Thermal nucleation: bridging simulations and analytics

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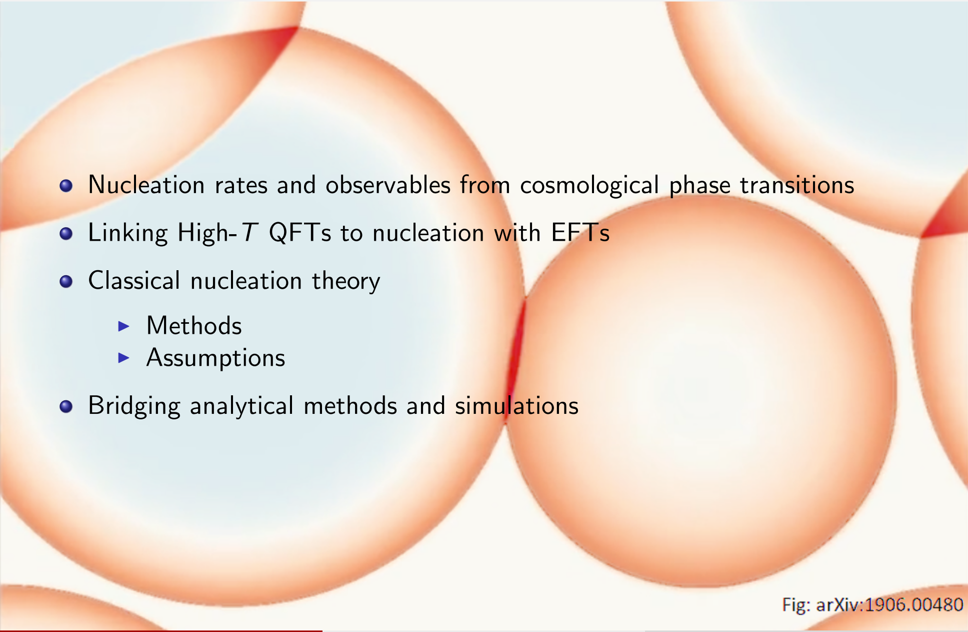
Fig: arXiv:1906.00480

Overview of the talk



- Nucleation rates and observables from cosmological phase transitions
- Linking High-T QFTs to nucleation with EFTs
- Classical nucleation theory
  - Methods
  - Assumptions
- Bridging analytical methods and simulations

Fig: arXiv:1906.00480

Nucleation

- Scalar field trapped in a metastable phase
- Cannot escape homogeneously
  - (Infinite energy)
- Escaping locally with bubble nucleation

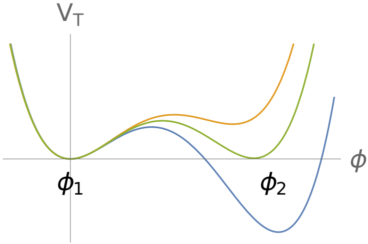


Figure: Potential with a transition

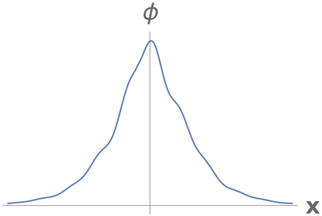


Figure: Cross section of a bubble

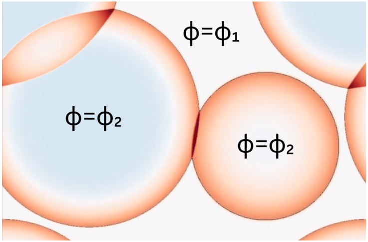


Figure: Growing bubbles [Cutting et al. '20]

Nucleation rates and cosmological phase transitions

[Enqvist, Ignatius, Kajantie, Rummukainen '92]

- Nucleation rate, Γ, changes in time
  - Sensitive to temperature
  - Supercooling
- Setting of a transition
  - Duration of a transition, Δt
  - Transition temperature, T_n

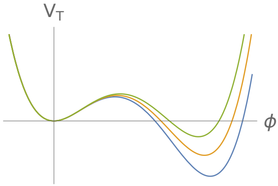


Figure: T-dependent potential

$$\Gamma(T) = A(T)\, e^{-S_\mathrm{cb}(T)}$$

$$\Gamma(t) = \alpha\, e^{\beta\cdot(t-t_n)}$$

Duration of a phase transition

- Duration of a transition, Δt

$$\Delta t \sim \left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\Gamma\right)^{-1}$$
$$= \beta^{-1}, \qquad \Gamma(t) = \alpha\, e^{\beta\cdot(t-t_n)}$$

- Δt shifts spectrum diagonally
  - Wider shocks
  - More energy

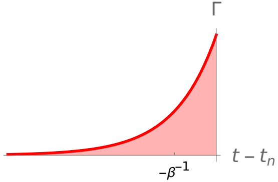


Figure: T-dependent potential

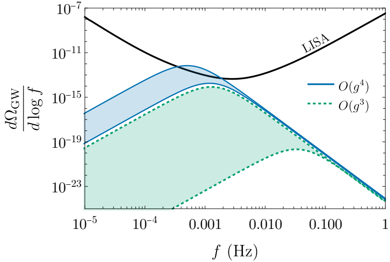


Figure: GW uncertainties from nucleation rates [Gould, Tenkanen '21]

Temperature of a phase transition

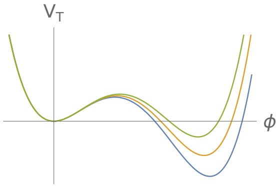


Figure: T-dependent potential

- Transition temperature, T_n
  - $\mathcal{O}(1)$ fraction of the universe in the new phase

$$\Delta t^4\,\Gamma \sim 1 \Rightarrow T_n$$

- T_n affects through the potential
  - Thermodynamics
  - Wall velocity

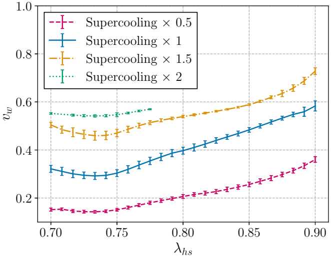


Figure: Wall velocities different levels of supercooling in xSM [WallGo Investigates '25]

Analytics vs. simulations

- Analytical result by Langer [Langer '67, '69, '74]
- First tests on the lattice: $\times e^{\mathcal{O}(10)}$ [Borsanyi et al. '00]
  - Simulation from Gaussian distribution
  - 2+1 dimensions
- Moore-Rummukainen-Tranberg method: $\times e^{\mathcal{O}(10)} - \times e^{\mathcal{O}(100)}$ [MR '00, MRT '01, Gould, Güyer, R '22]
  - Realistic models
  - Close to Langer's rate in assumptions
- $\times 1/8$ [Pîrvu, Shkerin, Sibiryakov '24]
  - Still Gaussian
  - 1+1 dimensions
  - Even closer with high damping and noise

