dN}\right)^2$$

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- Therefore, if we limit interest to the regime relevant for the large-scale CMB observations which spans $N_1 < N < N_2$ with $\Delta N \equiv N_2 - N_1 = \mathcal{O}(1)$, we have

$$\frac{\Delta\phi}{m_{\mathrm{Pl}}} = \int_{N_1}^{N_2} dN \sqrt{\frac{r}{8}} \sim \left(\frac{r}{0.01}\right)^{1/2}$$

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- If we ever detect $r \sim 0.01$ or larger, the field excursion is super-Planckian!
- This raises an important question in inflation model building

III. Quantum solutions of perturbations

1. Quantization of scalar perturbation
   a. Expansion in terms of operators
   b. Choice of vacuum states
   c. Particle creation from vacuum
2. Solutions of scalar perturbation
   a. Asymptotic solutions
   b. General solutions
3. Tensor perturbation
4. Power spectrum
5. Simple example: Quadratic potential

- Consider a simple model with a quadratic potential

$$V(\phi) = \frac{1}{2}m^2\phi^2$$

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- The derivatives of the potential are $V' = m^2\phi$ and $V'' = m^2$
- First we must check if we can have 60 e-folds: From slow-roll approximation

$$N = \int_i^f H dt = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi$$

$$\approx \frac{1}{m_{\mathrm{Pl}}^2} \int_{\phi_f}^{\phi_i} \frac{V}{V'} d\phi = \frac{1}{m_{\mathrm{Pl}}^2} \int_{\phi_f}^{\phi_i} \frac{\phi}{2} d\phi$$

$$= \frac{\phi_i^2 - \phi_f^2}{4 m_{\mathrm{Pl}}^2}$$

SR approximation

$$\frac{V}{V'} = \frac{m^2\phi^2/2}{m^2\phi} = \frac{\phi}{2}$$

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- Thus the initial point for 60 e-folds $\phi_i = \phi_{60}$ is

$$\phi_{60} = \sqrt{4Nm_{\mathrm{Pl}}^2 + \phi_f^2}\bigg|_{N=60} \approx \sqrt{240}m_{\mathrm{Pl}}$$

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- We can approximate $N = \phi^2/(4m_{\rm Pl}^2)$

- Now we proceed to compute perturbation quantities
    1. Amplitude of scalar power spectrum

$$\mathcal{P}_{\mathcal{R}} = \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}}\right)^2 \approx \frac{V^3}{12\pi^2 m_{\rm Pl}^6 V'^2} = \frac{m^2\phi^4}{96\pi^2 m_{\rm Pl}^6} \approx \frac{1}{8\pi^2}\frac{m^2}{m_{\rm Pl}^2}\left(\frac{\phi^2}{4m_{\rm Pl}^2}\right)^2$$

$$\sim \left(10\frac{m}{m_{\rm Pl}}\right)^2 \sim 2\times 10^{-9}$$

$$\therefore\ m \sim 5\times 10^{-6} m_{\rm Pl} \sim 10^{13}\,\mathrm{GeV}$$

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  2. Spectral index of scalar power spectrum: Using $\eta = \dot{\epsilon}/(H\epsilon) \approx 2m_{Pl}^2 V''/V + 4\epsilon$

$$n_{\mathcal{R}} \approx 1 - 2\epsilon - \eta \approx 1 - 2\left(\frac{4m_{\rm Pl}^2}{\phi^2}\right) \sim 1 - \frac{2}{N} \sim 0.96$$

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The simple relation $n_{\mathcal{R}} - 1 \propto 1/N$ is common to the models with power-law potential

3. Amplitude of tensor power spectrum

$$\mathcal{P}_h = \frac{8}{m_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2 \approx \frac{2V}{3\pi^2 m_{\rm Pl}^4} = \frac{4}{3\pi^2}\left(\frac{m}{m_{\rm Pl}}\right)^2\left(\frac{\phi^2}{4m_{\rm Pl}^2}\right) \sim 2\times 10^{-10}$$

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4. Spectral index of tensor power spectrum

$$n_h = -2\epsilon \approx -\frac{4m_{\mathrm{Pl}}^2}{\phi^2} \approx -\frac{1}{N} \sim -0.017$$

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5. Tensor-to-scalar ratio

$$r = 16\epsilon = -8n_h \sim \frac{8}{N} \sim 0.1$$

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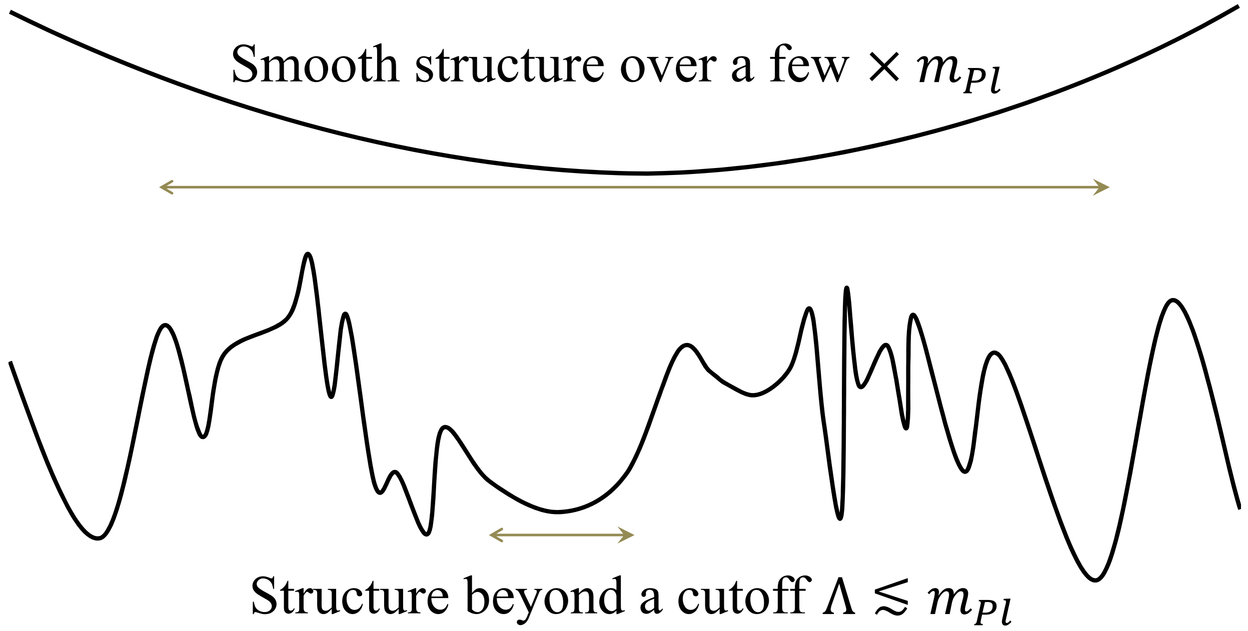
$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\phi} + \sum_{n>4} c_i \frac{\mathcal{O}_n}{\Lambda^{n-4}}$$



- The dimension-n operators $\mathcal{O}_n$ include not only ϕ but also its derivatives, e.g.

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- Thus, we expect many sub-Planckian structure that may well interrupt otherwise successful large-field inflation with super-Planckian field excursions (e.g. "η-problem" in inflation model building in supergravity)
- This is in tension with observations!