Korean Student Combinatorics Workshop 2025

August 20–24, 2025
Gyeongju, Korea

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IBS 이산수학그룹
Discrete Mathematics Group

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About

KSCW

The 2025 Korean Student Combinatorics Workshop (KSCW 2025) brings together advanced undergraduates, graduate students, and early-career postdoctoral researchers from across Korea to foster community, share open problems, and initiate collaborative research in combinatorics and algorithms. Participants will engage in focused problem-solving sessions and networking events designed to inspire new research directions and strengthen professional connections. First organized in 2024, KSCW is now in its second year.

Organizing committee

Kyungjin Cho	POSTECH
Seokbeom Kim	KAIST / IBS DIMAG
Hyunwoo Lee	KAIST / IBS ECOPRO
Jaehyeon Seo	Yonsei University

Timetable

Wednesday, Aug 20

17:00–17:30	**Registration**
17:30–18:50	**Welcome remarks**
19:00–21:00	**Dinner**

Thursday, Aug 21

09:30–10:50	**Opening, Introduction**		
11:00–11:50	IT	**Seunghun Lee** IBS DIMAG	Oriented Matroids: Scope & Perspectives
12:00–13:30	**Lunch**		
13:30–16:20	**Open Problem Session**		
16:30–18:50	**Group Discussion**		
19:00–21:00	**Dinner**		

Friday, Aug 22

09:30–10:20	CT	**Gunwoo Kim** KAIST / IBS DIMAG	Unifying Islands of Tractability for Maximum Independent Set
10:30–12:00	**Group Discussion**		
12:00–13:30	**Lunch**		
13:30–14:20	IT	**Hojin Chu** KIAS	Things I worry about as a Postdoc
14:30–16:00	**Group Discussion**		
16:00–18:50	**Excursion**		
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Saturday, Aug 23

09:30–10:20	CT	**Shinwoo An** POSTECH	Approximation Algorithm for the Geometric Multimatching Problem
10:30–12:00	**Group Discussion**		
12:00–13:30	**Lunch**		
13:30–14:20	CT	**Homoon Ryu** Ajou University	Linear-Time Computation of the Frobenius Normal Form for Symmetric Toeplitz Matrices via Graph-Theoretic Decomposition
14:30–19:00	**Group Discussion**		
19:00–21:00	**Dinner**		

Sunday, Aug 24

09:30–10:00	CT	**Jeewon Kim** KAIST	Progress on Covering a Hexagon with Seven Unit Triangles
10:10–10:40	CT	**Jihyo Chae** Yonsei University	Capturing Additive Structures via Gowers Norms: From Arithmetic Progression Counts to Inverse Theorems
10:50–12:00	**Final Discussion, Progress Report**		
12:00–13:30	**Lunch**		

List of Abstracts

In each session, the abstracts are listed in the order of presentations.

Invited Talks

Oriented Matroids: Scope & Perspectives

Seunghun Lee, IBS DIMAG

This is a very brief introduction to oriented matroids based on personal recollection, with hope that the theory gets wider attention from younger generation of mathematics in Korea.

Things I worry about as a Postdoc

Hojin Chu, KIAS

In this talk, I'll share a few of these thoughts, from research focus to future planning, in the hope that they resonate with others in similar stages. There won't be any magic answers, but maybe a few ideas that help make the uncertainty a bit more manageable.

Contributed Talks

Unifying Islands of Tractability for Maximum Independent Set

Gunwoo Kim, KAIST / IBS DIMAG

The MAXIMUM INDEPENDENT SET (MIS) problem is NP-complete and remains so for every minor-closed graph class that includes all planar graphs (Garey and Johnson [SIAM J. Appl. Math.'77]). On the other hand, due to the Grid Theorem of Robertson and Seymour [JCTB'95], every minor-closed graph class that excludes a planar graph has bounded treewidth, and MIS can be solved in polynomial time on such graph classes. A drawback of this result is that such graph classes are necessarily sparse. Since MIS can be efficiently solved on dense bipartite graphs, there has been interest in extending the tractability of MIS to larger graph classes. I will provide an overview of these attempts and introduce a new parameter called *odd cycle packing (*OCP*)-treewidth*, which unifies the tractability of MIS on the classes of bounded *odd cycle packing (*OCP*) number* and bounded *bipartite treewidth*.

This is based on joint work with Mujin Choi, Maximilian Gorsky, Caleb McFarland, Sebastian Wiederrecht.

Approximation Algorithm for the Geometric Multimatching Problem

Shinwoo An, POSTECH

Matching is one of the most fundamental combinatorial objects in both graph theory and combinatorial optimization. Moreover, a lot of variants of have been studied in both exact and approximation settings. In the first part of my talk, I will present several classical graph matching algorithms and illustrate some intuitions and techniques behind them.

Then I will focus on the geometric setting. Suppose we are given a set of points in a metric space, and we are asked to find a matching between them. Here, our underlying graph can be defined as a complete graph. This geometric matching problem provides both theoretical insights and real-world applications. In the second part of my talk, I will cover geometric variants of the aforementioned matching-like problems. This part shows that geometric tools can significantly improve the running time of the classical graph matching algorithms.

Linear-Time Computation of the Frobenius Normal Form for Symmetric Toeplitz Matrices via Graph-Theoretic Decomposition

Homoon Ryu, Ajou University

We introduce a linear-time algorithm for computing the Frobenius normal form (FNF) of symmetric Toeplitz matrices by utilizing their inherent structural properties through a graph-theoretic approach. Previous results of the authors established that the FNF of a symmetric Toeplitz matrix is explicitly represented as a direct sum of symmetric irreducible Toeplitz matrices, each corresponding to connected components in an associated weighted Toeplitz graph. Conventional matrix decomposition algorithms, such as Storjohann's method (1998), typically have cubic-time complexity. Moreover, standard graph component identification algorithms, such as breadth-first or depth-first search, operate linearly with respect to vertices and edges, translating to quadratic-time complexity solely in terms of vertices for dense graphs like weighted Toeplitz graphs. Our method uniquely leverages the structural regularities of weighted Toeplitz graphs, achieving linear-time complexity strictly with respect to vertices through two novel reductions: the α-type reduction, which eliminates isolated vertices, and the β-type reduction, applying residue class contractions to achieve rapid structural simplifications while preserving component structure. These reductions facilitate an efficient recursive decomposition process that yields linear-time performance for both graph component identification and the resulting FNF computation. This work highlights how structured combinatorial representations can lead to significant computational gains in symbolic linear algebra.

Progress on Covering a Hexagon with Seven Unit Triangles

Jeewon Kim, KAIST

John Conway and Alexander Soifer showed that an equilateral triangle T with side length $n + \epsilon$ can be covered using $n^2 + 2$ unit equilateral triangles. They also conjectured that using $n^2 + 1$ triangles is not enough.

As a more approachable version of this problem, we ask: Can a regular hexagon with side length $1 + \epsilon$ be covered by just 7 unit equilateral triangles? This simplified question reflects core aspects of the Conway–Soifer conjecture.

In this talk, I will present our progress on this problem, including our use of computer-assisted search and the insights it led to. This is joint work with Jineon Baek.

Capturing Additive Structures via Gowers Norms: From Arithmetic Progression Counts to Inverse Theorems

Jihyo Chae, Yonsei University

The Gowers uniformity norm has played a significant role in additive combinatorics as a measure of randomness associated with solution sets of certain linear configurations. In this talk, I introduce the notion of Gowers uniformity norms and demonstrate how they capture additive structures in a given set of integers. Gowers norms capture additive structures both through k-term arithmetic progression counts and through their connection to polynomial Freiman–Ruzsa via Gowers inverse theorems. I present applications of Gowers norms in both directions.

List of Selected Open Problems

Below are selected submissions of open problems, listed in alphabetical order by surname.

Note that "Matchings extend to Hamiltonian cycles in hypercubes" and "The Ruskey–Savage Conjecture" are on the same topic.

Characterization of proper chordal

Yeonsu Chang

Paul and Protopapas introduced the *proper chordal graphs*, which is a subclass of chordal graphs [1].

Let G be a n-vertex graph. A *tree-layout* of G is a triple (T, r, ρ_G) where T is a tree on a set V_T of n nodes rooted at r and $\rho_G : V(G) \to V_T$ is a bijection such that for every edge $xy \in E(G)$, either $\rho_G(x)$ is an ancestor of $\rho_G(y)$, or $\rho_G(y)$ is an ancestor of $\rho_G(x)$ in T. For $u, v \in V_T$, we write $u \prec_T v$ if u is an ancestor of v in T.

A graph G is a *proper chordal* if there exists a tree-layout (T, r, ρ_G) of G such that for every triple of vertices $x, y, z \in V(G)$ with $\rho_G(x) \prec_T \rho_G(y) \prec_T \rho_G(z)$, the following holds:

- If $xy, xz \in E(G)$, then $yz \in E(G)$.
- If $yz, xz \in E(G)$, then $xy \in E(G)$.

Forbidden induced subgraph

Paul and Protopapas showed that the class of proper chordal graphs lies strictly between proper interval graphs and strongly chordal graphs, and is incomparable to interval graphs. The characterizations of proper interval graphs, interval graphs, and strongly chordal graphs in terms of forbidden induced subgraphs are well known. Hence, it is natural to consider the characterization of proper chordal graphs as well.

Question 1. *Provide a characterization of proper chordal graphs in terms of forbidden induced subgraphs?*

Before addressing the above question, we may first consider the characterization of K_4-free proper chordal graphs.

An n-fan graph is a graph obtained from the path $v_1 v_2 \dots v_{n+1}$ by adding a vertex v_0 adjacent to all vertices of the path. It is easy to observe that the 5-fan graph is not a proper chordal graph, whereas the 4-fan graph is a proper chordal graph and has a unique tree-layout.

Observation 2. *For every $k \geq 5$, the k-fan graph is not proper chordal graph.*

Using the observation above, we can show that the following graphs are not proper chordal. Consider two 4-fan graphs with vertex sets $\{u_0, u_1, \dots, u_5\}$ and $\{v_0, v_1, \dots, v_5\}$, where u_0 and v_0 are universal vertices. Then, the resulting graph is not proper chordal in each of the following cases:

- when there is a path between u_0 and v_0,
- when there is a path between u_0 and v_1,
- when there is a path between u_0 and v_2,
- when there is a path between u_1 and v_1,
- when there is a path between u_1 and v_2,
- when there is a path between u_2 and v_2.

Similarly, by considering the connection between two 4-fan graphs, we can construct graphs that are not proper chordal.

Question 3. *Provide a characterization of K_4-free proper chordal graphs in terms of forbidden induced subgraphs?*

Elimination ordering

It is well known that chordal graphs and strongly chordal graphs admit elimination orderings satisfying certain conditions. Hence, it is a natural question whether proper chordal graphs can also be characterized by elimination orderings satisfying certain conditions.

Let G be a n-vertex graph. For a set X of vertices in G, let $G[X]$ denote the subgraph of G induced by X. A vertex $v \in V(G)$ is *simplicial* if and only if $N[v]$ is a clique. The ordering $(v_1, v_2, \ldots, v_n)$ of $V(G)$ is a perfect elimination ordering if and only if for all $i \in [n]$ the vertex v_i is simplicial in G_i, where $G_i = G[\{v_{i+1}, v_{i+2}, \ldots, v_n\}]$. The graph G is chordal if and only if G admit a perfect elimination ordering.

The ordering $(v_1, v_2, \ldots, v_n)$ of $V(G)$ is a strong elimination ordering if and only if for all $i \in [n]$, $N_i[v_j] \subseteq N_i[v_k]$ when $v_j, v_k \in N_i[v_i]$ and $j < k$ where $N_i[v]$ is the closed neighborhood of v in G_i. The graph G is strongly chordal if and only if G admit a strong elimination ordering.

Question 4. *Is there an elimination ordering that characterizes the class of proper chordal graphs?*

Intersection model

A graph G is called an *intersection graph* for a finite family F of non-empty set if there is a one-to-one correspondence between F and $V(G)$ such that two sets in F have non-empty intersection if and only if their corresponding vertices in $V(G)$ are adjacent. We call F an intersection model of G.

For example,

- A graph G is chordal if and only if there exists an intersection model $\mathcal{F}$ such that $\mathcal{F}$ is the set of connected subgraphs of a tree.
- A graph G is strongly chordal if and only if there exists an intersection model $\mathcal{F}$ such that $\mathcal{F}$ is the compatible collection of subtrees of a rooted, weighted tree [2].
- A graph G is interval if and only if there exists an intersection model $\mathcal{F}$ such that $\mathcal{F}$ is the set of intervals on a real line.

Question 5. *Is there an intersection model that characterizes the class of proper chordal graphs?*

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Chromatic number of random subgraphs

Jigang Choi

For $p \in (0, 1)$, the random subgraph G_p of a graph G is the subgraph obtained by randomly including each edge of G independently with probability p.

Theorem 1 (Erdős [1])**.** *For any $t, g \in \mathbb{N}$, there is a graph G with $\chi(G) \geq t$, $girth(G) \geq g$.*

In 1968, Erdős and Hajnal suggested the following conjecture(This conjecture is still an open problem).

Conjecture 2 (Erdős, Hajnal [2])**.** *For any $t, g \in \mathbb{N}$, there is $k = k(t, g) \in \mathbb{N}$ such that every graph G with $\chi(G) \geq k$ contains a subgraph G' with $\chi(G') \geq t$, $girth(G') \geq g$.*

Above Erdős's theorem was proved using random graphs, and for that reason, there is hope that this conjecture might also be resolved by studying random subgraphs of graphs. As a result, interest in the chromatic number of random subgraphs has also increased.

There is a well-known fact about the chromatic number of random subgraphs that is if $\chi(G) = k$, then $\mathbb{E}[\chi(G_{\frac{1}{r}})] \geq k^{\frac{1}{r}}$ for any positive integer r. This can be proved using the fact that if $G = G_1 \cup \cdots \cup G_r$, then $\chi(G) \leq \chi(G_1) \cdots \chi(G_r)$. However, this approach relies on r being an integer. As a result, nothing is known for non-integer values of r. So, Bukh, Krivelevich, and Narayanan suggested the following open problem in their paper [3] and this is the open problem I have chosen to study.