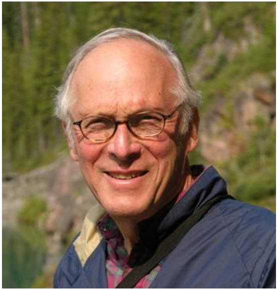


Figure: J. S. Langer

Towards nucleation from high-T QFTs [Gould & Hirvonen '21]

- Motivation for high-T
- Usual 1-loop construction based on Linde
- Effective field theories
- Physical picture of classical fields

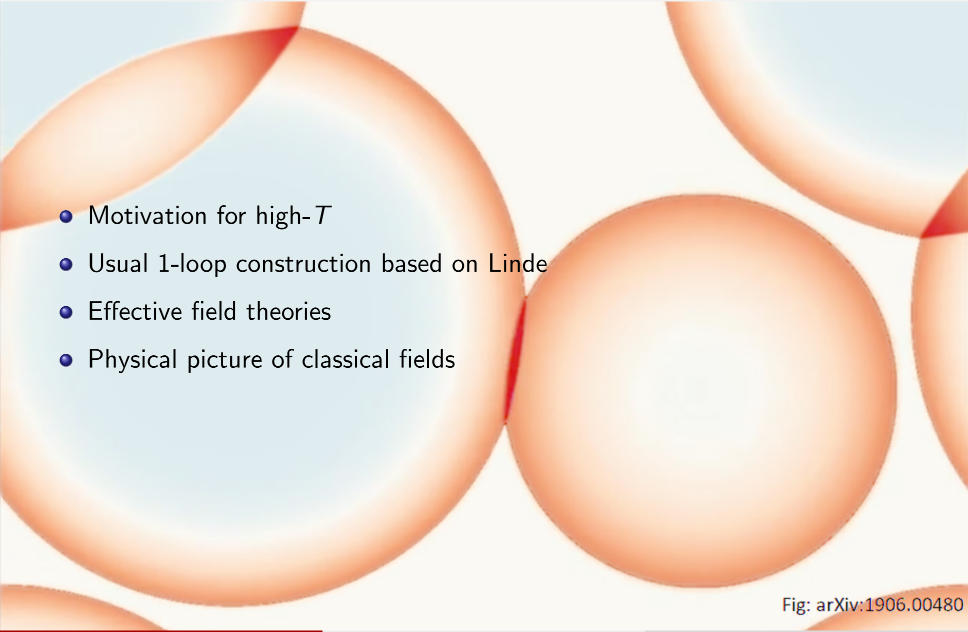


Fig: arXiv:1906.00480

Why high-T?

- Universe cooling initiates transitions
- Thermal effects leading order in effective potential
- Thermal corrections suppressed by couplings in perturbative models
  - High-T hierarchy enhances corrections

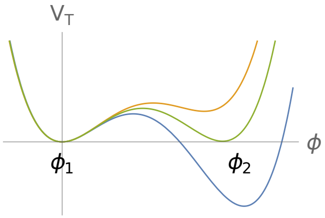


Figure: Thermally dependent effective potential

$$-\mu^2 \sim \frac{\lambda T^2}{24} \quad \& \quad \frac{\lambda}{4\pi} \ll 1$$

$$\Rightarrow T^2 \gg -\mu^2$$

Linde's rate

- Coleman and Callan: vacuum decay
  - Imaginary part of the energy
- Linde: quantum tunneling from thermal state
  - Imaginary part of the free energy

$$\Gamma = 2\mathrm{Im}E$$

$$\Gamma \stackrel{?}{=} 2\mathrm{Im}F$$

Motivation from a quantum particle

- Escape of a quantum particle
  [Affleck '80]
- Low-T
  - Tunneling from a thermal state
  - Analogous to Linde
- High-T
  - Over-barrier escape
  - Classical result
  - (NO analytic continuation)
- Linde's wrong in high-T

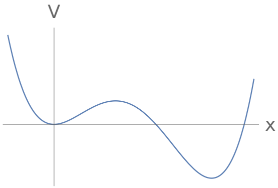


Figure: Particle potential

$$T < \frac{|\lambda_-|^{\frac{1}{2}}}{2\pi} : \qquad \Gamma = 2\mathrm{Im}F$$

$$T > \frac{|\lambda_-|^{\frac{1}{2}}}{2\pi} : \qquad \Gamma = \frac{|\lambda_-|^{\frac{1}{2}}}{\pi T}\mathrm{Im}F$$

Effective description for nucleation

$$e^{-S_{\text{eff}}} = \sum_{\begin{bmatrix}\text{constrained}\\ \text{microscopic variables}\end{bmatrix}} e^{-S_{\text{E}}}$$

- Thermal effects initiate transition
  $\rightarrow$ Integrate over thermal flucts
- Bubbles not coarse-grained over

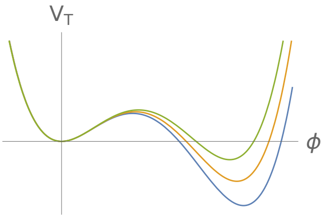


Figure: Effective potential with a first-order transition

Usual recipe built on [Linde '81]

- One-loop effective potential, $V_{\text{1-loop}}$
- Contains relevant thermal flucts ✓
- Does not discriminate flucts
  $\rightarrow$ Bubbles coarse grained over ✗
- Problems
  - Potential complex ✗
  - Diverging derivative expansions ✗
- Twisted approximation of EFT methods

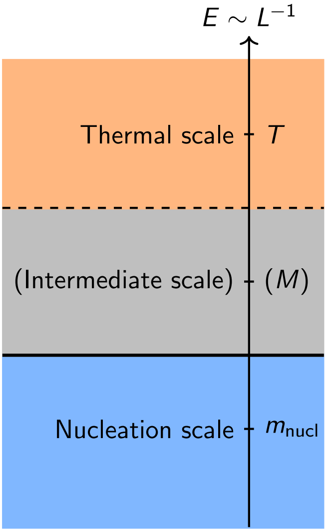
$$V_{\text{1-loop}} = V_{\text{bare}} + V_{\text{1-loop}}^{\text{CW}} + V_{\text{1-loop}}^{T}$$

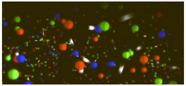
$$\Gamma = T^4 e^{-S_3}$$

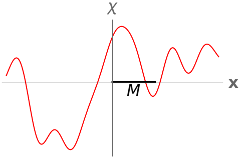
$$V_{\text{1-loop}}(\bar{\phi}) = -\frac{T}{V} \ln \int \mathcal{D}\phi' \, e^{-S_{\text{E}}[\bar{\phi}+\phi']} \Big|_{\text{1-loop}}$$

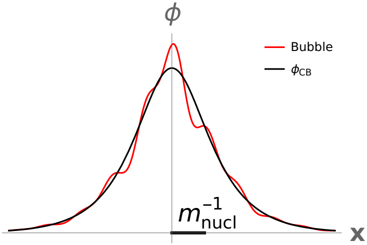
Scales in high-temperature nucleation

- Particles
- Fields









Integrating out thermal scale [Hirvonen '22]

- Thermal equilibrium assumed
- Bubbles in $\phi_{0,\mathrm{IR}}$ ✓
- Integrating out thermal scale ✓
  - Matsubara modes heavy, $M_n^2 = m^2 + (2\pi n T)^2$ [Kajantie et al. '95, Braaten & Nieto '95]
- Usual computation...
  - ...but expand in light quantities, m^2, k^2, before integration [Burgess '20, Hirvonen '22]
  - Leading bits were correct ✓
- DRalgo [Ekstedt, Schicho & Tenkanen '23]

$$Z = \int \prod_n \mathcal{D}\phi_n \, e^{-S_\mathrm{E}}$$

$$= \int \mathcal{D}\phi_{0,\mathrm{IR}} \, e^{-S_\mathrm{eff}} ,$$

$$S_\mathrm{eff} = -\ln \int \mathcal{D}\phi_{0,\mathrm{UV}} \prod_n \mathcal{D}\phi_{n\neq 0} \, e^{-S_\mathrm{E}}$$

$$\frac{1}{(p+k)^2 + m^2} \to \frac{1}{p^2} - \frac{m^2}{p^4} - \frac{2k\cdot p}{p^2} + \ldots$$

Demystifying effective field theories

$$
\begin{aligned}
S_{\text{eff}}[\phi_{\text{IR}}] &= -\ln \int \mathcal{D}\phi_{\text{UV}}\, e^{-S_0[\phi_{\text{IR}}]-S_0[\phi_{\text{UV}}]-S_{\text{I}}[\phi_{\text{IR}}+\phi_{\text{UV}}]} \\
&= S_0[\phi_{\text{IR}}] - \ln \int \mathcal{D}\phi_{\text{UV}}\, e^{-S_0[\phi_{\text{UV}}]}\left(1 - S_{\text{I}}[\phi_{\text{IR}}+\phi_{\text{UV}}] + \dots\right) \\
&= S_0[\phi_{\text{IR}}] - \frac{\pi^2 V T^3}{90} + \left\langle S_{\text{I}}[\phi_{\text{IR}}+\phi_{\text{UV}}]\right\rangle_{\text{UV,0,c}} + \dots \qquad S_{\text{I}}[\phi] = \int \mathrm{d}\tau \mathrm{d}^3\mathbf{x}\, \frac{\lambda\phi^4}{4!} \\
&= \underbrace{S_0[\phi_{\text{IR}}] + S_{\text{I}}[\phi_{\text{IR}}]}_{\text{Original action for } \phi_{\text{IR}}} + \underbrace{\int \mathrm{d}\tau \mathrm{d}^3\mathbf{x}\, \frac{\lambda}{4}\left\langle {\phi_{\text{UV}}}^2\right\rangle_{\text{UV,0,c}} {\phi_{\text{IR}}}^2}_{\text{Additional mass term for } \phi_{\text{IR}}} + \dots
\end{aligned}
$$

UV part of the correlator

$$\left\langle {\phi_{\mathrm{UV}}}^2 \right\rangle_{\mathrm{UV},0,\mathrm{c}} = \int_{\Lambda_{\mathrm{IR}}} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\mathbf{p}^2 + m^2} + \sumint_P{}' \frac{1}{P^2 + m^2}\,, \qquad P = (\omega_n, \mathbf{p})$$

$$= \int_{\Lambda_{\mathrm{IR}}} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\mathbf{p}^2} - m^2 \int_{\Lambda_{\mathrm{IR}}} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\mathbf{p}^4} + \dots$$

$$+ \sumint_P{}' \frac{1}{P^2} - m^2 \sumint_P{}' \frac{1}{P^4} + \dots$$

$$\left\langle {\phi_{\mathrm{UV}}}^2 \right\rangle_{\mathrm{UV},0,\mathrm{c}} = \sumint_P \frac{1}{P^2} - m^2 \sumint_P \frac{1}{P^4} + \dots$$

$$= \frac{T^2}{12} - \frac{2m^2}{(4\pi)^2} \left(\underbrace{\frac{1}{2\epsilon}}_{\substack{\text{cancels the mass}\\ \text{counterterm}}} + \underbrace{\ln\left(\frac{\mu}{T}\right)}_{\substack{\text{cancels the}\\ \text{mass running}}} \right) + \dots$$

3d effective field theory

$$
\begin{aligned}
S_{\text{eff}}[\phi_{\text{IR}}] &\approx S_0[\phi_{\text{IR}}] + S_{\text{I}}[\phi_{\text{IR}}] + \int \mathrm{d}\tau \mathrm{d}^3\mathbf{x} \frac{1}{2}\left[\frac{\lambda T^2}{24} - \frac{m^2}{(4\pi)^2}\left(\frac{1}{2\epsilon} + \ln\left(\frac{\mu}{T}\right)\right)\right]{\phi_{\text{IR}}}^2 \\
&= \beta \int \mathrm{d}^3\mathbf{x}\left(\frac{1}{2}(\boldsymbol{\nabla}\phi_{\text{IR}})^2 + \frac{m_{\text{eff}}^2}{2}\phi_{\text{IR}}^2 + \frac{\lambda}{4!}\phi_{\text{IR}}^4\right), \qquad \bar{\phi} \equiv \sqrt{\beta}\phi_{\text{IR}} \\
&= \int \mathrm{d}^3\mathbf{x}\left(\frac{1}{2}\left(\boldsymbol{\nabla}\bar{\phi}\right)^2 + \frac{m_{\text{eff}}^2}{2}\bar{\phi} + \frac{\lambda T}{4!}\bar{\phi}^4\right)
\end{aligned}
$$