Problem 3 (Bukh, Krivelevich, Narayanan [3])**.** *Find any lower bound of $\mathbb{E}[\chi(G_p)]$ where $p \in (\frac{1}{3}, \frac{1}{2})$ that is better than $[\chi(G)]^{\frac{1}{3}}$.*

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Separating system

Hojin Chu

Given a set A and a family $\mathcal{F}$ of sets, we say $\mathcal{F}$ is a *strongly-separating system* for A, or that $\mathcal{F}$ *strongly-separates* A, if for every two elements $a, b \in A$, there is a set $S \in \mathcal{F}$ satisfying $a \in S$ and $b \notin S$. Similarly, we say that $\mathcal{F}$ is a *weakly-separating system* for A, or that $\mathcal{F}$ *weakly-separates* A, if for every two elements $a, b \in A$, there is a set $S \in \mathcal{F}$ satisfying $|S \cap \{a, b\}| = 1$. We call A a *ground set* and call $\mathcal{F}$ a *separation*. The notion of separation was introduced by Rényi [1] in 1961. It is easy to see that the smallest weakly-separating system for a set of size n has size $\lceil \log_2 n \rceil$, and that the smallest strongly-smallest separating system has size $(1 + o(1)) \log_2 n$.

Separating path system

Our focus is on *separating path system*, where the ground set is the set of edges of given graph G and the members of the separating system are paths in G. Separating path systems arise naturally in the study of network design. In the context of extremal graph theory, this variant was proposed by Gyula Katona in the 5th Emléktábla Workshop (2013). Given a graph G, we denote the minimum number of weak (resp. strong) separation of G by $wsp(G)$ (resp. $ssp(G)$). Let $wsp(n)$ (resp. $ssp(n)$) be the maximum of $wsp(G)$ (resp. $ssp(G)$) over all n-vertex graphs G. Katona asked to determine $wsp(n)$. Falgas–Ravry–Kittipassorn–Korándi-Letzter-Narayanan [3], who focused on the weak version, and Balogh–Csaba–Martin–Pluhár [2], who focused on the strong version, conjectured the following.

Conjecture 1. *There exists a constant c such that every graph on n vertices has a (weakly- or strongly-)separating path system of size at mose cn. That is, $wsp(n) = O(n)$ and $ssp(n) = O(n)$.*

The conjecture was settled by Bonamy-Botler-Dross-Naia-Skokan [4], who proved that $ssp(n) \leq 19n$.

Some researchers considered path separation in various classes of graphs, including random graphs, complete graphs, and trees.

- $ssp(K_n) \leq 2n + 4$ by BCMP (2016)
- $ssp(K_n) = (1 + o(1))n$ by FMS (2025)
- $ssp(K_n) \leq n + 9$ and identifying n when $ssp(K_n) = n + \alpha$
- $ssp(K_{n,n}) \simeq 1.16n$

- $ssp(K_{m,n}) \geq (\sqrt{4+6n/m}-2)m$ if $n/2 \leq m \leq n$; $ssp(K_{m,n}) = n$ if $m < n/2$

Question 2. *An upper bound for $K_{m,n}$.*

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Matchings extend to Hamiltonian cycles in hypercubes

Ho Kim

The n-dimensional hypercube Q_n is known as Hamiltonian for all $n \geq 2$.

In 1993, Ruskey and Savage [4] conjectured the following.

Conjecture 1 (F. Ruskey, C. Savage [4]). *Let M be a matching in Q_n. Then there is an edge set $S \subseteq E(Q_n)$ such that $M \cup S$ is a Hamilitonian cycle.*

Fink [2, 3] verified the conjecture when M is a perfect matching. Also, Fink pointed out that the conjecture is true for $n = 2, 3, 4$.

The other developments toward this conjecture focused on the case when $|M|$ is small. Dvořák solved the conjecture when $|M| \leq 2n - 3$ in a stronger form.

Theorem 2 (Dvořák [1]). *For $n \geq 2$, let $E \subseteq E(Q_n)$ with $|E| \leq 2n - 3$. Then there exists a Hamiltonian cycle of Q_n containing E if and only if the subgraph induced by E is a linear forest.*

The most recently, Wang and Zhang [5] solved the conjecture when $|M| \leq 3n - 10$.

Roughly speaking, those linear improvements use the following method. To show that the conjecture holds when $|M| \leq an - b$, since there are n ways to separate Q_n into two vertex disjoint copies of Q_{n-1}, by the pigeonhole principle, there is a separation such that there are at most $a - 1$ edges of M between the two copies of Q_{n-1}. We obtain a Hamiltonian cycle from each copy of Q_{n-1} using induction, and carefully construct a Hamiltonian cycle of Q_n using $a - 1$ edges and the Hamiltonian cycles of Q_{n-1} we obtained.

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Small weak ε-net

Dohyeon Lee

For $\varepsilon \in (0, 1)$, a finite set $N \subset \mathbb{R}^d$ is a weak ε-net for a finite set $X \subset \mathbb{R}^d$ if N intersects every convex set K with $|K \cap X| \geq \varepsilon|X|$ [9]. Weak ε-net connects transversal and fractional transversal number, and has many applications in discrete and computational geometry (e.g., see this post) including the proof of the famous (p, q)-theorem [5].

Weak ε-net theorem asserts that there is a constant $k = k(d, \varepsilon)$ depends on d and ε, such that any finite set $X \subset \mathbb{R}^d$ has a weak ε-net of size at most k [6]. Then naturally the following question arises:

Question 1. *Given d and ε, what is the optimal (minimum) value of k?*

Formally, we ask for the value of a function $f : \mathbb{Z}_{>0} \times (0, 1) \to \mathbb{Z}_{>0}$, where

$$f(d, \varepsilon) = \min\{c : \text{any finite set } X \subset \mathbb{R}^d \text{ has a weak } \varepsilon\text{-net of size at most } k\}.$$

The best known bounds are $f(d, \varepsilon) = O(\varepsilon^{-\alpha_d})$ where $\alpha_d \approx d - 0.5$ [10] and $f(d, \varepsilon) = \Omega(\varepsilon^{-1} \log^{d-1} \varepsilon^{-1})$ [7]. There is a large gap between the upper and the lower bound, and finding correct asymptotic is a major open problem in combinatorial geometry, but this problem is super difficult and not doable in short workshop.

Calculating the exact value of f is even more difficult, but one may ask when ε is large. Since $f(d, \bullet)$ is a decreasing for fixed d, we may regard problem 1 as the following:

Question 2. *Given d and k, what is the infimum of ε such that any finite set $\mathbb{R}^d$ admits a weak ε-net of size at most k?*

Formally, we seek the value of $g : \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0} \to (0, 1]$, where

$$g(d, k) = \inf\{\varepsilon : \text{any finite set } X \subset \mathbb{R}^d \text{ has a weak } \varepsilon\text{-net of size at most } k\}.$$

We denote $g(d, k)$ by ε_k^d, $g(2, k)$ by ε_k, and set $g(d, 0) = 1$. The famous center point theorem [8] asserts that $\varepsilon_1^d = \frac{d}{d+1}$, and it is easy to show that $\varepsilon_k^1 = \frac{1}{k+1}$ for any k. However, other values are unknown except ε_2^2. The following is table of known bounds for $d = 2$.

From ε_2, all upper bounds come from recursive inequality using mass partition [1, 2, 3, 11]. Specifically, for nonnegative integers r, s and positive integer d, we have

[1]Unpublished results of the writer.

	ε_0	ε_1	ε_2	ε_3	ε_4	ε_5
Upper bound Lower bound	1	2/3 [8]	4/7 [3]	8/15 [3] 6/13[1]	1/2 [2] 1/3[1]	10/21 [2] 2/7[1]

Table 1: Upper and Lower bound of ε_k $(= \varepsilon_k^2)$.

$$\varepsilon_{r+ds+1}^d \le \frac{\varepsilon_r^d(1+(d-1)\varepsilon_s^d)}{1+\varepsilon_r^d(1+(d-1)\varepsilon_s^d)}[1], \tag{0.1}$$

and for nonnegative integers r_1, r_2, r_3, and s, we have,

$$\varepsilon_{r_1+r_2+r_3+3s+1} \le \frac{1}{2}\left(\frac{1}{\varepsilon_{r_1}} + \frac{1}{\varepsilon_{r_2}} + \frac{1}{\varepsilon_{r_3}}\right)^{-1} + \frac{1}{2}\varepsilon_s[2]. \tag{0.2}$$

Note that we obtain the upper bound of ε_2, ε_3 by plugging $(d, r, s) = (2, 1, 0)$, $(2, 2, 0)$ to 0.1, and ε_4, ε_5 by plugging $(r_1, r_2, r_3, s) = (0, 0, 0, 1)$, $(1, 0, 0, 1)$ to 0.2.

On the other hand, lower bounds of ε_2 and ε_3 are tight or close to the upper bound, but they are obtained by ad hoc case analysis [1, 3][1]. There are general lower bound constructions for ε_k^d for all k, d[1], and there is a recursive formula

$$\varepsilon_{a+b+1}^d \ge \frac{\varepsilon_a^d \varepsilon_b^d}{\varepsilon_a^d + \varepsilon_b^d}[4], \tag{0.3}$$

but the values obtained by them are far from upper bounds. We end the section with the following problems:

Problem 3.

- *Improve the upper and lower bounds in Table 1. In particular, determine whether ε_3 and ε_4 are less, same, or larger than $\frac{1}{2}$.*
- *Find a general lower bound construction better than known constructions.*
- *Improve the upper bound for $d = 3$.*

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Generalized coloring numbers on digraphs

Myounghwan Lee

Weak coloring number and *strong coloring number* are important tools in understanding the structure of sparse graphs, offering a generalization of degeneracy. For instance, Nešetřil and Ossona de Mendez's framework on sparsity [4][2] laid the foundation for these parameters, and they have since been used to give combinatorial characterizations of nowhere dense classes [8]. Their boundedness implies other useful properties, such as low tree-depth colorings and separator theorems, making them powerful in both structural and algorithmic graph theory. For more details, we refer to see the following paper [7].

Definition 1 (Basic notions)**.**

The *length* of a path P is the number of edges in P.

Let L be a linear ordering of the vertices of a (di)graph G. We denote by $\leq_L$ the total order on $V(G)$ induced by the ordering L. For vertices v and w of G, we write $v <_L w$ if $v \leq_L w$ and $v \neq w$. ◀

Definition 2 (Weak coloring number and strong coloring number)**.**

Let $k \in \mathbb{N} \cup \{\infty\}$. For vertices v and w of G, w is *weakly k-reachable* from v with respect to L if $w \leq_L v$ and there is a path P of length at most k from v to w such that for every vertex u of P, $w \leq_L u$. We denote by $\mathsf{WReach}_k[G, L, v]$ the set of all weakly k-reachable vertices from v in L. The *weak k-coloring number* of L is the maximum size of $\mathsf{WReach}_k[G, L, v]$ over all vertices v of G. The *weak k-coloring number* of G, denoted by $\mathsf{wcol}_k(G)$, is the minimum weak k-coloring number over all linear orderings of the vertices of G.

For vertices v and w of G, we say that w is *strongly k-reachable* from v in L if $w \leq_L v$ and there exists a path P of length at most k from v to w such that for every internal vertex u of P, $v \leq_L u$. Let $\mathsf{SReach}_k[G, L, v]$ be the set of all vertices that are strongly k-reachable from v in L. The *strong k-coloring number* of L is the maximum size of $\mathsf{SReach}_k[G, L, v]$ over all vertices v of G. The *strong k-coloring number* of G, denoted by $\mathsf{scol}_k(G)$, is the minimum strong k-coloring number over all possible linear orderings of the vertices of G. ◀

When $k = \infty$, one can observe that weak coloring number and strong coloring number are strongly related to *treewidth* and *treedepth*, respectively.

[2] Also, I recommend referring to [6].

Theorem 3 (Theorem 1.17 and Theorem 1.19 in Lecture Note Chapter 1 [6])**.**

For every graph G, $\mathrm{wcol}_\infty(G) = \mathrm{td}(G)$ *and* $\mathrm{scol}_\infty(G) = \mathrm{tw}(G) + 1$*.* ◀

This implies that there is no function f such that $\mathrm{wcol}_\infty(G) \leq f(\mathrm{scol}_\infty(G))$ for every graph G, since a path on $n \geq 2$ vertices has treedepth $\lceil \log(n+1) \rceil$ but treewidth 1. However, if $k \neq \infty$, there is a function f $\mathrm{wcol}_\infty(G) \leq f(\mathrm{scol}_\infty(G))$ for every graph G.

Proposition 4 (Proposition 4.8 [4])**.**

For every $k \in \mathbb{N} \cup \{\infty\}$ and every graph G, $\mathrm{scol}_k(G) \leq \mathrm{wcol}_k(G)$*. For every $k \in \mathbb{N}$ and every graph G,* $\mathrm{wcol}_k(G) \leq (\mathrm{scol}_k(G))^k$*.*

We may consider the directed analogues of these notions.

Definition 5 (One-sided weak and strong coloring number)**.** [3]

Let $k \in \mathbb{N} \cup \{\infty\}$. For vertices v and w of G, w is *weakly $(k, \rightarrow)$-reachable* from v with respect to L if $w \leq_L v$ and there is a directed path P of length at most k from v to w such that for every vertex u of P, $w \leq_L u$. We denote by $\mathrm{WReach}_k^{\rightarrow}[G, L, v]$ the set of all weakly $(k, \rightarrow)$-reachable vertices from v in L. The *weak $(k, \rightarrow)$-coloring number* of L is the maximum size of $\mathrm{WReach}_k^{\rightarrow}[G, L, v]$ over all vertices v of G. The *weak $(k, \rightarrow)$-coloring number* of G, denoted by $\mathrm{wcol}_k^{\rightarrow}(G)$, is the minimum weak k-coloring number over all linear orderings of the vertices of G.