- Expansion parameter: $\frac{\lambda T}{m_{\text{eff}}}$ $(\gg \lambda)$
- One loop: $-\frac{1}{12\pi} m_{\text{eff}}^3$

Physical picture of nucleation at high-T

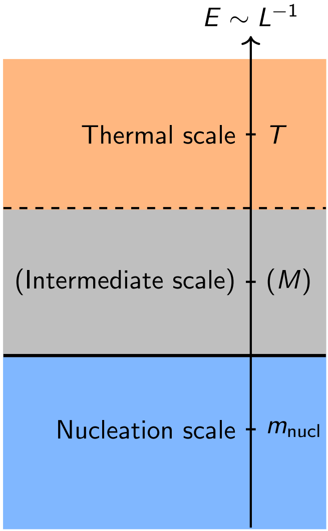
- Classical field theory
  - Large occupation numbers
  - Classical statistical path integral
- Classical nucleation theory
- (Linde: quantum tunneling with temperature)

$$n_{\mathrm{b}} = \frac{1}{e^{E/T} - 1} \approx \frac{T}{m_{\mathrm{nucl}}} \gg 1$$

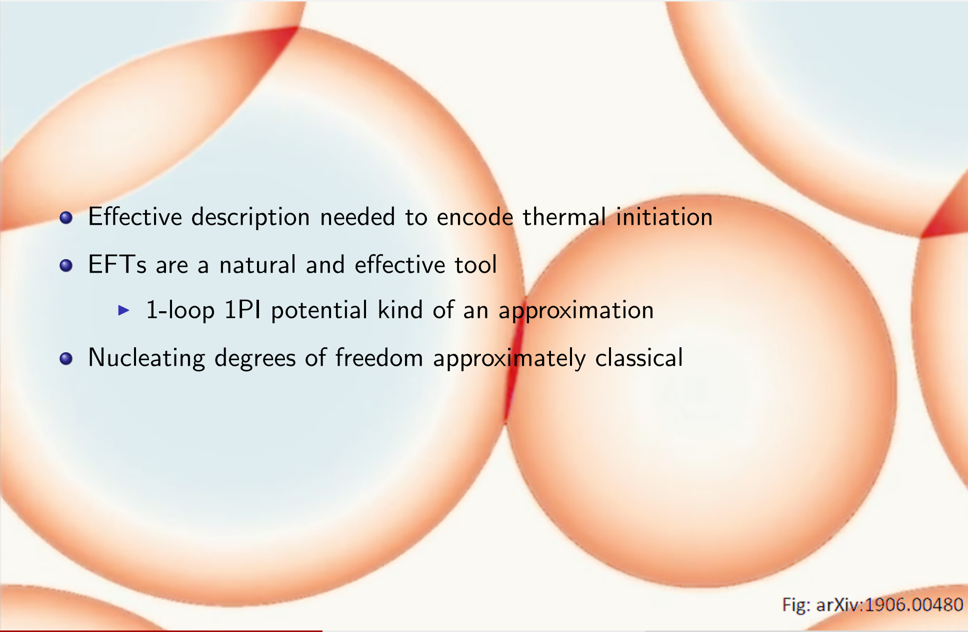
$$Z = \int \mathcal{D}\phi_{\mathrm{IR}} \exp\left(-\frac{1}{T} \int \mathrm{d}^3\mathbf{x} \mathscr{L}_{\mathrm{eff}} \right)$$

Beyond equilibrium high-T

- Thermal scale out of equilibrium due to nucleation [Hirvonen '24]
  - Boltzmann particles
  - Equilibrium part factorizes ✓
- Breaking high-T hierarchy
  - Rather general for perturbative nucleating fields*
  - Other fields can become heavy [Navarrete, Paatelainen, Seppänen & Tenkanen '25]



Summary of road from high-T QFTs towards nucleation



- Effective description needed to encode thermal initiation
- EFTs are a natural and effective tool
  - 1-loop 1PI potential kind of an approximation
- Nucleating degrees of freedom approximately classical

Fig: arXiv:1906.00480

Classical nucleation theory

- Setup & assumptions
- Methods
    - Direct wait-and-see method
    - MRT lattice method
    - Perturbative Langer's rate

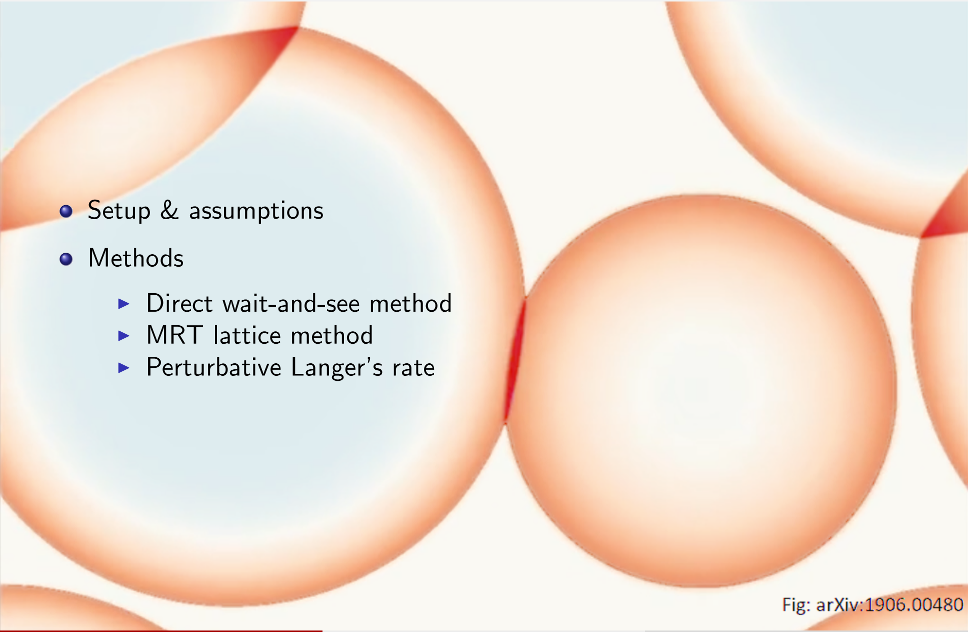


Fig: arXiv:1906.00480

Setup for classical nucleation theory [Kramers '40]

- Two phases
  - Metastable: thermal population
  - Stable: empty
- Nucleation rate
  - Leak from meta to stable

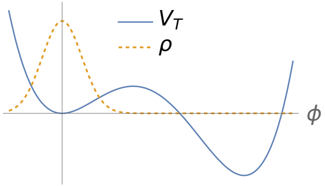


Figure: Schematic figure of an equilibrium distribution in the metastable state

Time evolution

- Evolution of the distribution
- Liouville equation
  - Equivalent to field equation
    - (Lattice)

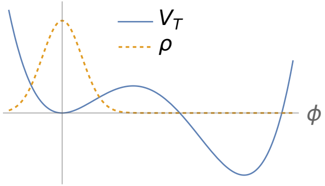


Figure: Initial distribution

$$\partial_t \rho = \int \mathrm{d}x \left[-\pi \frac{\delta \rho}{\delta \phi} + \left(\partial_x^2 \phi - V'(\phi) \right) \frac{\delta \rho}{\delta \pi} \right] \qquad \Leftrightarrow \qquad \partial_t^2 \phi = \partial_x^2 \phi - V'(\phi)$$

Wait-and-see method

1. Draw field configurations from a distribution
    - Gaussian 1+1 [Pîrvu, Shkerin, Sibiryakov '24]
    - Fully thermal 1+1 [Hirvonen, Gould '25]
    - Vacuum 3+1 [Dutka, Jung & Shin '25, Dutka '25]
2. Evolve field configurations
3. Count nucleation events

- Large suppressions, $e^{-\mathcal{O}(100)}$, impossible

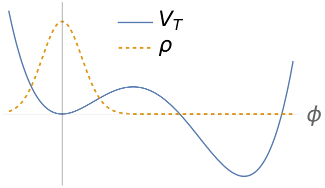


Figure: Initial distribution

$$\partial_t^2 \phi = \partial_x^2 \phi - V'(\phi)$$

Equilibrium assumption [Kramers '40]

- Equilibrium in meta

$$\rho_{\text{eq}} = \frac{1}{Z_{\text{meta}}} e^{-\beta H}$$

- Unaffected by nucleation
- Sources nucleation

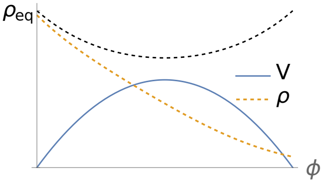


Figure: Distribution function in equilibrium in meta

1. Equilibrium distribution of an order parameter $\langle\phi\rangle_{\text{lattice}}$ $(\sim \theta_{\text{op}})$
   - Approximation of the critical bubble
   - Exponential suppression
2. Simulate from near critical bubble
   - Real-time dynamics of fields
3. Single out trajectories from meta to stable

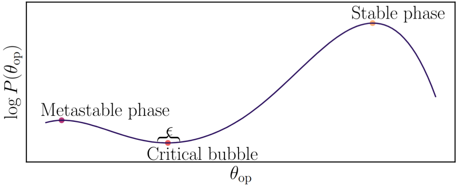


Figure: Equilibrium distribution and origin of nucleation trajectories [Gould, Kormu & Weir '24]

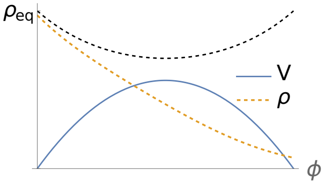


Figure: Meta to stable equilibrium trajectories

Assumption of perturbativity [Kramers '40, Langer '67, '69, '74]

- Linearizing the Liouville equation
  - Around the critical bubble:

$$\phi = \phi_{\text{cb}} + \Delta\phi\,,$$

$$\nabla^2\phi_{\text{cb}} - V'_{\text{nucl}}(\phi_{\text{cb}}) = 0$$

  - Linde problem, $\lambda T/m_{\text{eff}}$
- Equilibrium in meta
  - Steady state

$$\int \mathrm{d}x\left[-\pi\frac{\delta\rho}{\delta\phi} + \left(\nabla^2 - V''(\phi_{\text{cb}})\right)\Delta\phi\frac{\delta\rho}{\delta\pi}\right] = 0$$

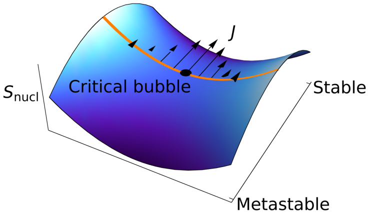


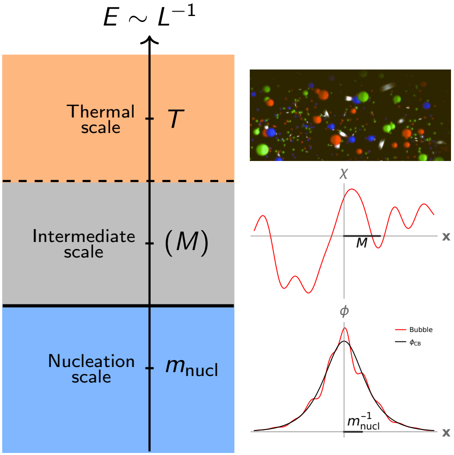
Figure: Critical bubble and nucleating flux in configuration space

Finding critical bubble [Gould & Hirvonen '21]

- Intermediate scale
  - Strong effects on nucleation
  - e.g. gauge bosons becoming heavy
- Integrate out to obtain V_{nucl}

$$\nabla^2 \phi_{\mathrm{cb}} - V'_{\mathrm{nucl}}(\phi_{\mathrm{cb}}) = 0$$

- Treated in equilibrium



- Statistical part
  - Exponential suppression, S_{CB}
  - Fluctuations around critical bubble
  - Boltzmann weighted nucleation area
- Dynamical part, κ
  - Exponential growth rate, $\phi = \phi_{\mathrm{CB}} + \delta\phi\, e^{\kappa t}$