For vertices v and w of G, we say that w is *strongly $(k, \rightarrow)$-reachable* from v in L if $w \leq_L v$ and there exists a directed path P of length at most k from v to w such that for every internal vertex u of P, $v \leq_L u$. Let $\mathrm{SReach}_k^{\rightarrow}[G, L, v]$ be the set of all vertices that are strongly $(k, \rightarrow)$-reachable from v in L. The *strong $(k, \rightarrow)$-coloring number* of L is the maximum size of $\mathrm{SReach}_k^{\rightarrow}[G, L, v]$ over all vertices v of G. The *strong $(k, \rightarrow)$-coloring number* of G, denoted by $\mathrm{scol}_k^{\rightarrow}(G)$, is the minimum strong $(k, \rightarrow)$-coloring number over all possible linear orderings of the vertices of G. ◀

Weak $(1, \rightarrow)$-coloring number and strong $(1, \rightarrow)$-coloring number are equal to the out-degeneracy. Similar to undirected graphs, when $k = \infty$, weak $(k, \rightarrow)$-coloring number and strong $(k, \rightarrow)$-coloring number are strongly related to *cycle rank* and *Kelly-width*.[4]

Theorem 6 (Lemma 3.3 in [3] and Lemma 9.2 in [1])**.**

For every digraph G, $\mathrm{wcol}_\infty^{\rightarrow}(G) = \mathrm{cr}(G) + 1$ *and* $\mathrm{scol}_\infty^{\rightarrow}(G) = \mathrm{Kellyw}(G) + 1$*.* ◀

[3]Two-sided version is also considered in [2] and [5].

[4]Cycle rank and Kelly-width are directed analogues of treedepth and treewidth, respectively. Several variants of directed analogues of treewidth have been proposed, including DAG-width and directed treewidth.

This also implies that there is no function f such that for any digraph G, $\mathrm{wcol}_\infty^{\rightarrow}(G) \leq f(\mathrm{scol}_\infty^{\rightarrow}(G))$. (Consider the digraph obtained from an undirected path on n vertices by replacing each edge by a directed digon.) We are left with the following question.

Question 7. *Let k be a fixed non-negative integer. Is there a function f such that for any digraph G, $\mathrm{wcol}_k^{\rightarrow}(G) \leq f(\mathrm{scol}_k^{\rightarrow}(G))$?* ◀

We say a class $\mathcal{C}$ of digraphs has *bounded reachable expansion* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{wcol}_k^{\rightarrow}(G) \leq f(k)$ for all $k \in \mathbb{N}_{\geq 1}$ and $G \in \mathcal{C}$.

Question 8. *Which digraph classes have bounded reachable expansion?* ◀

Question 9. *Are there other natural characterizations of digraph classes of bounded reachable expansion?* ◀

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The Ruskey–Savage Conjecture

Seongbin Park

A matching in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges in M share a common endpoint. Each vertex is incident to at most one edge of the matching. If every vertex of G is covered by exactly one edge of M, then M is called a perfect matching.

A Hamiltonian cycle in a graph G is a cycle that visits every vertex of G exactly once before returning to its starting point.

Definition 1. For a positive integer $n \geq 2$, the n-dimensional hypercube Q_n is the simple, undirected graph defined as follows:

- The vertex set of $Q_n(V(Q_n))$ is the set of all 2^n subsets of $\{1, 2, \ldots, n\}$.
- Two vertices $u, v \in V(Q_n)$ are adjacent, i.e., $(u, v) \in E(Q_n)$, if and only if their corresponding subsets differ by exactly one element.

The study of Hamilton cycles in Q_n can be traced back to Gros (1872) [1] and Gray (1953), and has applications in various areas as image processing, data compression or information retrieval.

Ruskey and Savage [2] conjectured the following which is still open:

Question 2 (Ruskey and Savage [2])**.** *Does every matching M of the n-dimensional hypercube Q_n extend to a Hamiltonian cycle, for all $n \geq 2$?*

This conjecture can be shown to be true for $n = 2, 3, 4, 5$ using a computer program. [3, 4] However, no proof or counterexample is known for $n \geq 6$.

Kreweras [5] posed a more specific question: whether every *perfect matching* of Q_n can be extended to a Hamiltonian cycle. This conjecture was later proved by Fink [3]. Fink's result further shows that any perfect matching can be paired with another perfect matching to form a Hamiltonian cycle of Q_n.

Theorem 3 (Fink [3])**.** *For $n \geq 2$ and every perfect matching M of $K(Q_n)$, there exists a perfect matching N of Q_n such that $M \cup N$ forms a Hamiltonian cycle of $K(Q_n)$.*

Recent progress has been made by Fink and Mütze (2024) [4], who showed that any matching of Q_n spanning at most five coordinate directions can be extended to a Hamiltonian cycle. While this significantly generalizes known partial results, the full Ruskey–Savage conjecture remains open.

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Additional Information

Restaurants

Lunch A restaurant in the hotel

Dinner

Wednesday, August 20	서민식당	숯불직화구이, 양념쭈꾸미볶음
Thursday, August 21	소솜당	김치찜, 석갈비, 불제육, 냉모밀, 오징어땡전
Friday, August 22	복길 경주본점	전복솥밥, 전복죽, 한우불고기솥밥, 고등어구이
Saturday, August 23	보문갈비	갈비, 토장찌개, 더덕구이

Excursion

Visit Gyeongju National Museum (국립경주박물관)
⟶ Stroll around Cheomseongdae Observatory (첨성대 일대)
⟶ Dinner at Bokgil, Hwangridan-gil (복길 경주본점)

List of Participants

Shinwoo An	POSTECH
Ingyu Baek	Yonsei University
Yeonsu Chang	Hanyang University
Jihyo Chae	Yonsei University
Kyungjin Cho	POSTECH
Jigang Choi	KAIST
Hojin Chu	KIAS
Taehyun Eom	Chonnam National University
Sanghwa Han	POSTECH
Musung Kang	SNU
Gunwoo Kim	KAIST / IBS DIMAG
Ho Kim	KAIST / IBS DIMAG
Hyungu Kim	KAIST
Jeewon Kim	KAIST
Seokbeom Kim	KAIST / IBS DIMAG
Dohyeon Lee	KAIST / IBS DIMAG
Hyunwoo Lee	KAIST / IBS ECOPRO
Jeongin Lee	Hanyang University
Myounghwan Lee	Hanyang University
Seunghun Lee	IBS DIMAG
Hyemi Park	Hanyang University
Seongbin Park	POSTECH
Homoon Ryu	Ajou University
Jaehyeon Seo	Yonsei University
Hyeonjun Shin	POSTECH
Chanho Song	POSTECH

List of Submitted Open Problems

Below are unselected submissions of open problems, listed in alphabetical order by surname.

Low Sensitivity Hopsets for Planar Digraphs

Shinwoo An

Given a weighted directed graph $G = (V, E, w)$, a β-shortcut set is a set $H \subset V \times V$ of edges, such that (1) Every edge $(s, t) \in H$ is in transitive closure of G, and (2) For every (s, t) vertex pair of G, there is a s-t path in $G \cup H$ using β edges (hops). A hopset is a weighted variant of shortcut set, defined as follows. A (β, ϵ)-hopset is a set $H \subset V \times V$ of edges, such that (1) Every edge $(s, t) \in H$ has weight $\mathrm{dist}_G(s, t)$, and (2) for every (s, t) pair, there is a s-t path in $G \cup H$ using β hops, whose weight is at most $(1 + \epsilon)\mathrm{dist}_G(s, t)$. While these structures are widely studied over the past few decades, a new concept of low sensitivity shortcut/hopset has been recently introduced in [1].

Let H be a shortcut/hopset. For an edge (s, t) of H, let $P(s, t)$ be the shortest s-t path in G. We assume that such a path is unique for every pair (s, t). Then, a *sensitivity* of a vertex $v \in V$ is defined as the number of paths $P(s, t)$ from $(s, t) \in E(H)$ intersecting v. Then *sensitivity* of shortcut/hopset H is defined by the maximum sensitivity among the vertices of G. One can view a collection $\{P(s, t)\}$ as a collection of single flows, and then the sensitivity notion becomes congestion on the flow-type problems. Recall that low-congestion flow plays the central role in designing efficient dynamic algorithms for flow-type problems, as one vertex change roughly makes congestion-size flow computation in the dynamic setting. Hence, one might expect further application of low-sensitivity shortcut/hopset as a toolbox for designing dynamic graph algorithms. In fact, traditional shortcut/hopset data structures are widely used throughout distance oracles and path-reporting oracles for various settings.

As a initial study, [1] gives the upperbound result as follows: for a shortcut set, it is possible to add a shortcut set satisfying hopbound $\beta = O(\sqrt{n}\log^3 n)$ with sensitivity $O(\log n)$. For the $(\beta, 0)$-hopset, the bound becomes $\beta = O(\sqrt{n}\log n)$ and sensitivity $O(\sqrt{n}\log n)$. I believe we can further improve their result if we address “nice" graph classes. For instance, what happens if the graph has bounded treewidth or bounded degree? In fact, introduce graph measure as a parameter to design efficient hopset/spanner/spanning tree is a widely-used research direction. As a result, I want to suggest the following open problem. **Can we beat the previous upper bound if the given graph G is a planar digraph?**

Notes. In the traditional setting, the goal of shortcut/hopset problems are to add as small number of edges as possible to bound the hopbound as much as possible. For undirected graphs, the shortcut problem is relatively easy: star graphs consist of $O(n)$ edges satisfying the hopbound by two. Hence, the world of directed graphs provides much more challenging and intriguing scenarios.

For the traditional shortcut/hopset on directed graphs, the best-known construction is adding $\tilde{O}(n)$ edges to bound the hop distance by $\tilde{O}(n^{1/3})$ [2]. Although [1] already mentioned that the construction of [2] does not work for the low sensitivity setting, we can reference them as a starting intuition.

References

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[2] Kogan, Shimon and Merav Parter. "New diameter-reducing shortcuts and directed hopsets: Breaking the barrier." Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA). Society for Industrial and Applied Mathematics, 2022.

Entropy method for counting graph homomorphism

Ingyu Baek

Sidorenko's conjecture states the following:

Conjecture 1. *For any bipartite graph H, $t(H,G) \geq t(P_1,G)^{e(H)}$ holds for arbitrary graph G.*

Various methods were introduced to prove Sidorenko's conjecture for certain bipartite graphs, although the general conjecture stays widely open.

The information entropy is defined as follows.

Definition 2. Let X be a random variable over a finite support, F. Then, $\mathbb{H}(X) = \sum_{x \in F} -p(X = x) \log p(X = x)$ is called the *entropy* of a random variable X. If Y is another random variable over a finite support F', the conditional entropy is defined as follows:

$$\mathbb{H}(X|Y) = \sum_{y \in F'} p(Y = y) \mathbb{H}(X|Y = y),$$

where $\mathbb{H}(X|Y = y) = \sum_{x \in F} -p(X = x|Y = y) \log p(X = x|Y = y)$.

If $X = (X_1, X_2, \cdots, X_k)$ and $Y = (Y_1, Y_2, \cdots, Y_t)$, the usual notation is $\mathbb{H}(X) = \mathbb{H}(X_1, X_2, \cdots, X_k)$ and $\mathbb{H}(X|Y) = \mathbb{H}(X_1, X_2, \cdots, X_k|Y_1, \cdots, Y_t)$.

Information entropy is an extremely useful tool in extremal combinatorics, usually connected to the size of the set we would like to consider via the following inequality.

Lemma 3. *If X is a random variable over F, then*

$$\mathbb{H}(X) \leq \log |F|$$

where the equality holds if and only if $X \sim Unif(F)$.

Also, the following properties are widely used.

Proposition 4. *Let X,Y and Z be random variables.*

1. $\mathbb{H}(X) \geq 0$ *and* $\mathbb{H}(X|Y) \geq 0$.

2. $\mathbb{H}(X, Y) = \mathbb{H}(X|Y) + \mathbb{H}(Y)$.

3. $\mathbb{H}(X|Y) \le \mathbb{H}(X)$ *where the equality holds if X and Y are independent with each other. If X and Y are independent, then $\mathbb{H}(X|Y) = \mathbb{H}(X)$ and thus $\mathbb{H}(X, Y) = \mathbb{H}(X) + \mathbb{H}(Y)$.*

4. $\mathbb{H}(X|Y, Z) \le \mathbb{H}(X|Z)$ *where the equality holds if X and Y are conditionally independent given Z.*

The basic method to use entropy for proving that certain bipartite graphs have Sidorenko property is as follows:

Let $F = V(G)$. Define a random tuple $(X_v)_{v\in V(H)}$, where each X_v is supported on $V(G)$, and $(X_v)_{v\in V(H)}$ is supported on $Hom(H, G)$, i.e., we define a random algorithm, that samples H from G, in a sense of homomorphism. Then, as $\log|Hom(H, G)| \ge \mathbb{H}((X_v)_{v\in V(H)})$, we can prove $|Hom(H, G)| \ge n^{v(H)}p^{e(H)}$ by analyzing the algorithm and show $\mathbb{H}((X_v)_{v\in V(H)}) \ge \log n^{v(H)}p^{e(H)}$.

For instance, if H is a bipartite graph such that it has a vertex that is complete to the other part, then it is known that H is Sidorenko. If $H = (A, B)$ is a bipartite graph such that a fixed vertex $k \in A$ is complete to B, then we may define the following probabilistic algorithm:

1. Sample X_k in $V(G)$, where the probability of $X_v = t$ is proportional to $deg(t)$ for all $t \in V(G)$.

2. Pick $(X_b)_{b\in B}$, where each X_b is chosen uniformly at random among neighbors of X_k.

3. Now we sample $(X_a)_{a\in A\setminus k}$. We define the probability distribution of each X_a's, as a conditionally independent copy of X_k conditioned on $(X_b)_{b\in N(a)}$.