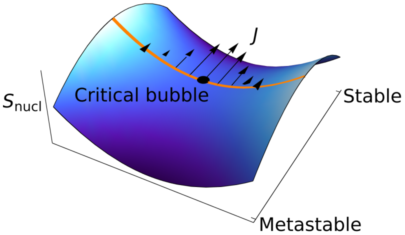
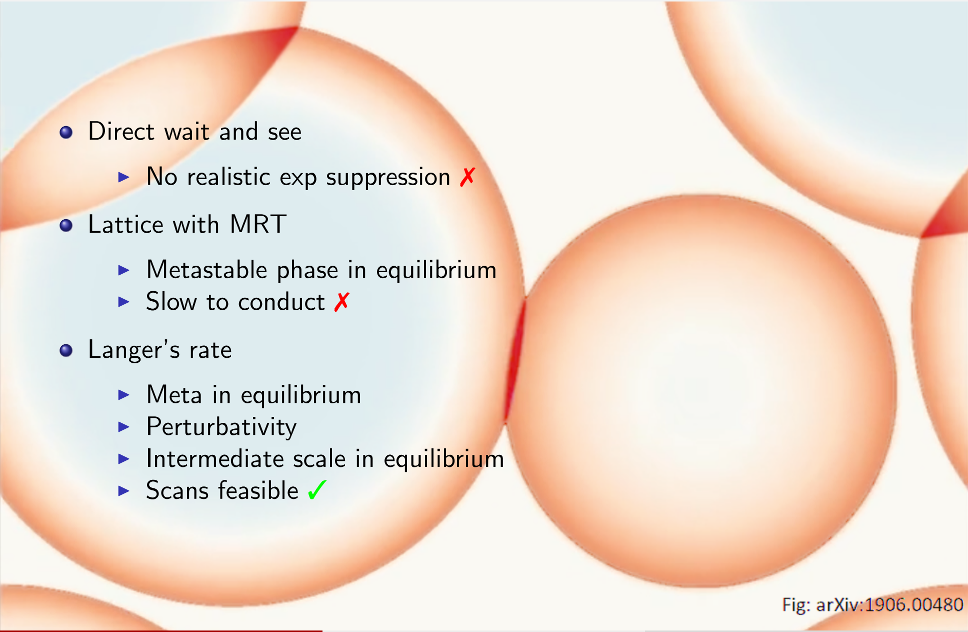


Figure: Critical bubble and nucleating flux in configuration space

$$\frac{\Gamma}{V} = \underbrace{\frac{\kappa}{2\pi}}_{\text{dyn.}} \underbrace{\left(\frac{S_{\mathrm{CB}}}{2\pi}\right)^{\frac{3}{2}} \left|\frac{\det\left(-\nabla^2 + V''_{\mathrm{meta}}\right)}{\det'\left(-\nabla^2 + V''_{\mathrm{CB}}\right)}\right|^{\frac{1}{2}}}_{\text{equilibrium fluctuations}} \underbrace{e^{-S_{\mathrm{CB}}}}_{\text{crit. bub.}}$$

Method summary



- Direct wait and see
  - No realistic exp suppression ✗
- Lattice with MRT
  - Metastable phase in equilibrium
  - Slow to conduct ✗
- Langer's rate
  - Meta in equilibrium
  - Perturbativity
  - Intermediate scale in equilibrium
  - Scans feasible ✓

Fig: arXiv:1906.00480

Bridging the gaps

- Analytics and MRT method
  - Computing full Langer's rate
  - Equilibrium intermediate scale

- Analytics and wait-and-see method
  - Off-equilibrium intermediate scale
  - Non-equilibrium metastable phase

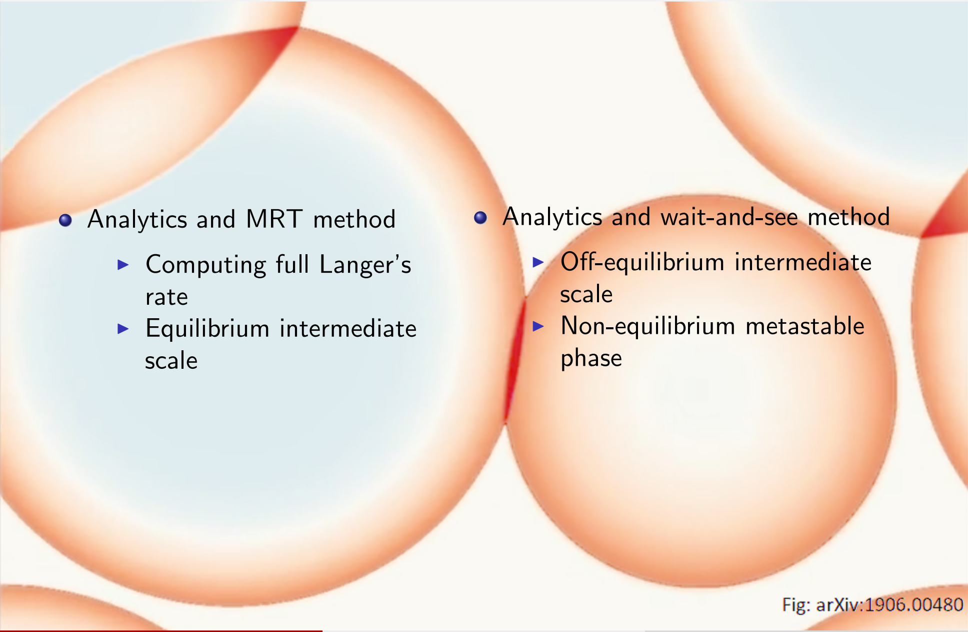


Fig: arXiv:1906.00480

Prefactor

$$\frac{\sqrt{|\lambda_-|}}{2\pi}\left|\frac{\det\left(-\nabla^2+V''_{\text{meta}}\right)}{\det'\left(-\nabla^2+V''_{\text{CB}}\right)}\right|^{\frac{1}{2}} \overset{?}{\sim} m^4_{\text{nucl}} \overset{!}{\not\approx} T^4$$

- Can be significant
- Can signal non-perturbativity
  - Linde problem
  - Thin-wall limit
    (see [Gould & Hirvonen '21])

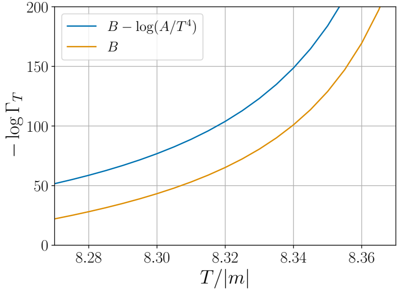


Figure: Action vs. dets [Ekstedt, Gould & Hirvonen '23]

$$\frac{\sqrt{|\lambda_-|}}{2\pi}\left|\frac{\det\left(-\nabla^2+V''_{\text{meta}}\right)}{\det'\left(-\nabla^2+V''_{\text{CB}}\right)}\right|^{\frac{1}{2}} \overset{?}{\sim} m^4_{\text{nucl}} \overset{!}{\not\approx} T^4$$

- Built-in `CosmoTransitions`
  Wainwright '11
- Only for one background field
- No mixing
  - Still order-of-magnitude estimate
- `bubbledet.readthedocs.io`
- Improvement requests to me and Oliver Gould

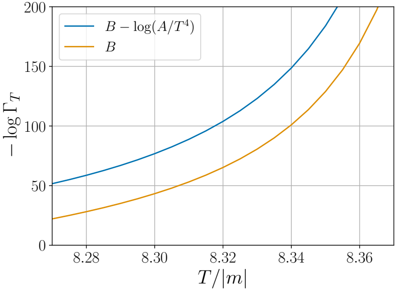


Figure: Action vs. dets [Ekstedt, Gould & Hirvonen '23]

Lattice and full Langer's rate [Gould, Kormu & Weir '25]

- Real singlet model, $V_{3d}(\phi) = \sum_{n=1}^{4} a_n \phi^n$, with MRT
- One loop drastic improvement, $\times e^{-20}$
- Nucleation temperature accurate

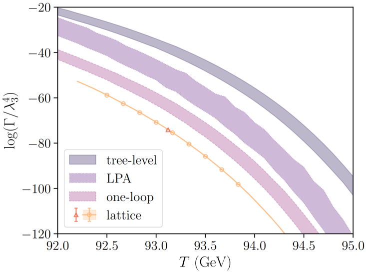


Figure: Nucleation rate and supercooling ($T_c \approx 98.5$GeV) [Gould, Kormu & Weir '25]

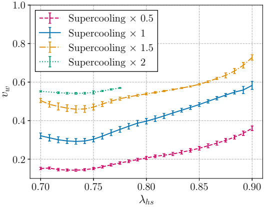
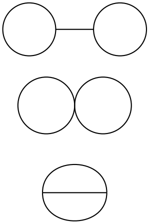


Figure: Wall velocities and supercooling [WallGo Investigates '25]

Beyond Langer

- Thermodynamic quantities need and have high precision
- Analytical results for rate [Ekstedt '22]
- How big effects, when important?
  - Comparison against lattice of [Gould, Kormu, Weir '24]



Mistreated intermediate scale?

- Ugly result
  - Missing a scale?
- Possibly
  - Found an intermediate scale in [Gould, Tenkanen '23]

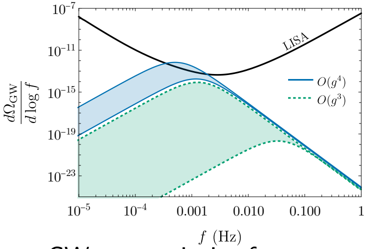


Figure: GW uncertainties from nucleation rates [Gould, Tenkanen '21]

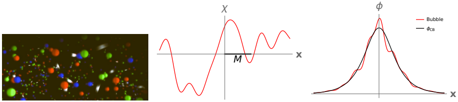
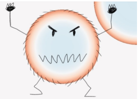


Figure: Scales

Intermediate scale can be grumpy



- Radiatively induced phase transition
  - Intermediate scale
- All scales known, seemingly pristine computation [Ekstedt '22]
  - Gauge invariant approach [Hirvonen et al '21, Löfgren et al '21]
  - Higher loop corrections worsen results
- Possibly due to the critical bubble requiring special care

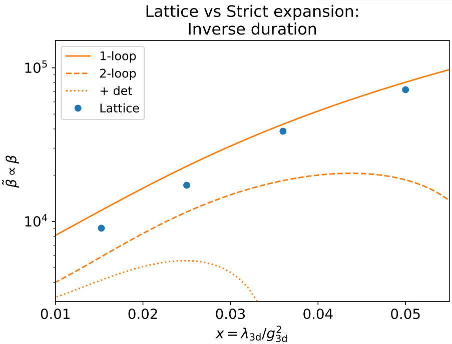


Figure: Inverse duration predictions

(Lattice: [Gould, Güyer, Rummukainen '22] & [Moore, Rummukainen '00])

Classically scale-invariant model: Different improvements

[Kierkla, Schicho, Świeżewska, Tenkanen, van de Vis (future)]

- Thermal scale to two loops
- Intermediate scale
  - Two-loop
  - Gradient expansion vs. fluctuation determinant
- Nucleation scale
  - Determinant (negligible)

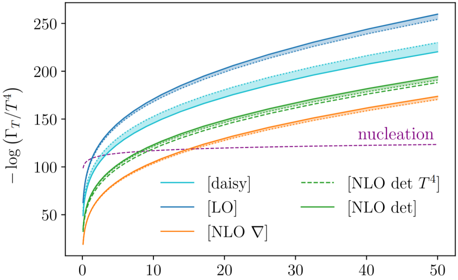


Figure: Nucleation rate improvements

Off-equilibrium intermediate scale

- Effects unknown
- Off-equilibrium thermal scale
  [Hirvonen '24]
  - Linearly infrared divergent
  - Intermediate scale sizable?
- Simplified comparison
  - 1+1 wait-and-see
  - Analytic counterpart

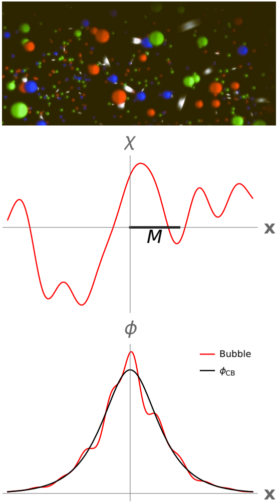
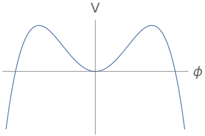


Figure: Thermal scales

Evidence for non-equilibrium dynamics [Pîrvu, Shkerin, Sibiryakov '24]



- Real scalar
  - 1+1 dimensions
  - $\hat{T}$: temperature
  - $\hat{\eta}$: noise and dissipation
- Wait and see does not assume equilibrium
  - Rate smaller to Langer, $\times 1/8$
- Gaussian initialization

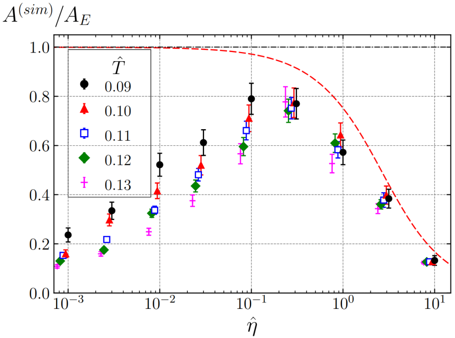


Figure: Dots: $\dfrac{\text{Measured rate}}{\text{Langer's rate with } \hat{\eta}=0}$ Red line: $\dfrac{\text{Langer's rate}}{\text{Langer's rate with } \hat{\eta}=0}$

Matching to Langer [Hirvonen & Gould '25]