It is clear that the support of the distribution of $(X_v)_{v\in V(H)}$ is $\mathrm{Hom}(H, G)$, and thus we have the following equation.

$$
\begin{aligned}
&\log|\mathrm{Hom}(H, G)| \ge \mathbb{H}((X_v)_{v\in V(H)}) = \mathbb{H}((X_a)_{a\in A}|(X_b)_{b\in B}) + \mathbb{H}((X_b)_{b\in B}) \\
&= \sum_{a\in A\setminus k} \mathbb{H}(X_a|(X_b)_{b\in B}) + \mathbb{H}(X_k|(X_b)_{b\in B}) + \mathbb{H}((X_b)_{b\in B}) \\
&= \sum_{a\in A\setminus k} \mathbb{H}(X_a|(X_b)_{b\in N(a)}) + \mathbb{H}(X_k, (X_b)_{b\in B}) \\
&= \sum_{a\in A\setminus k} \big(\mathbb{H}(X_a, (X_b)_{b\in N(a)}) - \mathbb{H}((X_b)_{b\in N(a)})\big) + \mathbb{H}(X_k, (X_b)_{b\in B}) \\
&\ge \sum_{a\in A\setminus k} ((\deg(a)+1)\log n + \deg(a)\log p - \deg(a)\log n) + (\deg(k)+1)\log n + \deg(k)\log p \\
&= \log n^{v(H)}p^{e(H)}.
\end{aligned}
$$

We refer the readers to Theorem 5.5.14 of [1] for a detailed explanation of each step and more breakdown about entropy method in combinatorics.

Our goal is to generalize this type of entropy analysis to the bipartite graphs that does not have a vertex complete to the other side, by manually adding the vertex k to our bipartite graph, and explicitly excluding the (conditional) entropy of k from our final analysis. For the toy example, I demonstrate the case for $H = C_6$, although it is already (by different methods) known that C_6 is Sidorenko. Let H^+ be the bipartite graph where a new vertex k is added to $H = C_6 = (A = \{1, 3, 5\}, B = \{2, 4, 6\})$ where $V(H) = [6]$ and k is complete to B.

We may similarly define random variables $(X_v)_{v \in V(H^+)}$ as above. Then we have the following:

$$\mathbb{H}((X_v)_{v\in V(H^+)}) = \sum_{a\in A} \left(\mathbb{H}(X_a, (X_b)_{b\in N(a)}) - \mathbb{H}((X_b)_{b\in N(a)})\right) + \mathbb{H}(X_k, (X_b)_{b\in B}).$$

On the other side,

$$\mathbb{H}((X_v)_{v\in V(H^+)}) = \mathbb{H}(X_k|(X_v)_{v\in V(H)}) + \mathbb{H}((X_v)_{v\in V(H)}) = \mathbb{H}(X_k|(X_b)_{b\in B}) + \mathbb{H}((X_v)_{v\in V(H)}).$$

Together we have:

$$\begin{aligned}
&\mathbb{H}((X_v)_{v\in V(H)}) = \sum_{a\in A} \left(\mathbb{H}(X_a, (X_b)_{b\in N(a)}) - \mathbb{H}((X_b)_{b\in N(a)})\right) + \mathbb{H}(X_k, (X_b)_{b\in B}) - \mathbb{H}(X_k|(X_b)_{b\in B}) \\
&= \sum_{a\in A} \left(\mathbb{H}(X_a, (X_b)_{b\in N(a)}) - \mathbb{H}((X_b)_{b\in N(a)})\right) + \mathbb{H}((X_b)_{b\in B}) \\
&= 3\mathbb{H}(K_{1,2}) - 3\mathbb{H}(X_2, X_4) + \mathbb{H}(X_2, X_4, X_6)
\end{aligned}$$

where at the last equality, we used the symmetry of C_6 and abused the notation to denote $\mathbb{H}(X_2, X_3, X_4) = \mathbb{H}(K_{1,2})$.

Let $A_2 := 2\log n - \mathbb{H}(X_2, X_4)$ and $A_3 := 3\log n - \mathbb{H}(X_2, X_4, X_6)$. Then, $\log|\mathrm{Hom}(H, G)| \geq \log n^{v(H)}p^{e(H)} + 3A_2 - A_3$. If $3A_2 - A_3 \geq 0$, then it is direct that H is Sidorenko. However, for some bipartite graph H and target graph G, it can be computationally checked that the "gained entropy" ($3A_2$ in above case) is smaller than the "lost entropy" (A_3 in above case).

Question 5. *Is $3A_2 - A_3 \geq -K$ for some absolute constant $K \geq 0$?*

If we positively answer the generalized version of the above question for arbitrary bipartite graph H and some $K = K(H) > 0$, it would imply Sidorenko's conjecture, via standard tensor power trick.

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Sets with no square differences

Jihyo Chae

Let A be a subset of $[N]$. The difference set $|A-A|$ is defined as the collection of all possible differences between elements in A. Two numbers a and b are said to have a square difference if $a-b=n^2$ for some $n\in\mathbb{N}$. How large can a set without any square differences be? The question was first raised by Lovász with his conjecture $|A|=o(n)$, and was answered positively by Sárcözy and Furstenberg independently in the late 1970s. To be specific, Sárcözy gave the upper bound $|A|\leq\frac{N}{(\log N)^{\frac{1}{3}+o(1)}}$ using techniques from analytic number theory. This question can be understood as part of a larger question in additive combinatorics: determining the arithmetic structures that are guaranteed to be present in dense subsets of $[N]$, as Wolf [2] mentions.

The approach of consecutive improvements basically follows the scheme of the circle method, the method used in [3] to attack this problem. However, the spirit of density increment from the proof of Szemerédi's regularity lemma appears as the key step for the improvement. The first improvement of Sárcözy's upper bound, $|A|\ll\frac{N}{(\log N)^{c\log\log\log\log N}}$, was proved by Pintz, Steiger, and Szemerédi [4] in 1988 by using Fourier analytic density increment. Bloom and Maynard improved this result by ($|A|\ll\frac{N}{(\log N)^{c\log\log\log N}}$) by refining the density increment itself by proving that large Fourier coefficients with a nice additive structure imply the density increment in some large arithmetic progression.

Theorem 1. *Let N be sufficiently large. If $A\subset\{1,\dots,N\}$ has no solutions to $a-b=n^2$ with $a,b\in A$ and $n\geq 1$, then*

$$|A|\ll\frac{N}{(\log N)^{c\log\log\log N}}$$

for some absolute constant $c>0$.

Lemma 2 (Large Fourier coefficents with the same denominator give density increment)**.** *Let $\nu,\alpha\in(0,1]$ and let $N,K,q\geq 1$ be such that $K<\frac{N}{2}$ and $\frac{\nu\alpha N}{(Kq^2)}$ is sufficiently large. Let $A\subset[N]$ be a set with no non-zero square differences and density $\alpha=|A|/N$, and*

$$\sum_{\frac{a}{q}\in\mathbb{Q}_{=q}}\int_{\mathfrak{M}(\frac{a}{q};N,K)}|\widehat{1_A}(\gamma)-\alpha\widehat{1_{[N]}}(\gamma)|^2d\gamma\geq\nu\alpha|A|.$$

Then there exists $N'\gg\frac{\nu\alpha N}{Kq^2}$ and a set $A'\subset[N']$ with no non-zero square differences such that the density $\alpha'=|A'|/N'$ satisfies

$$\alpha'\geq(1+\frac{\nu}{5})\alpha.$$

There was also a recent significant improvement $|A|\ll Ne^{-c\sqrt{\log N}}$ by Ben Green and Mehtaab Sawhney [6] obtained by making use of a global hypercontractivity result for functions defined on finite abelian groups, which is also in the language of Fourier analysis.

Question 3. *Can we improve the upper bound by taking a different approach?*

Since the most meaningful improvements for the upper bound have mostly been proved from refinements of Sárcözy's idea involving Fourier analysis, approaches from a completely different point of view, such as combinatorial methods, can be applied to discover a different scheme for improving the bound. As an example, Terrence Tao [5] has also suggested that a Fourier-free approach for the upper bound is possible in his blog, but with a looser bound.

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The Lower bounds for (Multicut) Mimicking Network

Kyungjin Cho

Mimicking problem for (G, T) aims to find a minor graph of G maintaining the minimum edge cut size between two parts $(A, T - A)$ for any $A \subset T$. Additionally, in the context of separating terminals into more than two parts, there exists a corresponding graph sparsification problem known as the *multicut-mimicking problem*. A *multicut-mimicking network* for terminals T in a normal graph G is a minor graph that preserves the size of the minimum multicut of any set of cut requests over T. Equivalently, it preserves the size of the minimum multiway cut of any partition of T.

It is already known that there exists a planar graph G and terminal set $T \subset V(G)$ where every mimicking network of (G, T) has at least $2^{|T|-2}$ edges [1]. Especially, there is a mimicking network of (G, T) with $\Theta(|T|^2)$ edges for any planar graph G and terminals $T \subset V(G)$ if terminals in T are located on the same face, which is optimal [2, 3]. However, parameterized by $k = \sum_{t \in T} \deg_G(t)$, it was represented that an instance (G, T) has a mimicking network with at most $O(k^4)$ edges for a general hypergraph G [4, 5]. Furthermore, there is a multicut-mimicking network with $k^{O(\log k)}$ hyperedges if G is a normal graph (not hypergraph) [6].

Problem 1. *Is there an instance (G, T) so that every multicut-mimicking network has at least $k^{\Omega(r)}$ hyperedges, where r is the rank of G which is the maximum cardinality of an hyperedge in G?*

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What I left behind at graduate student era

Taehyun Eom

Dennis Stanton's homework

At first, if you have any interest in the enumerative combinatorics, you'd better to check the Dennis Stanton's document "Some Problems"

Area statistic for defective boards

For a given lattice path in a rectangle board, we may define area statistic as the number of squares that exists on the upper left side of the path. This area statistic defines Gaussian polynomials, which is the q-analog of binomial coefficients. Then, suppose some tiles of the lattice are defective, so we do not count given tiles as area. Now, the question is simple:

Is it possible to detect the position of defective tiles by comparing generating functions from given area statistic and defective area statistic?

However, actually, for square board cases, there exist some trivial cases that have same generating function but different defect positions by diagonal based reflection.

So we need to change the question:

When two distinct defect tiles give the same generating functions?

At first, if there exists only one defect, the trivial reflective case above is only such a case. But from two defects, there exist some non-trivial cases.

For example, $(3k+2)\times 3m$ and $3k\times 3m$ boards have some special cases, which can be proved by involution. Also, 2×6 and 2×7 boards have also have new sets of defects give same generating functions. But it is not done that there exist no other cases.

Also, it seems that there exist some patterns for the number of exceptional cases for 3 or more defects, but nothing is proven rigorously.

If you have any interest, I can provide a document with 23 pages.

Bijective proof for the unimodality (γ-positivity) of the descent statistic of involutions

For descent statistic on involutions in S_n, it is already known that if we write its generating function into

$$\sum_{\iota} q^{\mathrm{des}(\iota)} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} r_i q^i (1+q)^{n-2i-1}$$

form, then each r_i is non-negative integers. Is it possible to find a bijective proof of this?

If you have any interest, I can provide a document with 8 pages.

Forbidden sub-co-walks

Consider some password-making rules, such as forbidding consecutive letter sequence or repeated letter sequence. If we want to count the number of passwords which respect such rules, we may consider a directed graph which defines the next letter or the same letter. Then, the number of such password is the number of vertex sequences without walks of certain length.

From technical reason, we will consider the directed graph D as the complement of given graph and restrict the length of co-walk. We will use De Bruijn technique to count such vertex sequences.

For given D, we may define $\mathcal{L}^{(k)}(D)$ be the direct graph where vertex sequences $v_0 \cdots v_k$ are vertices and arcs $(u_0 \cdots u_k, v_0 \cdots v_k)$ where $u_0 \sim v_0 = u_1 \sim v_1 \cdots v_{k-1} = u_k \sim v_k$. This is indeed, generalized line digraph. Lastly, let $\theta_m^{(k)}(D)$ be the number of length $m+1$ vertex sequences where the length of co-walk is bounded by k. From its definition, $\theta_m^{(0)}(D)$ is nothing but the number of walks.

Then, if we define $\delta^{(k)}(D) = \mathcal{L}^{(k)}(J_{|V(D)|}) - \mathcal{L}^{(k)}(J_{|V(D)|} - D)$, we have

$$1_{n^{k+1}}^T \delta^{(k)}(D)^m 1_{n^{k+1}} = \theta_{m+k}^{(k)}(D).$$

Hence, the number of such passwords is strongly related to $\rho(\delta^{(k)}(D))$ where ρ is the spectral radius. This is based on Perron-Frobenius theorem.

Also, since $\delta^{(k)}$ has too many vertices, we can consider some reductions $\tilde{\delta}^{(k)}$ which do not see full sequence, but only some end parts.

Moreover, we may consider some probabilistic approximation. The number of sequences is not Markov chain because it fails to be independent, but if we assume it to be independent we only need $|V(D)| \times |V(D)|$ matrix to analyze it. Without long explanation, it is equivalent to consider a diagonal matrix Q where each diagonal element is $q_a = 1 - \frac{1_n^T (J_n - D)^k e_a}{n^k}$ and define

$$\hat{\delta}^{(k)}(D) = D + (I_n - Q)(J_n - D) = J_n - Q(J_n - D)$$

With these definitions, we may consider many questions.

- $\mathcal{L}^{(k+1)}(D)$ is isomorphic to $\mathcal{L}^{(1)}(\mathcal{L}^{(k)}(D))$ except for some isolated vertices. Then, is it possible to define a proper operator for δ works similarly to this?

- Is it possible to define η satisfies $\rho(\eta^k(D)) = \rho(\delta^{(k)}(D))$?

- Can δ be defined for multigraphs also?

- How about both sub-work and sub-co-works are bounded? What if both direction and reversed direction? Since we may think D as 2-edge-colored graphs, so what if we generalize it as multi-colored cases? How about closed walks? How about binomial coefficients and inclusion-exclusion principle?

- If $D \neq 0$ then we can prove that

$$\rho(\delta^{(k)}(D)) = \lim_{m\to\infty} \theta_m^{(k)}(D)^{1/m} \geq n^{1-\frac{2}{k+1}}.$$

  Hence, for any f such that $\lim_{m\to\infty} f(m) = \infty$, we have $\theta_m^{(f(m))}(D)^{1/m} \to n$. Then, is it possible to find some polynomial bounds p_k (may depend on D) satisfies

$$\frac{\theta_{m+k}^{(k)}(D)}{\rho(\delta^{(k)}(D))^m} \leq p_k(m)$$

  where $p_{f(m)}(m)^{1/m} \to 1$? For irreducible or eventually irreducible D? For some good conditions on f? Is such polynomials possible to be not depend on k?