- Same setup
  - Fully thermalized metastable phase
  - $\hat{\eta} = 0$
- First quantitative match!
  - Two-loop uncertainty
- Gaussian initialization wonky
  - Pertinent differences in low friction
  - High friction OK

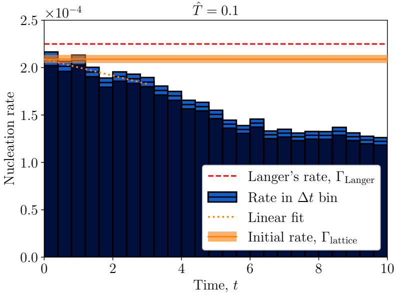


Figure: Nucleation rate in time

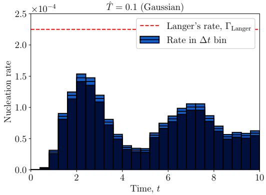


Figure: Gaussian initialization

Outlook of non-equilibrium rates

- Langer's rate found
- Deviation of $\times 1/8$ or $-\ln 8 \approx -2$
- Realistic cases
  - Suppression: $e^{-\mathcal{O}(10)} \to e^{-\mathcal{O}(100)}$
  - Tentatively closer to Langer

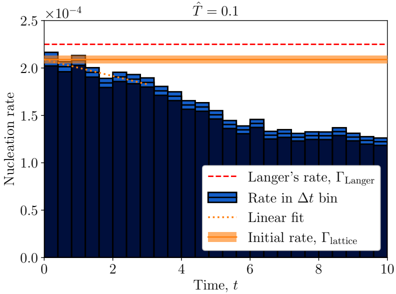
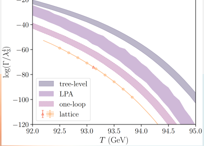
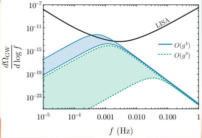


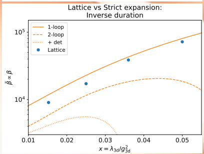
Figure: Nucleation rate in time

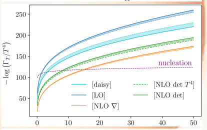
Summary

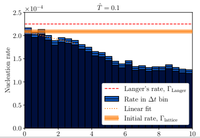
- EFTs backbone of modern computations
- One-loop rate at finger tips!
- Handling the intermediate scale in and out of equilibrium
- Langer's rate confirmed on the lattice, but rate time dependent
- Robust perturbative predictions and understanding in the horizon?





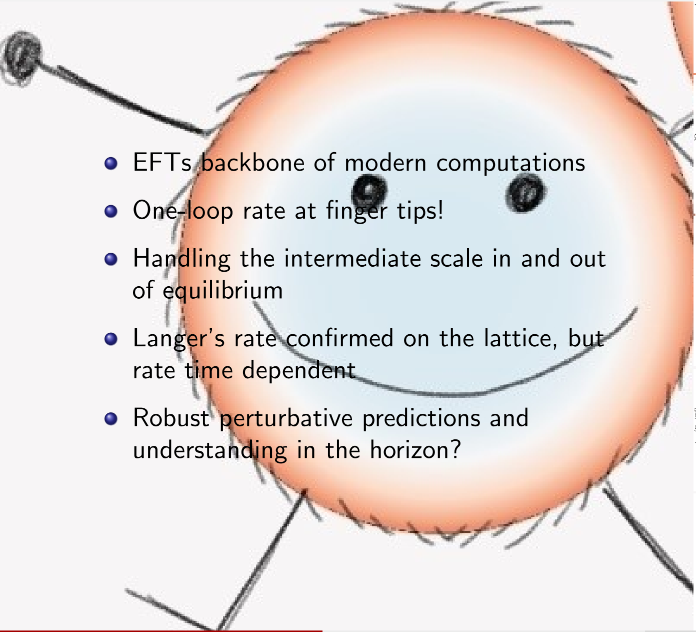






6.00480

Thanks for listening!



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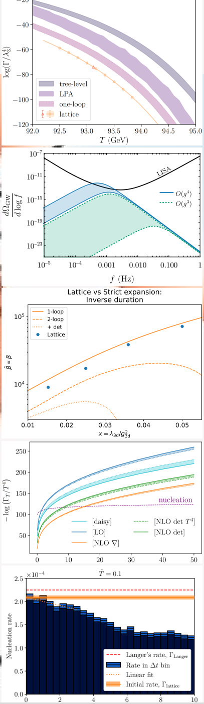

Local action term

$$\int \mathrm{d}^d\mathbf{x}\mathrm{d}^d\mathbf{y}\, \phi_{\mathrm{IR}}(\mathbf{x})\phi_{\mathrm{IR}}(\mathbf{y})\langle\chi(\mathbf{x})\chi(\mathbf{y})\rangle_{\mathrm{UV,0,c}}$$

$$= \int \mathrm{d}^d\mathbf{x}\mathrm{d}^d\mathbf{y}\, \phi_{\mathrm{IR}}(\mathbf{x})\phi_{\mathrm{IR}}(\mathbf{y}) \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2 + M^2}, \quad \mathbf{k} \text{ is an external momentum } \ll M$$

$$= \int \mathrm{d}^d\mathbf{x}\mathrm{d}^d\mathbf{y}\, \phi_{\mathrm{IR}}(\mathbf{x})\phi_{\mathrm{IR}}(\mathbf{y}) \sum_{n=0}^{\infty} \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \left(\frac{-p^2}{M^2}\right)^n$$

$$= \int \mathrm{d}^d\mathbf{x}\mathrm{d}^d\mathbf{y}\, \phi_{\mathrm{IR}}(\mathbf{x})\phi_{\mathrm{IR}}(\mathbf{y}) \sum_{n=0}^{\infty} \left(\frac{\boldsymbol{\nabla}_{\mathbf{x}}^2}{M^2}\right)^n \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

$$= \sum_{n=0}^{\infty} \int \mathrm{d}^d\mathbf{x}\mathrm{d}^d\mathbf{y}\, \frac{\boldsymbol{\nabla}_{\mathbf{x}}^{2n}\phi_{\mathrm{IR}}(\mathbf{x})}{M^{2n}} \phi_{\mathrm{IR}}(\mathbf{y}) \underbrace{\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}_{=\delta(\mathbf{x}-\mathbf{y})}$$

$$= \sum_{n=0}^{\infty} \int \mathrm{d}^d\mathbf{x} \frac{1}{M^{2n}}\, \phi_{\mathrm{IR}} \boldsymbol{\nabla}^{2n} \phi_{\mathrm{IR}}$$