- From above bound, we may consider a normalization

$$\left[\frac{\rho(\delta^{(k)}(D))}{n^{1-\frac{2}{k+1}}}\right]^{\frac{k+1}{2}}.$$

  How about the limit of this value?

- If D is an outer-degree-regular graph, then we can prove that

$$\rho(\delta^{(k)}(D)) = \rho\left(\begin{bmatrix} d & d & \cdots & d \\ n-d & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & n-d & 0 \end{bmatrix}\right).$$

  What is the exact formula for this value? For general graph, if we take d as the mean outer degree, then can this approximate real value efficiently?

- If $\delta^{(k)}(D)$ is irreducible, then so is $\delta^{(k+1)}(D)$. Also, this irreducibility strongly related to have a source or sink. For reduction $\tilde{\delta}^{(k)}(D)$, how about its irreducibility? How about ranks? Also, is there other strong reductions rather than for the case outer-degree-regularity?

- Is $\rho(\hat{\delta}^{(k)}(D))$ really good approximation to $\rho(\delta^{(k)}(D))$? How about the expectation $E[\rho(\delta^{(k)}(D))]$ for random binary matrix? Is there some meaningful limit distribution?

- We also can define $\hat{\delta}^{(k)}(D)$ by 1) make blocks on $\delta^{(k)}(D)$, 2) count the number of 1 in each block, 3) replace block to $\frac{\#}{n^k}$. From this, we can ask the following question: For equidivided block matrix B, is it possible to make good approximation when computing ρ, by take ρ for each block and then take ρ again?

- From above block-ρ computation, it derives other question. Easily, we can check for binary $n \times n$ matrix with exactly one 1, $E[\rho] = \frac{1}{n}$. Also, for two 1's, $E[\rho] = \frac{2}{n}$. Lastly, for three 1's, we have
$$E[\rho] = \frac{3}{n}\left(1 - \frac{3-\sqrt{5}}{(n+1)(n^2-2)}\right).$$
From this, for k 1's, is this true that $nE[\rho] \to k$ as $n \to \infty$?

If you have any interest for detail, you may check this presentation file.

Nonexistence of Rogers-Ramanujan like identity, beyond the Alder

q-Pochhammer symbol is defined as
$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

With this symbol, our famous Rogers-Ramanujan identities are
$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$
and
$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

Also, Euler proved that
$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+2k-1)}{2}}}{(q;q)_n} = \frac{1}{(q^k;q)_k (q^{2k+1};q^2)_\infty}$$
for $k \geq 1$ and it is easy to check
$$\sum_{n=0}^{\infty} \frac{q^{kn}}{(q;q)_n} = \frac{1}{(q^k;q)_\infty}.$$

From [1], Alder proved that there exists no other proper degree 2 polynomial p satisfies

$$\sum_{n=0}^{\infty} \frac{q^{p(n)}}{(q;q)_n} = \frac{1}{(1-q^{a_0})(1-q^{a_1})(1-q^{a_2})\cdots}$$

and from [2], there exist some polynomials $G_{k,n}(q)$ such that

$$\frac{1}{(q;q^{2k+1})_\infty \cdots (q^{k-1};q^{2k+1})_\infty (q^{k+2};q^{2k+1})_\infty \cdots (q^{2k};q^{2k+1})_\infty} = \sum_{n=0}^{\infty} \frac{G_{k,n}(q)}{(q;q)_n}$$

and

$$\frac{1}{(q^2;q^{2k+1})_\infty \cdots (q^{2k-1};q^{2k+1})_\infty} = \sum_{n=0}^{\infty} \frac{G_{k,n}(q)q^n}{(q;q)_n}$$

Then, we may ask that for an integer-valued polynomial $p(n)$ with degree at least 3, how

$$\sum_{n=0}^{\infty} \frac{q^{p(n)}}{(q;q)_n}$$

be. Here, this generating function has combinatorial meaning if given polynomial is convexly strictly increasing which means

$$0 = p(0) < p(1) < p(2) < \cdots$$

and

$$p(1) - p(0) \le p(2) - p(1) \le \cdots .$$

Note that this condition does not affect polynomials of degree 3, and with a computer, we can check that there exists some patterns for degree 3 polynomials, so it seems that there exist no sequences satisfies

$$\sum_{n=0}^{\infty} \frac{q^{p(n)}}{(q;q)_n} = \frac{1}{(1-q^{a_0})(1-q^{a_1})(1-q^{a_2})\cdots}.$$

Can we prove this rigorously, for any polynomial with degree at least 3?

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Mahjong Waiting Structure Theorem

We may consider mathematical Mahjong as following way:

- We have a set of tiles.
- Blocks are multisets of tiles.
- Categories are sets of blocks.
- For categories $C_1, \cdots$, we may define winning index $c_1, \cdots$ as non-negative integers.

Then, winning hand is a multiset of tiles admits a block decomposition such that the number of block in category C_i is exactly c_i.

For usual Mahjong, we have two categories; the first category contains blocks such as

$$111, 123, 222, 234, 333, 345, \cdots$$

and the second category contains

$$11, 22, 33, \cdots .$$

Then, winning index is $(\bullet, 1)$ where Taiwanese Mahjong admits $(5, 1)$ and other Mahjongs usually admits $(4, 1)$.

In Mahjong, player's hand have two phases; one is $n-1$ tile phase and other is n tile phase. Basically, player's hand is in $n-1$ phase, and each turn, player take one tile so it temporarily becomes n phase. If it is a winning hand, then player wins. If not, player discard a tile so hand returns to $n-1$ phase.

Therefore, it is important to analyze the hand in $n-1$ phase. Here, we may define ready hand and ready block decompositions, waiting block and waiting tiles easily. Now, general Mahjong has tiles only 1 to 9, but we may extend it to any integers.

From the structure of blocks and categories, one hand may admit several ready block decomposition simultaneously so it has very many waiting tiles.

For example, 1112345678999 admits

- 11/123/456/789/99→1,9
- 111/23/456/789/99→1,4
- 111/234/56/789/99→4,7

- 111/234/567/89/99→7,(10)
- 111/234/567/8/999→8
- 111/234/5/678/999→5
- 111/2/345/678/999→2
- 11/12/345/678/999→(0),3
- 11/123/45/678/999→3,6
- 11/123/456/78/999→6,9

Among these ready block decompositions, we may define minimal conversions, such as 11/123 ⇔ 111/23.

We can make a list of minimal converions contains 1 as the minimum as follows.

- 11/xx ⇔ xx/11 (exchange waiting block and complete block)
- 1/111 ⇔ 11/11
- 1/123 ⇔ 23/11, 2/123 ⇔ 13/22, 3/123 ⇔ 12/33
- 11/123 ⇔ 23/111, 22/123 ⇔ 13/222, 33/123 ⇔ 12/333
- 1/234 ⇔ 4/123
- 1/222 ⇔ 12/22, 2/111 ⇔ 12/11, 1/333 ⇔ 13/33, 3/111 ⇔ 13/11
- 13/234 ⇔ 34/123, 12/345 ⇔ 45/123, 12/234 ⇔ 24/123
- 1/123/123 ⇔ 22/33/111, 2/123/123 ⇔ 11/33/222, 3/123/123 ⇔ 11/22/333
- 34/11/234 ⇔ 13/44/123, 24/11/234 ⇔ 12/44/123
- 11/234/234 ⇔ 44/123/123
- 12/123/123 ⇔ 33/111/222, 13/123/123 ⇔ 22/111/333, 23/123/123 ⇔ 11/222/333
- 111/222/333 ⇔ 123/123/123
- 1/345/456/456 ⇔ 13/66/444/555, 6/123/123/234 ⇔ 46/11/222/333

and conversion schemes

- 1/345/678/$\cdots$/wxy/zzz $\Leftrightarrow$ 13/zz/456/789/$\cdots$/xyz
- z/111/234/$\cdots$/vwx $\Leftrightarrow$ xz/11/123/$\cdots$/uvw
- 11/123/456/$\cdots$/wxy/zzz $\Leftrightarrow$ zz/111/234/567/$\cdots$/xyz

Is this list the full list of minimal conversions?

Also, if we make some cyclic structure on mahjong, which means 891,912 can be also considered as a block where we only use tiles 1 to 9. In this case we also have some minimal conversions such as 123/456/789 $\Leftrightarrow$ 234/567/891. Then, how the list extended?

A variation of the NIM Game

Suppose that there exist some piles of objects. Now, each player choose a nonempty set of nonempty piles, then take one object from each chosen pile. Then, who has the winning strategy for each initial state and what is the winning strategy?

About Representability

For a multiset X of positive integers, a tuple $a_1, \cdots, a_k$ is representable by X if there exists a partition $A_1, \cdots, A_k, A_\infty$ of X such that a_i is the sum of elements in A_i. In other words, if $S(A)$ is the sum of elements, then $S(A_i) = a_i$.

Now, X is k-representable if every k-tuple $a_1, \cdots, a_k$ with $\sum a_i \leq S(X)$ is reprensetable.

Then, we have the following.

- If $(a_1, \cdots, a_k)$ is representable, then so is its refinements.
- If X is k-representable, $mk \leq S(X) \leq m(k+1)$ for some m, Y is 1-representable, and $S(Y) = m$. Then, $X + Y$ is $(k+1)$-representable.
- Let $\max X = M$. Then, if $X - \{M\}$ is k-representable and $S(X - \{M\}) \geq (M-1)k$, then X is k-representable.
- Suppose X is consist with $1 = x_0 < x_1 < \cdots < x_n$ where each x_i has multiplicity m_i. Then, X is k-representable if $m_i \geq k(\lceil \frac{x_{i+1}}{x_i} \rceil - 1)$ for every i.

Now, is there any way to determine given set is k-representable or not completely?

Convex Fair Partitions of Convex Polygons

Han Sanghwa

A *convex fair partition* of a convex polygon is a division of the polygon into n convex pieces such that all parts have equal area and equal perimeter. This elegant geometric problem was first proposed by Nandakumar and Ramana Rao in 2007, and since then it has stimulated rich mathematical exploration across discrete geometry, topology, and algorithmic partitioning.

Formally, given a convex polygon P and a positive integer n, can we partition P into n convex pieces $P_1, \ldots, P_n$ such that for all i, j, we have

$$\text{Area}(P_i) = \text{Area}(P_j), \quad \text{Perimeter}(P_i) = \text{Perimeter}(P_j)?$$

This is the **Convex Fair Partition Problem**.

The problem is trivially solvable when $n = 2$ by a straight-line cut and has been proven solvable for $n = 2^k$ (powers of two) by topological methods. In particular, the $n = 3$ case was resolved using equivariant topology techniques [1]. Moreover, the general case when $n = p^k$ for prime p has also been settled [2].

However, it remains an open problem whether a convex fair partition exists for *all* positive integers n.

Open problem. Does every convex polygon admit a convex fair partition into n pieces for every positive integer n?

Recent progress. A recent algorithmic approach by Campillo et al. (2024) proposes a numerical method for constructing convex fair partitions for arbitrary n. Their algorithm uses an initial centroidal Voronoi tessellation followed by iterative flow-based correction to balance both area and perimeter across all regions [3]. Though experimentally effective for $n \leq 50$, this method does not provide a formal existence proof.

Potential directions.

- Is there an existence proof for arbitrary n, possibly via equivariant topology or measure theory?
- Can one minimize the total cut length among all fair convex partitions?

- What happens in higher dimensions: is there a fair convex partition into *n* convex polyhedra of equal volume and surface area?

Applications. The problem lies at the intersection of fair division, computational geometry, and convex analysis. Potential applications range from territory subdivision in geographic systems to load balancing in spatial networks.

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Quantum Walks

Musung Kang

The time-independent Schrödinger equation is given by

$$\frac{dT(t)}{dt} = -i\hbar^{-1}BT(t),$$

where B is a constant matrix. For $B \in \{A(G), L(G)\}$, we consider B a *graph Adjacency or Laplacian matrix*, representing two possible choices:

$$A(G) := (\text{Adjacency matrix of undirected loopless graph G}), \textit{ and}$$
$$L(G) := (\text{Laplacian matrix of graph } G).$$

Let X be a graph with n vertices and see

$$H_X(t) := \exp(iBt)$$

which is a solution of the Schrödinger equation up to a constant time dilation. A one-variable (semi)group becomes a quantum physical density matrix in a given time. If there exists $t > 0$ such that $|e_i^T H_X(t) e_i| = 1$ for every $1 \leq i \leq n$, then X is called a *periodic graph*.

Problem 1. *Find a class of periodic on both the Adjacency and the Laplacian matrices.*

For example, the smallest non-trivial graph P_2 is periodic on both the Adjacency and the Laplacian matrices. Moreover, a general problem is

Problem 2. *Is there a canonical definition for continuous quantum walks on Hypergraphs?*

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Shortest Path Reconfiguration **for unit disk graphs**

Gunwoo Kim

Let G be an undirected graph and $s, t \in V(G)$. We say that two s-t shortest paths P and Q are *reconfigurable* in G if there is a sequence of s-t shortest paths $(P_0, P_1, \ldots, P_k)$ for $k \in \mathbb{N}$, where $P_0 = P$ and $P_k = Q$ such that P_{i-1} and P_i for $i \in [k]$ differ only in one vertex.

> Shortest Path Reconfiguration
> **Input:** A graph $G = (V, E)$ and two shortest paths P and Q.
> **Question:** Are P and Q reconfigurable in G?

Combinatorial reconfiguration problems have received attention in the last decade. It asks whether we can reach a desired configuration from an initial one without breaking a desired condition. For instance, there are reconfiguration problems in the context of satisfiability problems [2], independent sets [3, 5], vertex colorings [4], and matchings [5].

Shortest Path Reconfiguration is PSPACE-complete in general [1] and remains so even when restricted to bipartite graphs [1], and graphs of bounded bandwidth, pathwidth, and treewidth [9]. The problem is W[1]-hard when parameterized by the length of the shortest s-t paths (even when restricted to bounded degereracy graphs) [10]. However, the problem is FPT when parameterized by the treedepth, the cluster deletion number, the modular-width of the input graph [10]. Surprisingly, there is a polynomial time algorithm for certain graph classes such as planar graphs [6], grid [7], claw-free graphs and chordal graphs [1], permutation graphs, circle graphs, circular-arc graphs [8]. Hence, we may pose the following question.

Question 1. *Is there a polynomial time algorithm (or an approximation algorithm) for unit disk graphs?*

This problem is particularly interesting since it captures the rerouting problem across a network of telecommunication towers. Each tower has a coverage radius, and two towers interact if they are within the coverage radius of each other.

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Cluster the points

Hyungu Kim

Given a set of finite points S, we denote the *convex hull* of S, the smallest convex set containing S, by $\mathrm{conv}(S)$.

In $\mathbb{R}^2$, we are given a set of six points on the unit circle, say, $S = \{a, b, c, d, e, f\}$ in clockwise order. If the lines ad, be and cf do not intersect at a single point, we may choose a partition of S into three pairs S_1, S_2 and S_3 such that

$$P = \bigcup \mathrm{conv}(S_i \cup S_{i+1})$$

(where we interpret indicies modulo 3) divides $\mathbb{R}^2$ into a bounded triangular region and an unbounded region. Either $S_1 = \{a, b\}, S_2 = \{c, d\} S_3 = \{e, f\}$ or $S_1 = \{b, c\}, S_2 = \{d, e\} S_3 = \{f, a\}$ will work.

In some sense, we partitioned the six points into three clusters where each cluster works like a vertex of 2-dimensional polytope, i.e. triangle.

Problem 1. *Given integer d and a $d(d+1)$ point set S in $\mathbb{R}^d$, under what conditions on S can we make a partition $S_1, \dots, S_{d+1}$ of S such that $|S_i| = d$ for all i and*

$$P = \bigcup_{I \subsetneq [d+1]} \mathrm{conv}\left(\bigcup_{i \in I} S_i\right)$$

divides $\mathbb{R}^d$ into one d-dimensional bounded region and one unbounded region?

There is a configuration of seven points in $\mathbb{R}^2$ where we can use any six of them as our set S to form such triagle-like partition: a regular heptagon. [1]

Problem 2. *Can we find a set of $d(d+1)+1$ points S' in $\mathbb{R}^d$ such that we can use any $d(d+1)$ point subset as our set S in the previous problem?*

Problem 3. *Can we also make all the faces behave well? i.e., for each integer $2 \le k \le d$,*

$$H_{k-1}\left(\bigcup_{I \subsetneq K} \mathrm{conv}\left(\bigcup_{i \in I} S_i\right)\right) \neq 0$$

for every $K \subset [d+1]$ with $|K| = k$?

Problem 4. *Can we do the same for $n(d+1)+k$ points for integers n and k? i.e., find a set S' of $n(d+1)+k$ points such that for every subset S with $|S| = n(d+1)$, we can find a partition $S_1, \dots, S_{d+1}$ with $|S_i| = n$, satisfying above conditions?*

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Hat guessing number of graphs

Jeewon Kim

Let G be a finite, simple graph. In the *hat guessing game*, each vertex of G is assigned a hat color from a set of q colors. Each player sees only the colors of its neighbors' hats and, without communication, simultaneously guesses its own color according to their predetermined strategy. The players win if at least one guess is correct on every assignment. The *hat guessing number* $HG(G)$ is the maximum q where players have a winning strategy.

Exact Hat Guessing Numbers

First, we can show $HG(K_n) = n$ with simple probabilistic method. The following are some graphs with known hat guessing numbers.

Theorem 1 (Feige [9])**.**

$$HG(T) = 2 \quad \textit{for every tree } T \textit{ with at least 2 vertices}$$

Theorem 2 (Szczechla [3])**.** *For all $n \geq 3$,*

$$HG(C_n) = \begin{cases} 3, & 3 \mid n \textit{ or } n = 4, \\ 2, & \textit{otherwise.} \end{cases}$$

Definition 3. A cactus graph is a connected graph in which every pair of cycles share at most one vertex.

Theorem 4 (Chizewer et al. [12])**.** *Let G be a cactus graph.*

$$HG(G) = \begin{cases} 4, & G \textit{ contains at least two triangles,} \\ 3, & \textit{Not case 1 and } G \textit{ contains at least two cycles or a cycle of length 4 or divisible by 3,} \\ 2, & G \textit{ is a pseudotree with at least one edge and no cycle of length 4 or divisible by 3} \end{cases}$$

Theorem 5 (Konstantin, Latyshev [15])**.** *The above theorem exactly classifies all graphs with $HG(G) \leq 2$.*

Definition 6. The *book graph $B_{d,n}$* is obtained by adding n nonadjacent common neighbors to the complete graph K_d. The *windmill graph $W_{k,n}$* is defined as n disjoint copies of K_k glued together at a single vertex.

Theorem 7 (He, Ido, Przybocki [5]). *For sufficiently large n,*

$$HG(B_{d,n}) = 1 + \sum_{i=1}^{d} i^i, \quad HG(W_{k,n}) = \begin{cases} 2k-2, & n \geq \lceil \log_2(2k-2) \rceil, \\ d^n, & k = d^n - d^{n-1} + 1. \end{cases}$$

Problem 8. *Figure out the exact hat guessing number or any meaningful bound for another class of graphs.*

Complete Bipartites

Theorem 9 (Gadouleau, Georgiou [11]).

$$HG(K_{m,n}) \leq \min(m+1, n+1), \; HG(K_{q-1,(q-1)^{q-1}}) \geq q$$

This implies $\Omega(\log n) = HG(K_{n,n}) \leq n+1$.

Theorem 10 (Alon et al. [1]). *For $r \geq 2$, the complete r-partite graph $K_{n,\ldots,n}$ satisfies*

$$HG\big(K_{n,\ldots,n}\big) \; = \; \Omega\big(n^{\frac{r-1}{r} - o(1)}\big)$$

This implies $\Omega(n^{\frac{1}{2} - o(1)}) = HG(K_{n,n})$.

In [1], the authors present results on many special cases of graphs. Their main strategy is to convert the strategies into polynomials over finite fields and apply the Combinatorial Nullstellensatz to ensure a non-vanishing point which implies losing strategy:

The guessing strategy of v_i is a function f_i, whose variables are the colors assigned to the neighbors of v_i. Consequently, we can write $f_i = f_i(x) = f_i(x_{i_1}, \ldots, x_{i_{d_i}})$, where $v_{i_1}, \ldots, v_{i_{d_i}}$ are the neighbors of v_i. One can easily see that a vertex v_i guesses its color correctly if and only if $x_i - f_i = 0$, and $f := (f_1, \ldots, f_n)$ forms a proper guessing strategy for G if and only if the function

$$F(x) = \prod_{i=1}^{n} (x_i - f_i)$$

vanishes on Q^n, where for $Q = [q]$, f_i is an $\mathbb{R}$-valued function with d_i variables, and for a prime power q and $Q = \mathbb{F}_q$, f_i is an $\mathbb{F}_q$-valued function with d_i variables.

Still, the asymptotic behavior of $HG(K_{n,n})$ is still not known exactly.

Question 11. *$HG(K_{n,n}) = \Omega(n^{1/2})$? or $HG(K_{n,n}) = o(n^{1/2})$?*

Degree and degeneracy

The following theroem is an application of Lovász Local Lemma.

Theorem 12 (Folklore, proof in [1]). *If Δ is the maximum degree of G, then*

$$HG(G) \leq e\,\Delta$$

We also have lower bounds with maximum degree.

Theorem 13 (Latyshev, Kokhas [18]). *For any positive integer k there exists a graph G such that*

$$HG(G) = \Delta(G) + k$$

Theorem 14 (Latyshev, Kokhas [18]). *There exists a sequence of graphs G_n such that*

$$\Delta(G_n) \to \infty,\ HG(G_n)/\Delta(G_n) = 4/3$$

Problem 15. *Reduce the $4/3$ and e gap.*

Problems about minimum degree is not studied well.

Question 16 (Alon et al. [1]). *Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ s.t. if G has minimum degree at least $f(d)$, then $HG(G) \geq d$?*

What about d-degenerate graphs?

Definition 17 (He, Li [6]). Let $G_d(N)$ to be the graph obtained from the complete rooted N-ary tree with depth d, by drawing an edge between every vertex and each of its ancestors (if the edge does not exist already).

Ordering the vertices by their original depth, we see that $G_d(N)$ is d-degenerate.

Definition 18. *Sylvester's sequence* $(s_n)_{n=0}^{\infty}$ is defined by $s_0 = 2$ and $s_n = s_0 \cdots s_{n-1} + 1 = s_{n-1}^2 - s_{n-1} + 1$ for $n \geq 1$, satisfies $s_n = \lfloor E^{2^{n+1}} + \frac{1}{2} \rfloor$ for some $E \approx 1.264$.

Theorem 19 (He, Li [6]). *For any $d \geq 1$ and all N sufficiently large in terms of d,*

$$HG(G_d(N)) = s_d - 1$$

Conjecture 20. *There exists $f : \mathbb{N} \to \mathbb{N}$ such that every d-degenerate graph G satisfies*

$$HG(G) \ \leq \ f(d)$$

This would imply that the hat guessing number of any graph can be upper bounded by a function of its Hadwiger number (the order of a largest clique minor in G), as graphs with Hadwiger number k are known to be $O(k\sqrt{logk})$-degenerate. Look at [7].

Actually, we have a stronger conjecture with Hadwiger numbers.

Conjecture 21 (Bosek et al.[2]). *Let $h(G)$ denote the Hadwiger number of G. Then every graph G satisfies*

$$HG(G) \leq h(G)$$

We don't know yet about degeneracy, but we do have bounds for "strong degeneracy".

Definition 22 (Knierim et al. [17]). Let $d \geq 1$ be an integer and G a graph. We say that a vertex $v \in V(G)$ is *d-removable in G* if $d_G(v) \leq d$ and if $|\{w \in N_G(v) | d_G(w) > d\}| \leq 1$. We say that a graph G is *strongly d-degenerate* if every non-empty subgraph G' of G contains a vertex which is d-removable in G'.

Theorem 23 (Knierim et al. [17]). *Every strongly d-degenerate graph G satisfies $HG(G) \leq (2d)^d$.*

Planar graphs

It is natural to expect small hat guessing numbers for planar graphs, since they are 5-degenerate. Here are some results on planar graphs. However, it is not known a lot.

Conjecture 24. *Hat guessing number of planar graphs is bounded above by some universal constant.*

However, there is a lower bound for planar graphs, and there is an upper bound for some subclasses of planar graphs.

Theorem 25 (Knierim et al. [17]). *Every outerplanar graph G satisfies*

$$HG(G) \leq 40$$

This bound is better than simply using the fact that outerplanar graphs are strongly 4-degenerate along with the bound on strong degenerate graphs.

Definition 26 (Bradshaw [4]). Layered planar graphs ar planar graphs that can be obtained by beginning with a 2-connected outerplanar graph G_1, and then for $1 \leq i \leq \tau$, adding a 2-connected outerplanar graph G_{i+1} to some interior face of G_i and adding non-crossing edges between G_i and G_{i+1}. We also use the term *layered planar graph* to describe a subgraph of such a planar graph.

Theorem 27 (Bradshaw [4]). *If G is a layered planar graph, then*

$$\log_2 \log_2 \log_2 \log_2 \log_2 HG(G) < 149$$

Problem 28. *Find another subclass of planar graphs with bounded hat guessing number.*

In [8], the authors construct a planar graph to obtain a lower bound of hat guessing number for planar graphs.

Theorem 29 (Latyshev, Kokhas [8]). *There exists a planar graph G such that*

$$HG(G) = 22$$

Their construction has 546 vertices.

Problem 30. *Construct a planar graph G with $HG(G) > 22$.*

Random Graphs

Theorem 31 (Alon, Chizewer [7]). *With high probability, the random graph $G \sim \mathcal{G}(n, 1/2)$ satisfies*

$$n^{1-o(1)} \leq HG(G)$$

The authors use the result from [5] and use the fact that the random graph contains a a sufficiently large book graph as a subgraph with high probability.

Before this result, the best known lower bound was $(2 + o(1)) \log_2 n$, which is trivial since almost all graphs contain a clique of size $(2 + o(1)) \log_2 n$.

Theorem 32 (Wang, Yaojun [14]). *Let $0 < p < 1$ be a constant. With high probability, the random graph $G \sim \mathcal{G}(n, p)$ satisfies*

$$n^{1-o(1)} \leq HG(G) \leq (1 - o(1))n$$

For the lower bound, they improve the technique from [7]. For the upper bound, they use the relation between chromatic number and hat guessing number, and some known bounds of chromatic number of random graphs.

Question 33 (Alon, Chizewer [7]). *When $G \sim \mathcal{G}(n, 1/2)$, is $HG(G) = o(n)$ with high probability?*

Others

There are not much literature on algorithms or complexity of computing $HG(G)$. It is not even known if computing $HG(G)$ is NP-hard. There is no known approximation algorithm for $HG(G)$.

2025/07/30, I.M.J. McInnis just released a senior thesis *Slavic Techniques for Hat Guessing Algorithms* [16]. The author uses some variants of hat guessing games to obtain algorithmic results on the hat guessing game.

There are some generalizations to directed graphs [10], [11], allowing multiple guesses [2] [8] [4], fractional hat guessing numbers [13], and unbalanced hat guessing numbers (hatness function)[19],[12].

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Unavoidable substructures in large strongly connected digraphs

Seokbeom Kim

The well-known Ramsey's theorem states the following.

Theorem 1 ([6])**.** *For any integer $n \geq 1$, there is an integer $N \geq 1$ satisfying the following. Every graph with at least N vertices contains a clique or a stable set of size n.*

Motivated by this result, there have been extensive studies to find 'unavoidable' substructures of graphs with a desired property: That is, what can we say more if we assume more structural properties?

Let us state this question more rigorously. A *graph property* $\mathcal{P}$ is a class of graphs that is closed under isomorphisms (that is, if $G \in \mathcal{P}$ and G' is isomorphic to G, then $G' \in \mathcal{P}$) *and has infinitely isomorphism types* (this is not a standard definition, but we add this condition as we are not interested in graph properties without this condition). Given two graph properties $\mathcal{P}$, $\mathcal{P}'$ and a graph containment relation $\preceq$ on graphs, we write $\mathcal{P} \preceq \mathcal{P}'$ if for every $G \in \mathcal{P}$ there is $G' \in \mathcal{P}$ such that $G \preceq G'$. If $\mathcal{P} \preceq \mathcal{P}'$ as well as $\mathcal{P}' \preceq \mathcal{P}$, we say $\mathcal{P}$ and $\mathcal{P}'$ are *equivalent* and write $\mathcal{P} \equiv \mathcal{P}'$.

Given a graph property $\mathcal{P}$ and a graph containment relation $\preceq$, we say a finite set $\{\mathcal{U}_1, \ldots, \mathcal{U}_t\}$ of graph properties with $\mathcal{U}_i \subseteq \mathcal{P}$ is $(\mathcal{P}, \preceq)$-*unavoidable* (or simply *unavoidable*) if:

- $\mathcal{U}_i \not\preceq \mathcal{U}_j$ whenever $i \neq j$; and
- for every graph property $\mathcal{Q} \subseteq \mathcal{P}$, there is i such that $\mathcal{U}_i \preceq \mathcal{Q}$.

For instance, if we let $\mathcal{K} = \{K_t : t \geq 1\}$ and $\overline{\mathcal{K}} = \{\overline{K_t} : t \geq 1\}$, then Ramsey's theorem says that $\{\mathcal{K}, \overline{\mathcal{K}}\}$ is a $(\mathcal{P}, \preceq)$-unavoidable set when $\mathcal{P}$ is the family of all graphs and $\preceq$ is a subgraph relation. Observe that the unavoidable set is unique up to equivalence, that is, if $\{\mathcal{U}_1, \ldots, \mathcal{U}_t\}$ and $\{\mathcal{V}_1, \ldots, \mathcal{V}_s\}$ are two unavoidable sets, then $t = s$ and, by possibly reindexing, we have $\mathcal{U}_i \equiv \mathcal{V}_i$ for each $i = 1, \ldots, t$.

By using previously defined notions, we can state the main question as follows.

Question 2. *Given a graph property $\mathcal{P}$ and a graph containment relation $\preceq$, what is the $(\mathcal{P}, \preceq)$-unavoidable set?*

Question 2 has been answered for:

- connected graphs and induced subgraph relation [3, Proposition 9.4.1];
- 2-connected graphs and the topological minor relation [3, Proposition 9.4.2];
- 3-connected graphs and the minor relation [5];
- 4-connected graphs and the minor relation [5];
- connected and co-connected graphs and the induced subgraph relation [2]; and
- prime graphs and the induced subgraph relation [1].

Note that most of the previous research focused on the connectivity of graphs.

Naturally, one might ask the analogous questions for digraphs. Indeed, by using Ramsey's theorem (and possibly Stearns' theorem [7]), it can easily be deduced that every digraph with a large number of vertices contains an edgeless digraph, a bidirected complete graph, or a transitively oriented complete graph as a subdigraph. Thus, it is natural to ask the following question.

A digraph is *strongly connected* if, for any two distinct vertices u and v, there is a directed path from u to v. For an integer $k \geq 1$, a digraph is *strongly k-connected* if it has at least $k+1$ vertices and deleting at most k distinct vertices does not break the strong connectedness.

Problem 3. *Let $\mathcal{S}$ be the class of strongly connected digraphs. Find the $(\mathcal{S}, \preceq)$-unavoidable set for some digraph containment relation $\preceq$.*

Possible suggestions for $\preceq$ are the subdigraph relation, the topological minor relation, and the *butterfly-minor relation*. Here, the butterfly-minor relation is defined as follows. Let D be a digraph. We say an edge $e = (u, v) \in E(D)$ is *butterfly-contractible* if v is the only out-neighbour of u or u is the only in-neighbour of v. The *butterly-contraction* is an operation that, given a butterfly-contractibe edge $e \in E(D)$, merges two vertices u, v into a single vertex x_{uv}, deletes the edge (u, v), replaces all edges in the form (s, v) and (u, t) by (s, x_{uv}) and (x_{uv}, t), respectively, and deletes all multiple edges. Finally, we say that a digraph F is a *butterfly-minor* of D if F is obtained from D by a sequence of vertex deletions, edge deletions, and butterfly-contractions.

Now, we move our attention to *tournaments*, which is an orientation of a complete graph. The structure of tournaments has received enormous attention because it has a natural containment relation, called a *subtournament* relation, which is similar to the induced subgraph relation for graphs. The most famous Ramsey-type theorem would be Stearns' theorem, which states that acyclic tournaments are unavoidable objects for all tournaments.

Theorem 4 ([7])**.** *Every tournament with at least 2^{k-1} vertices contains an acyclic tournament on k vertices.*

Question 2 has been answered only when $\mathcal{P}$ is the class of strongly connected tournaments [4] or prime tournaments [4], so we consider the following question.

Problem 5. *Find the* $(\mathcal{P}, \preceq)$*-unavoidable set when* $\mathcal{P}$ *is the family of all* 2*-connected tournaments and* $\preceq$ *is the subtournament relation.*

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Monochromatic Empty Triangles

Hyunwoo Lee

Consider n points configuration $\mathcal{P}$ in $\mathbb{R}^2$, which is in a general position, that is, no three distinct points of $\mathcal{P}$ lie on the same line. Now, we consider an arbitrary red/blue coloring of the points in $\mathcal{P}$.

Definition 1. Assume a red/blue coloring on $\mathcal{P}$ is given. We say a set of distinct three points $p_1, p_2, p_3 \in \mathcal{P}$ forms a *monochromatic* triangle if all the colors of p_1, p_2, p_3 are the same. We say they form an *empty* triangle if the convex hull of $\{p_1, p_2, p_3\}$ does not contain any other points of $\mathcal{P}$.

Conjecture 2 (Aichholzer et al. [1])**.** *Let $\mathcal{P} \subseteq \mathbf{R}^2$ be an n-point configuration without a colinear triple. Then there exists a universal constant $c > 0$ such that the following holds:*
For all red/blue coloring of $\mathcal{P}$, there are at least cn^2 monochromatic empty triangles.

The current best bound is $\Omega(n^{4/3})$, which is proven by Pach and Tóth [2]

Their proof is very short and elegant. They used the key lemmas on the convex hull of the point configurations iteratively.

The first key lemma is *discrepancy lemma*, which says that if the number of red points and the blue points are sufficiently different, then we can find many monochromatic empty triangles.

The second key lemma is *order lemma*, which says that if we have a set of triangles that share a common monochromatic side, then one can find many monochromatic empty triangles. The proof can be done by applying Dilworth's theorem in a very elegant way.

I suggest improving Pach-Tóth's result on Conjecture 2 during KSCW2025.

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Erdős–Pósa property of subdivision class

Jeongin Lee

The packing problem and the covering problem are two of the most general questions in graph theory. The Erdős–Pósa property characterizes the cases when the optimal solutions of these two problems are bounded by functions of each other.

A class F of graphs has the induced Erdős–Pósa property if there exists a bounding function $f: N \to R$ such that for every graph G and every positive integer k, G contains either k pairwise vertex-disjoint induced subgraphs that belong to F, or a vertex set of size at most $f(k)$ hitting all induced copies of graphs in F.

Theorem 1 (Erdős–Pósa Theorem, [3])**.** *There is a function $f(k) = O(k \log k)$ such that for every graph G and every positive integer k, G contains either k vertex disjoint cycles, or a vertex set X of size at most $f(k)$ such that $G - X$ has no cycle.*

Theorem 2 ([8], [1], [4])**.** *There is a function $f(k,l) = O(kl + k \log k)$ such that for every graph G and every positive integer k, G contains either k vertex-disjoint cycles of length at least l, or a vertex set X of size at most $f(k,l)$ such that $G - X$ has no such cycle.*

Robertson and Seymour proved that when packing and covering H-minors for any fixed graph H, the planarity of H is equivalent to the Erdős–Pósa property.

Theorem 3 (Robertson, Seymour, [9])**.** *The family of H-expansions has the Erdős–Pósa property if and only if H is planar.*

However, planarity may not be equal to the Erdős–Pósa property in some other class. Representatively, there is a subdivision class, and there are interesting results for this.

Theorem 4 ([2])**.** *Either G contains k edge-disjoint K_4-subdivisions or there is a set $Y \subseteq E(G)$ of size $O(k^8 \log k)$ such that $G - Y$ does not contain any K_4-subdivision.*

At this time, we can ask the same question about the vertex.

Problem 5. *Do K_4-subdivisions have the Erdős–Pósa property?*

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Hajós' Conjecture

Hyemi Park

Hajós' Conjecture was proposed by **György Hajós** in the early 1950s, addressing the relationship between the *chromatic number* and the *maximum degree* of a graph.

Conjecture 1. *[1] Every graph G of chromatic number at least n contains a subdivision of the complete graph* K_n.

The conjecture holds for small values: it is trivial for $n \leq 2$, and was proven by Dirac in 1952 for $n = 3$ and $n = 4$ [2]. However, in 1979, Catlin constructed counterexamples for all $n \geq 7$, disproving the conjecture in general [3]. The cases $n = 5$ and $n = 6$ remain open.

Question 2. *Is is true for the case* $n = 5$ *of* **Hajós' Conjecture***?*

Theorem 3. *[4] Every 4-connected graph contains a subdivision of* K_5^-

In paper [4], 4-connectivity is shown to guarantee the existence of a subdivision of K_5^-. It provides a clue toward understanding 1.

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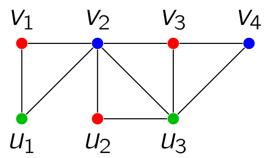
Nonrepetitive graph colorings

Jaehyeon Seo

A *nonrepetitive coloring* is a vertex coloring of a graph where no path of even order has its first half and the second half with the same color pattern. We call a path *repetitively colored* if it satisfies this forbidden pattern. The *nonrepetitive chromatic number* $\pi(G)$ is the minimum number of colors in a nonrepetitive coloring of a graph G. For example, in the below figure, the paths $v_1v_2v_3v_4$ and $v_1u_1v_2u_2u_3v_4$ are repetitively colored while $v_1u_1v_2u_3v_4v_3$ is not.



See [3] for a comprehensive survey.

TL; DR

- Find a new way to obtain a nonrepetitive coloring of a grid.
- How do operations on graphs (e.g., the Cartesian product and the strong product) affect their nonrepetitive chromatic numbers?
- Provide bounds on the variants of the nonrepetitive chromatic number when G is a cycle or a tree.

Nonrepetitive colorings are interesting for several reasons [3, §1 Introduction]:

- They bring together two major areas—combinatorics of words and graph coloring—in a very natural way.
- Several advanced techniques have been used: Lovász Local Lemma, entropy compression, layered treewidth, product structure theorems, ... In some cases, the methods themselves were developed with motivations from the nonrepetitive colorings.
- They are deeply related to graph sparsity. Indeed, graph classes with bounded expansion can be characterised in terms of nonrepetitive colorings.
- They are one of the most illustrative examples of the use of the Lovász Local Lemma. In addition, they were one of the first applications of the 'entropy compression' method used in the constructive proof of the Lovász Local Lemma (which is an important recent

development in algorithmic graph theory), and showed that this new method yields better results than those obtained using the Lovász Local Lemma.

I hope that at least one of those reasons is related to your interest.

There are several variants of $\pi(G)$, and below are two of them which are very important in the sense that they are necessary to obtain results on $\pi(G)$. A vertex-colored walk of even order is *repetitively colored* if the first half and the second half have the same color pattern.

- A *walk-nonrepetitive coloring* is a vertex coloring of a graph whose repetitively colored walk is always of the (trivial) form $v_1 \cdots v_k v_{k+1} \cdots v_{2k}$ with $v_i = v_{k+i}$ for all $1 \leq i \leq k$. Let $\sigma(G)$ be the minimum number of colors in a walk-nonrepetitive coloring.

- A *stroll-nonrepetitive coloring* is a vertex coloring of a graph without repetitively colored stroll; a *stroll* is a walk $v_1 \cdots v_{2k}$ with $v_i \neq v_{k+i}$ for all $1 \leq i \leq k$. Let $\rho(G)$ be the minimum number of colors in a stroll-nonrepetitive coloring.

It is straightforward to see that $\pi(G) \leq \rho(G) \leq \sigma(G)$. Below are the bounds on π and σ for various graph classes, taken from [3, Table 1].

Graph class	π	σ
Paths	3	4
Cycles	$[3, 4]$	$[4, 5]$
Pathwidth k	$[k+1, 2k^2+6k+1]$	$[\Delta+1, (2k^2+6k+1)(k\Delta+1)]$
Trees	4	$[\Delta+1, 4\Delta]$
Outerplanar	$[7, 12]$	$\Theta(\Delta)$
Treewidth k	$[\binom{k+2}{2}, 4^k]$	$O(\min\{4^k k\Delta, k^2\Delta^2\})$
Planar	$[11, 768]$	$\Theta(\Delta)$
Euler genus g	$[\Omega(g^{3/5}/\log^{1/5} g), O(g)]$	$O(g\Delta)$
Excluded minor	$\Theta(1)$	$\Theta(\Delta)$
Excluded topological minor	$\Theta(1)$	$\Theta(\Delta)$
Max degree Δ	$[\Omega(\Delta^2/\log\Delta), (1+o(1))\Delta^2]$	?

Paths and grids

Thue [2] proved that $\pi(P_n) = 3$ for all $n \geq 4$, where P_n is the path of order n. This was done using the Thue–Morse sequence, which is the binary sequence obtained as follows: starting from 0, iteratively append the complement of the word generated so far. The first few iterations are given below.

$$0 \quad 01 \quad 0110 \quad 01101001 \quad 0110100110010110 \quad \cdots$$

Construct the ternary sequence by setting the i-th entry as the difference (in $\mathbb{Z}_3$) of the $(i+1)$-th and i-th entries of the Thue–Morse sequence. This results in the following:

$$1021\ 2010\ 2012\ 1021 \cdots$$

(the spacings are just for the readability.) Since the Thue–Morse sequence does not contain the patterns $0X0X0$ or $1X1X1$, the generated ternary sequence is nonrepetitive. Thus, $\pi(P_n) = 3$. In other words, if we color the edges of P_n using the Thue–Morse sequence, a nonrepetitive coloring of P_n can be obtained by coloring each vertex by the difference of two colors incident to it (with additional care for the end-vertices).

For graphs G_1 and G_2, let $G_1 \square G_2$ and $G_1 \boxtimes G_2$ be their Cartesian product and strong product, respectively. Below are some observations.

(1) If $H \subseteq G$, then $\pi(H) \leq \pi(G)$ and $\sigma(H) \leq \sigma(G)$.

(2) $G_1 \times G_2 \subseteq G_1 \boxtimes G_2$.

(3) $\sigma(G_1 \boxtimes G_2) \leq \sigma(G_1)\sigma(G_2)$, since walk-nonrepetitive colorings f_1 and f_2 of G_1 and G_2, respectively, yield a walk-nonrepetitive coloring $f(v_1, v_2) := (f_1(v_1), f_2(v_2))$ of $G_1 \boxtimes G_2$ (call f the *product* of f_1 and f_2.)

(4) $\sigma(P_n) = 4$ for all $n \geq 6$, which can be proved by modifying a non-repetitive coloring of paths using 3 colors.

Combining these yields

$$\pi(P_n \square P_n) \leq \sigma(P_n \square P_n) \leq \sigma(P_n \boxtimes P_n) \leq \sigma(P_n)^2 \leq 16,$$

where $P_n \square P_n$ is the $n \times n$ grid. However, this method of taking a product of colorings does not directly work for nonrepetitive colorings. In other words, the product of nonrepetitive colorings of G_1 and G_2 does not always provide a nonrepetitive coloring of $G_1 \boxtimes G_2$.

Recently, Tianyi Tao [1] showed that $5 \leq \pi(P_n \square P_n) \leq 12$. The nonrepetitive coloring which yields this upper bound is constructed by first coloring the grid with 8 colors using a walk-nonrepetitive coloring of a path, and next destroying all the repetitive paths by recoloring some vertices using the new 4 colors.

Problem 1 (Tianyi Tao, personal communication)**.** *Find an edge coloring of $P_n \square P_n$ which yields a nonrepetitive coloring of $P_n \square P_n$, in a similar way that the Thue–Morse sequence yields a nonrepetitive coloring of P_n.*

Others

I selected some open problems from [3] which require less background to understand the statements.

Problem 2 ([3]). *(1) (Open Problem 2.21) Provide bounds on $\pi(G_1 \square G_2)$ and $\pi(G_1 \times G_2)$ in terms of $\pi(G_1)$ and $\pi(G_2)$.*

(2) (Open Problem 3.9) Is $\rho(C_n) \leq 4$ for infinitely many n? Is $\sigma(C_n) \leq 4$ for infinitely many n?

(3) (Open Problem 4.4) Is there a constant c such that $\sigma(T) \leq \Delta(T) + c$ for every tree T?

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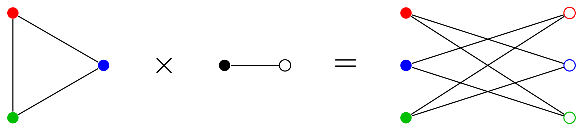
Comparing the number of proper colorings

Jaehyeon Seo

Recall that the *tensor product* $H_1 \times H_2$ of graphs H_1 and H_2 is the graph on the vertex set $V(H_1)\times V(H_2)$ where $(u_1, u_2)(v_1, v_2) \in E(H_1\times H_2)$ if and only if $u_1v_1 \in E(H_1)$ and $u_2v_2 \in E(H_2)$. Below is the example when $H_1 = K_3$ and $H_2 = K_2$.



TL; DR Prove the conjecture of Zhao [4]: for all graphs H and $q \geq 1$, the number of proper q-vertex-colorings of H squared is at most the number of proper q-vertex colorings of $H \times K_2$. In other words, if $c_q(H)$ is the number of proper q-colorings of H, then

$$c_q(H)^2 \leq c_q(H \times K_2).$$

Proving it completely would be very challenging. Finding *any* unknown example of H which satisfies the statement would be a meaningful progress.

Fix $q \geq 1$. Which graph has the largest number of proper q-(vertex-)coloring?

This question is very natural to ask, but it is ill-posed since $H \sqcup H$ always has more proper q-colorings than H. A suitable normalized (and slightly generalized) version is as follows.

Fix $q \geq 1$ and a graph class $\mathcal{H}$. Which $H \in \mathcal{H}$ maximizes $c_q(H)^{1/e(H)}$?

Galvin and Tetali [1] showed, among other results, that when $\mathcal{H}$ consists of the d-regular bipartite graphs, $K_{d,d}$ maximizes $c_q(H)^{1/e(H)}$. Suppose we have shown $c_q(H)^2 \leq c_q(H \times K_2)$ for all H and q. Then,

$$c_q(H)^{1/e(H)} \leq c_q(H \times K_2)^{1/(2e(H))} = c_q(H \times K_2)^{1/e(H\times K_2)} \leq c_q(K_{d,d})^{1/d^2},$$

whence $K_{d,d}$ maximizes $c_q(H)^{1/e(H)}$ among all the d-regular graphs. Here, the second inequality follows from the result of Galvin and Tetali together with the fact that $H \times K_2$ is always bipartite. In fact, the aforementioned sufficient condition was conjectured to be true by Zhao.

Conjecture 1 (Zhao [4]). *$c_q(H)^2 \leq c_q(H \times K_2)$ for all H and q.*

Zhao showed that the inequality holds whenever $q \geq (2n)^{2n+2}$, where $n := |V(H)|$.

Problem 2. *Improve the value $q_0(H)$ such that $c_q(H)^2 \leq c_q(H \times K_2)$ whenever $q \geq q_0(H)$.*

A very recent preprint of mine [2] made progress toward the conjecture. To explain the results, we need some definitions. Let G be a bipartite graph whose bipartition $V_1 \sqcup V_2$ is implicitly given, and let $a, b \geq 1$ be integers. Let A and B be color sets where $|A| = a$, $|B| = b$, and one is a subset of the other. Define $C_{a,b}(G)$ to be the number of proper colorings where the vertices in V_1 use the colors in A and the vertices in V_2 use the colors in B. For example, $C_{a,a}(G) = c_a(G)$ and $C_{2,a}(K_{1,s}) = 2(a-1)^s + 2(a-2)^s$ when $a \geq 2$. We define two properties for H, where the first is from Conjecture 1 and the second is its strengthening.

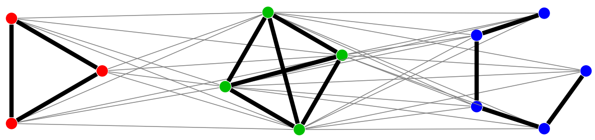
(P1) $c_q(H)^2 \leq c_q(H \times K_2)$ for all $q \geq 1$.

(P2) $c_a(H)\, c_b(H) \leq C_{a,b}(H \times K_2)$ for all $1 \leq a \leq b$.

Below are ways to construct (P1)- and (P2)-graphs. The first two are from [2], and the last two are not in the preprint but not hard to check.

(1) The complete multipartite graphs, the paths, and the even cycles are (P2).

(2) A blow-up of a bipartite graph *filled by* (P2)-graphs is (P1). Here is an example of a blow-up of P_3 filled by K_3, K_4, and P_5.



(3) Let H be (P1), and let H' be the graph obtained by adding a dominating vertex to H. Then H' is (P1).

(4) Let H be (P1), and let H' be a graph obtained by adding a new vertex v to H so that $N(v) \cap V(H)$ is a clique. Then H' satisfies (P1). In particular, every chordal graph is (P1).

Problem 3. *(1) Find more examples of (P1)- or (P2)-graphs. For graphs H with simple structures, this can be done by (almost) explicitly computing $c_q(H)$ and $c_q(H \times K_2)$ by hand.*

(2) Find more graph operations which generate (P1)- or (P2)-graphs.

In fact, Sah, Sawhney, Stoner, and Zhao [3] showed that $K_{d,d}$ maximizes $c_q(H)^{1/e(H)}$ among all the d-regular graphs, whose proof does not rely on Conjecture 1. Nonetheless, resolving Conjecture 1 would give us a new proof of the statement.

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Distributed Triangle Finding

Hyeonjun Shin

Recently, as modern computing systems grow larger and more complex, several abstract models have been proposed to better capture their behavior. For graphs, a particularly well-studied model is the *CONGEST* model.

CONGEST Model Consider an n-vertex graph $G = (V, E)$ representing a distributed network, where each vertex corresponds to a player who knows only the edges incident to it. Communication proceeds in *synchronous rounds*, and in each round, a player (vertex) can send $O(\log n)$-bit messages to each of its neighbors. This choice of $O(\log n)$ bound on the size of messages allows sending a node identifier within a single message.

In this model, the limited local view of each vertex and the constrained message size make certain graph problems significantly harder than in the traditional *RAM* model — even simple tasks such as determining whether the graph is triangle-free. I refer the interested reader to [9] for a detailed survey, particularly Section 2.2 for triangle detection. We will focus on the Triangle Detection problem.

Definition 1 (Triangle Detection)**.** If G contains no triangle as a subgraph, *all* players output *no*; otherwise, at least *some* players output *yes*.

This year, the lower bound for this problem was improved to $\Omega(\log \log n)$ rounds — for both deterministic and randomized algorithms — by Assadi and Sundaresan [1]. However, no polylogarithmic-round algorithm is currently known for the upper bound.

- Izumi and Le Gall [8] proposed randomize algorithms for triangle listing and detection in $O(n^{3/4} \log n)$ and $O(n^{2/3} \log^{2/3} n)$ rounds w.h.p., respectively. Their approach is to split the task of finding triangles, where ϵ-*heavy* triangles and another that looks for other triangles, where an ϵ-heavy triangle is one in which at least one of its edges appears in at least n^ϵ triangles. Specifically, heavy triangles can be detected within $O(n^{1-\epsilon})$ rounds by randomly sampling which edges to send, and can be listed in $O(n^{1-\epsilon/2})$ rounds by randomly hashing the edges sent to different neighbors.

- Chang et al. [3] showed that triangle listing (and thus also detection) can be solved in $\tilde{O}(n^{1/2})$ rounds w.h.p., using an expander decomposition. Their algorithm recursively partitions the edge set $E(G)$.

  Specifically, they decompose the edge set into three parts — E_m, E_s, and E_r — within $O(n^{1/2})$ rounds:

(1) Each connected component induced by E_m has minimum degree at least $n^{1/2}$ and conductance $\Omega(1/\text{polylog } n)$.

(2) The graph induced by E_s has arboricity at most $n^{1/2}$.

(3) The number of remaining edges in E_r is at most a constant fraction of the total number of edges.

They then enumerate triangles in E_s using an orientation derived from the decomposition algorithm, and enumerate triangles in E_m by simulating the triangle listing algorithm in another model (*CLIQUE* model[1]) [7]. The algorithm recurses on E_r.

- The approach using expander decomposition is refined by Chang and Saranurak [4], which solves the triangle listing in $\tilde{O}(n^{1/3})$ rounds, w.h.p.

- For deterministic algorithms, Chang and Aranurak [5] proposed a deterministic algorithm for expander decomposition and expander routing, which solves triangle detection in $O(n^{0.58})$ rounds and triangle listing in $n^{2/3} + o(1)$ rounds. Building upon this expander decomposition and routing, Censor-Hillel et al. [6] proposed $n^{1/3+o(1)}$-round deterministic algorithm for triangle listing. After that, the improved expander routing algorithm of Chang et al. [2] leads to an $\tilde{O}(n^{1/3})$-round deterministic algorithm, combining the framework of [6].

As observed, most existing results focus on triangle listing. Although the final result is nearly optimal for triangle listing, matching the known lower bound of $\Omega(n^{1/3}/\log n)$, no near-optimal algorithm is currently known for triangle detection itself.

Question 2. *Can triangle detection be solved in $O(\text{polylog } n)$ rounds in the CONGEST model?*

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Seymour's Second Neighborhood Conjecture

Chanho Song

Given a directed graph $G = (V, E)$, let $N^+(v)$ denote the out-neighbors of a vertex v, and let $N^{++}(v)$ denote the set of vertices that are reachable from v via a path of length two, excluding those already in $N^+(v)$.

Definition 1 (Second Neighborhood)**.** In a digraph G, the *first neighborhood* of a vertex v, denoted $N^+(v)$, is the set of all vertices u such that there is a directed edge $v \to u$. The *second neighborhood* $N^{++}(v)$ is the set of all vertices $w \notin N^+(v)$ such that there exists a vertex $u \in N^+(v)$ with an edge $u \to w$.

Conjecture 2 (Seymour's Second Neighborhood Conjecture)**.** *Every finite simple digraph with no directed 2-cycle (i.e., no pair of vertices u, v with both $u \to v$ and $v \to u$) contains a vertex v such that*

$$|N^{++}(v)| \geq |N^+(v)|.$$

Background and Known Results This conjecture was first proposed by Paul Seymour in 1990. It remains open in general but has been confirmed in several special cases.

- For tournaments (complete oriented graphs), the conjecture was posed by Dean and proved by Fisher [1] using Farkas' Lemma and averaging arguments. A combinatorial proof using median orders was later given by Havet and Thomassé [2], who showed that such a vertex exists.

- Kaneko and Locke [3] verified the conjecture for digraphs in which some vertex has out-degree at most six.

- Chen, Shen, and Yuster [4] proved that in any digraph, there exists a vertex v with $|N^{++}(v)| \geq r \cdot |N^+(v)|$, where $r \approx 0.657$, and they further improved this bound to approximately 0.678. Huang and Peng [5] proved that every oriented digraph contains a vertex u such that $|N^{++}(u)| \geq 0.715538 \cdot |N^+(u)|$.

- Recently, Daamouch et al. [7] proved that SSNC holds for tournaments with two disjoint stars removed. Additionally, they showed that if a tournament contains no two disjoint directed paths of length 2, then not just one but *at least two* vertices satisfy the second neighborhood condition.

Interestingly, this conjecture implies a special case of the well-known Caccetta–Häggkvist Conjecture.

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